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# Polylogarithms from diagrams with elliptic obstructions

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Numerical evaluations of 2-loop kites and 3-loop tadpoles with several elliptic obstructions lead to remarkable empirical evaluations in terms of polylogarithms, for which proofs are very hard to find, notwithstanding intensive use of packages such as HyperInt and MZIteratedIntegral. I describe the efficient methods by which puzzlingly simple results were obtained and hopes for demystifying them.

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## 1. Introduction

More than sixty years ago, the two-loop kite integral was found to have elliptic obstructions from 3-particle cuts. In particular, the two-loop electron propagator has a spectral function that contains the integral of an elliptic integral, tackled by Afaf Sabry [37] after work with Gunnar Källén [26] on the two-loop photon propagator. Thirty years later [13], I extended both results to univariate cases needed in unbroken gauge theories. For that work, dispersion relations [2, 5], pioneered by Hans Kramers and Ralph Kronig [28] and extended to particle physics by Murph Goldberger [24], were of the essence.

Closing the kite with a sixth propagator, one obtains a three-loop tetrahedral tadpole. In [14, 15, 21], I studied binary tadpoles, with unit or zero masses, discovering that all are reducible to multiple polylogarithms [11] of sixth roots of unity. These results were used by Matthias Steinhauser, whose computer-algebra package Matad [38] reduces every binary 3-loop tadpole, no matter what its subdivergences, to polylogarithmic constants.

In the present work, I give efficient dispersive methods for evaluating 2-loop kites and 3-loop tadpoles in generic multivariate cases. I also highlight 3 remarkable special cases in which totally massive tadpoles are empirically reduced to classical polylogarithms, notwithstanding the elliptic obstructions in their constituent kites.

This multivariate analysis is complementary to work on univariate $L$-loop equal-mass sunrise integrals [20, 30], with $L+1$ edges, in preparation for Stefano Laporta's result for the magnetic moment of the electron at 4 loops [31]. That subject was revolutionized by Spencer Bloch and Pierre Vanhove [7], who gave a modular parametrization of the 2-loop sunrise diagram in two spacetime dimensions. With Matt Kerr, they achieved a similar feat at 3 loops [8] where the Picard-Fuchs equation is a symmetric square of the equation at 2 loops, as had been observed by Geoff Joyce [25], long before, in the context of lattice Green functions for condensed matter problems. With David Bailey, Jon Borwein and Larry Glasser [4] I exploited this in elliptic evaluations of integrals of products of Bessel functions. Many conjectures [16–19] on Bessel moments were later proved by Yajun Zhou [39–43]. Further analysis in [1, 22, 27, 29, 35, 36] led to understanding of elliptic polylogarithms in 2-point functions with intermediate states containing at least 3 massive particles.

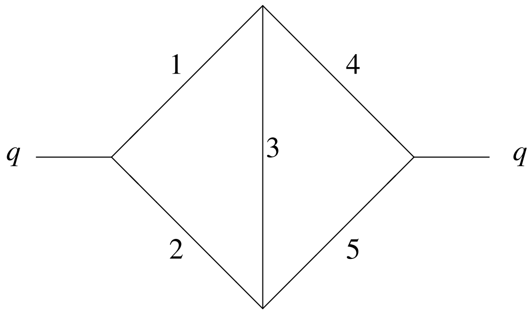
The remainder of this article is organized as follows. Section 2 reduces the generic kite to a single integral of the derivative $\sigma'(w^2)$ of the discontinuity of the kite with external energy $w$, against a logarithm, and also reduces the generic tadpole to a single integral of $\sigma'$, against a dilogarithm. Then the derivative $\sigma'$ is decomposed into three parts, with logarithmic terms, from 2-particle intermediate states, elliptic terms, from 3-particle intermediate states, and (possibly) an algebraic term arising from anomalous thresholds [2] in triangles within the kite. These results enable evaluations of kites and tadpoles to 100 digits in a second and to 600 digits in a minute. Section 3 gives remarkable empirical determinations of the finite parts of totally massive tadpoles in terms of classical polylogarithms, for which there was little expectation and there is still no proof. Section 4 offers comments and conclusions.



