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Auxiliary mass flow method for master integrals around non-analytic points

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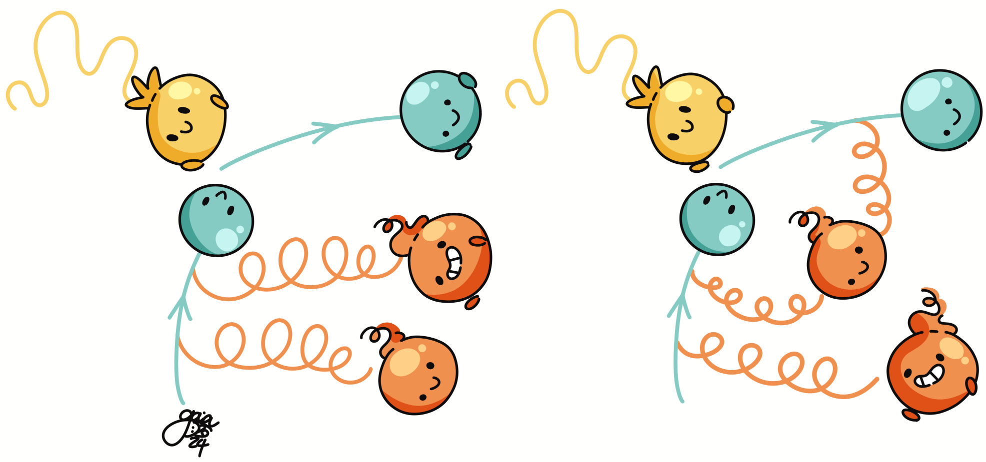
The calculation of phase-space integrals via reverse unitarity and differential equations often faces bottlenecks in fixing boundary conditions. In this talk we present a general and analytical method to derive boundary conditions for phase-space master integrals. Our strategy is based on the auxiliary mass flow method (AMFlow), but it is purely analytic. It is suited for the calculation of boundary conditions near the non-analytic endpoint region of phase space integrals, where a numerical approach is not feasible. We present some applications to DIS-like phase space integrals at two-loops.



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1. Introduction

The method of differential equations [1–8] is one of the most efficient strategies developed to evaluate Feynman integrals and is ubiquitous in state-of-the-art calculations. It consists of deriving a system of differential equation for the master integrals with respect to internal masses or external invariants. The resulting system of differential equations is then solved (analytically, numerically, or with hybrid approaches) to obtain the desired evaluation of the integrals. Independently of the strategy used to solve the system, two steps are always necessary: i) to find a generic solution by integrating the differential equation and ii) to match the result to some boundary conditions. In this talk we focus on the second step, i.e. how to fix boundary conditions, especially in an analytic and algorithmic way. In general, boundary conditions are obtained by first exploiting consistency conditions at the level of the differential equation. This is usually not sufficient to fix all of them, leaving some to