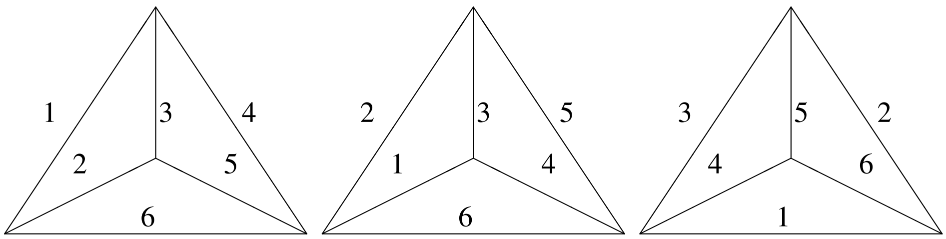
PoS(LL2024)086

## 2. Dispersive methods for kites and tadpoles

Consider the generic 2-loop scalar kite with 5 internal masses




and close the kite by integrating it against a sixth propagator $1/(q^2 - m_6^2)$ to obtain a tadpole




tadpole same tadpole same tadpole

with the symmetry group $S_4$ of the tetrahedron giving 12 elliptic obstructions from constituent kites with 3-particle intermediate states. The tadpole has a logarithmic divergence, regulated in $D = 4 - 2\varepsilon$ dimensions to give

$$T^{5,4,6}_{1,2,3} = \left(\frac{1}{3\varepsilon} + 1\right) 6\zeta_3 + 3\zeta_4 - F^{5,4,6}_{1,2,3} + O(\epsilon) \tag{1}$$

with a finite part $F$ that depends on the six ratios $m_k/\mu$, where $\mu$ is the scale of dimensional regularization. The rescaling $m_k \to \lambda m_k$ gives $F \to F + 12\zeta_3 \log(\lambda)$. Without loss of generality, choose $m_6$ to be the largest mass and set $\mu = m_6 = 1$.

With $\mu = m_6 = 1$, Schwinger parametrization gives the 5-dimensional integral

$$F^{5,4,6}_{1,2,3} = \int_0^\infty \mathrm{d}x_1 \ldots \int_0^\infty \mathrm{d}x_5 \, \frac{1}{U^2} \log\left(1 + \sum_{k=1}^{5} x_k m_k^2\right) \tag{2}$$

after setting $x_6 = 1$ in the Symanzik polynomial of the tetrahedron

$$\begin{aligned} U = {} & x_3(x_1x_2 + x_4x_5) + x_6(x_1x_4 + x_2x_5) + x_3x_6(x_1 + x_2 + x_4 + x_5) \\ & + x_2x_4(x_1 + x_3 + x_5 + x_6) + x_1x_5(x_2 + x_3 + x_4 + x_6). \end{aligned} \tag{3}$$

It may then be reduced to a single integral of a dilogarithm against the derivative of the discontinuity

$I(s + \mathrm{i}\epsilon) - I(s - \mathrm{i}\epsilon) = 2\pi\mathrm{i}\sigma(s)$ of a kite integral:

$$F^{5,4,6}_{1,2,3} = -\int_{s_0}^{\infty} \mathrm{d}s\, \sigma'(s)\, \mathrm{Li}_2(1-s), \tag{4}$$

$$I(q^2) = -\frac{q^2}{\pi^4}\int \mathrm{d}^4 l \int \mathrm{d}^4 k \prod_{j=1}^{5} \frac{1}{p_j^2 - m_j^2 - \mathrm{i}\epsilon} = \int_{s_0}^{\infty} \mathrm{d}s\, \sigma'(s) \log\left(1 - \frac{q^2}{s}\right), \tag{5}$$

$$(p_1,\, p_2,\, p_3,\, p_4,\, p_5) = (l,\, l-q,\, l-k,\, k,\, k-q), \tag{6}$$

$$s_0 = \min(s_{1,2}, s_{4,5}, s_{2,3,4}, s_{1,3,5}), \quad s_{j,k} = (m_j + m_k)^2, \quad s_{i,j,k} = (m_i + m_j + m_k)^2. \tag{7}$$

### 2.1 Non-elliptic contribution

The non-elliptic contribution from 2-particle intermediate states has the form

$$\sigma'_{\mathrm{N}}(s) = \Theta(s - s_{1,2})\sigma'_{1,2}(s) + \Theta(s - s_{4,5})\sigma'_{4,5}(s). \tag{8}$$

Denote the square root of the symmetric Källén function by

$$\Delta(a,b,c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)} \tag{9}$$

with abbreviations $\Delta_{j,k}(s) = \Delta(s, m_j^2, m_k^2)$ and $\Delta_{i,j,k} = \Delta_{j,k}(m_i^2)$. Then

$$D_{j,k}(s) = \frac{r}{s - (m_j - m_k)^2} \log\left(\frac{1+r}{1-r}\right), \quad r = \left(\frac{s - (m_j - m_k)^2}{s - (m_j + m_k)^2}\right)^{1/2} \tag{10}$$

provides the logarithms in

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) = \Re\left((s+\alpha)D_{4,5}(s) + L_{4,5} + \sum_{i=0,+,-} C_i \frac{D_{4,5}(s) - D_{4,5}(s_i)}{s - s_i}\right) \tag{11}$$

with constants

$$C_0 = -(m_1^2 - m_2^2)(m_4^2 - m_5^2), \quad C_\pm = \alpha s_\pm + \beta, \quad L_{4,5} = \log\left(\frac{m_4 m_5}{m_3^2}\right), \tag{12}$$

$$\alpha = \frac{(m_1^2 - m_4^2)(m_2^2 - m_5^2)}{m_3^2} - m_3^2, \quad \beta = \frac{(m_1^2 m_5^2 - m_2^2 m_4^2)(m_1^2 - m_2^2 - m_4^2 + m_5^2)}{m_3^2}, \tag{13}$$

$$s_0 = 0, \quad s_\pm = \frac{m_1^2 + m_2^2 - 2m_3^2 + m_4^2 + m_5^2 - \alpha}{2} \pm \frac{\Delta_{1,3,4}\Delta_{2,3,5}}{2m_3^2}, \tag{14}$$

where $s_\pm$ locate leading Landau singularities of triangles that form the kite.

### 2.2 Elliptic contribution

This comes from 3-particle intermediate states, giving

$$\sigma'_{\mathrm{E}}(s) = \Theta(s - s_{2,3,4})\sigma'_{2,3,4}(s) + \Theta(s - s_{1,3,5})\sigma'_{1,3,5}(s). \tag{15}$$

It contains complete elliptic integrals of the third kind of the form

$$P(n,k)=\frac{\Pi(n,k)}{\Pi(0,k)},\quad \Pi(n,k)=\int_0^{\pi/2}\frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{16}$$

with $\Pi(0,k)=(\pi/2)/\mathrm{AGM}\,(1,\sqrt{1-k^2})$ given by an arithmetic-geometric mean [10, 12].

With $s=w^2$, an integration over the phase space of particles 2, 3 and 4 determines

$$k^2=1-\frac{16m_2m_3m_4w}{W},\quad W=(w_+^2-m_+^2)(w_-^2-m_-^2) \tag{17}$$

with $w_\pm=w\pm m_2$ and $m_\pm=m_3\pm m_4$. Then I obtain

$$\sigma'_{2,3,4}(w^2)=\frac{4\pi m_3m_4}{\mathrm{AGM}\,(\sqrt{16m_2m_3m_4w},\sqrt{W})}\Re\left(\sum_{i=+,-}E_i\frac{P(n_i,k)-P(n_1,k)}{t_i-t_1}\right) \tag{18}$$

with coefficients and arguments given, as compactly as possible, by

$$E_\pm=\frac{m_2^2-m_3^2+m_5^2}{2m_5^2}\pm\left(\frac{m_4^2-m_5^2-w^2}{2m_5^2}\right)\frac{\Delta_{2,3,5}}{\Delta_{4,5}(w^2)}, \tag{19}$$

$$t_\pm=\frac{\gamma\pm\Delta_{2,3,5}\Delta_{4,5}(w^2)}{2m_5^2},\quad t_1=m_1^2,\quad n_i=\frac{(w_-^2-m_+^2)(t_i-m_-^2)}{(w_-^2-m_-^2)(t_i-m_+^2)}, \tag{20}$$

$$\gamma=(m_2^2+m_3^2+m_4^2-m_5^2+w^2)m_5^2+(m_2^2-m_3^2)(m_4^2-w^2). \tag{21}$$

An AGM procedure speedily evaluates $P(n,k)=\Pi(n,k)/\Pi(0,k)$ to high precision.

1. Initialize $[a,\,b,\,p,\,q]=[1,\,\sqrt{1-k^2},\,\sqrt{1-n},\,n/(2-2n)]$. Then set $f=1+q$.
2. Set $m=ab$ and then $r=p^2+m$. Replace $[a,\,b,\,p,\,q]$ by a vector of new values as follows: $[(a+b)/2,\,\sqrt{m},\,r/(2p),\,(r-2m)q/(2r)]$. Add $q$ to $f$.
3. If $|q/f|$ is sufficiently small, return $P(n,k)=f$, else go to step 2.

On the cut with $n\geq 1$, the principal value is $\Re P(n,k)=1-P(k^2/n,k)$.

### 2.3 Criterion for an anomalous contribution

Suppose that $s_{4,5}\geq s_{1,2}$. Then

$$\sigma'(s)=\sigma'_{\mathrm{N}}(s)+\sigma'_{\mathrm{E}}(s)+C_{\mathrm{A}}\frac{\Theta(s-s_{4,5})}{\Delta_{4,5}(s)}\Re\left(\frac{2\pi\mathrm{i}\Delta_{4,5}(s_-)}{s-s_-}\right) \tag{22}$$

with $C_{\mathrm{A}}\neq 0$ if and only if $(m_1+m_2)(m_3^2+m_1m_2)<m_1m_5^2+m_2m_4^2$ and at least one of $\Delta_{1,3,4}$ and $\Delta_{2,3,5}$ is imaginary, in which case $C_{\mathrm{A}}=\pm1$ is the sign of $\Im\Delta_{4,5}(s_-)$.

This value of $C_{\mathrm{A}}\in\{0,1,-1\}$ is determined by the high-energy behaviour

$$s^2\sigma'(s)=2L_3+\sum_{k=1,2,4,5}(L_k+m_k^2)+O\left(\frac{\log(s)}{s}\right),\quad L_k=m_k^2\log(s/m_k^2). \tag{23}$$



PoS(LL2024)086

### 2.4 Stringent tests for kites and tadpoles

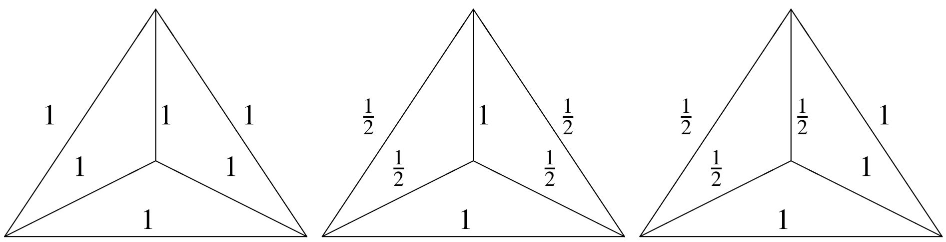
1. Elliptic terms do not depend on the order of phase-space integrations [23].
2. The derivative of the discontinuity of a kite satisfies the sum rule

$$\int_{s_0}^{\infty} \mathrm{d}s\, \sigma'(s) \log\left(\frac{s}{s_0}\right) = 6\zeta_3. \tag{24}$$

3. The high-energy behaviour (23) of $\sigma'(s)$ holds irrespective of anomalous thresholds.
4. Benchmarks for kites given by Stefan Bauberger and Manfred Böhm [6], to 6 decimal digits, and by Stephen Martin [32], to 8 decimal digits, are confirmed and then extended to 100 digits in less than a second.
5. The same tadpole is obtained by integrating over 6 distinct kites.
6. Binary tadpoles with $m_k \in \{0, 1\}$ agree with my previous reductions to poloylogs of sixth roots of unity [15].

## 3. Surprising reductions to polylogarithms

When all 6 masses are non-zero, there is no non-elliptic route. Yet in 3 cases, I found empirical reductions to polylogarithms, with masses as below.



binary perfect imperfect

### 3.1 A binary surprise

Dressings of the tetrahedron with zero or unit masses give rational linear combinations of 4 constants: $\zeta_4 = \pi^4/90$, $\mathrm{Cl}_2^2(\pi/3)$, $U_{3,1}$ and $V_{3,1}$, with $\mathrm{Cl}_2(\pi/3) = \Im\mathrm{Li}_2(\lambda)$, $\lambda = (1+\sqrt{-3})/2$, and reducible double sums

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \tfrac{1}{2}\zeta_4 + \tfrac{1}{2}\zeta_2 \log^2(2) - \tfrac{1}{12}\log^4(2) - 2\,\mathrm{Li}_4(\tfrac{1}{2}), \tag{25}$$

$$V_{3,1} = \sum_{m>n} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n} = -\tfrac{145}{432}\zeta_4 + \tfrac{1}{8}\zeta_2 \log^2(3) - \tfrac{1}{96}\log^4(3) + \tfrac{1}{32}\mathrm{Li}_4(\tfrac{1}{9}) - \tfrac{3}{4}\mathrm{Li}_4(\tfrac{1}{3}) + \tfrac{1}{3}\mathrm{Cl}_2^2(\pi/3). \tag{26}$$

With 5 unit masses, there was a non-elliptic route to my result

$$F^{(1,1,1)}_{(1,1,0)} = \tfrac{550}{27}\zeta_4 + 16V_{3,1} - \tfrac{8}{3}\mathrm{Cl}_2^2(\pi/3) \tag{27}$$

which Yajun Zhou and I proved, using `HyperInt` from Erik Panzer [33, 34]. More surprising is my very simple empirical result for the totally massive case

$$F^{(1,1,1)}_{(1,1,1)} \stackrel{?}{=} 16\zeta_4 + 8U_{3,1} + 4\mathrm{Cl}_2^2(\pi/3). \tag{28}$$

The closest we got to a proof was a double integral of products of logs for which `HyperInt` gave 1300 multiple polylogarithms of 12th roots of unity. We used `MZIteratedIntegral` from Kam Cheong Au [3] to handle 12th roots, yet still fell far short of proving (28).

### 3.2 A perfect surprise

I investigated a perfect tetrahedron with $\Delta_{i,j,k} = 0$ at all 4 vertices, eliminating all square roots. Here I also found an empirical reduction to classical polylogarithms, with help from Steven Charlton. Promoting subscripts and superscripts to masses values, I conjecture that, with $L = \log(2)$,

$$F^{(\frac{1}{2},\frac{1}{2},1)}_{(\frac{1}{2},\frac{1}{2},1)} \stackrel{?}{=} B = 6\left(2\zeta_4 - 3\mathrm{Li}_4(\tfrac{1}{4})\right) + 8\left(2\zeta_3 - 3\mathrm{Li}_3(\tfrac{1}{4})\right)L - 12\,\mathrm{Li}_2(\tfrac{1}{4})L^2 - 4L^4. \tag{29}$$

This is equivalent to an evaluation in classical polylogarithms of the integral of a trilog against complete elliptic integrals of the first and second kinds:

$$K(k) = \int_0^{\pi/2} (1 - k^2\sin^2\theta)^{-1/2}\mathrm{d}\theta, \quad E(k) = \int_0^{\pi/2} (1 - k^2\sin^2\theta)^{1/2}\mathrm{d}\theta, \tag{30}$$

$$Z(y) = \frac{y(1+y)K(k) + E(k)}{(1+y+y^2)\sqrt{1+y}}, \quad k^2 = 1 - y^3, \tag{31}$$

$$T(y) = \mathrm{Li}_3(u) - \tfrac{1}{2}\mathrm{Li}_2(u)\log(u), \quad u = \frac{y}{(1+y)^2}, \tag{32}$$

$$4\int_0^1 \mathrm{d}y\left(\frac{1}{y} - 1\right)T(y)Z(y) \stackrel{?}{=} B + 16\,\zeta_4 + 32\,U_{3,1} - 30\,\zeta_3\log(2). \tag{33}$$

### 3.3 A third surprise

In an imperfect case, I found empirically that

$$F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} \stackrel{?}{=} 10\zeta_4 - 4U_{3,1} + 10\mathrm{Cl}_2^2(\pi/3) + 3\zeta_3\log(2) - \tfrac{1}{2}B \tag{34}$$

also has a remarkable reduction to classical polylogarithms.

Combining the perfect and imperfect cases, I arrive at the conjecture

$$4\int_2^\infty \frac{\mathrm{d}w}{w}\left(\mathrm{Li}_2\left(1 - \frac{1}{w^2}\right) - \zeta_2\right)Y(w) \stackrel{?}{=} \zeta_4 - 4U_{3,1} + 7\zeta_3\log(2), \tag{35}$$

$$Y(w) = \frac{\Pi(0,k) - \Pi(n,k) - 6\,\Pi(\widehat{n},k)}{(w-1)\sqrt{w^2+2w}}, \tag{36}$$

$$k^2 = 1 - \frac{4}{(w-1)^2(w+2)}, \quad n = 1 - \frac{1}{(w-1)^2}, \quad \widehat{n} = 1 - \frac{2}{w(w-1)}, \tag{37}$$

with an integral of a dilogarithm against complete elliptic integrals of the third kind reduced to classical polylogarithms in a spectacularly simple result.

## 4. Comments and conclusions

I conclude with an open question: can totally massive 3-loop tadpoles be reduced to polylogarithms? A conservative answer is that such reductions are (surprisingly) possible in special cases, yet unlikely in the generic case of 6 distinct non-zero masses. Until we develop novel analytical methods to prove the three empirically reducible cases (28,29,34), speculation on the limits of reducibility seems premature. By way of hope for further progress, I remark that the Schwinger parametrization, in (2,3), does not look too frightening. It would be instructive to learn what transformations of variables lead to the quadrilogarithms in the alphabets so far discovered. A parallel strategy would be to transform the tadpoles to double integrals of products of logs with complicated arguments. Then obstructing square roots of quartics might be rationalized by a pair of Euler substitutions, as was achieved for the case (28) with 6 equal masses, where reduction to many multiple polylogarithms of level 12 is now proven, though the final far simpler answer seems to lie beyond the reach of packages such as `HyperInt` and `MZIteratedIntegral`. As so often in perturbative quantum field theory, exact answers are easier to guess, from numerical computation, than to prove by analysis.

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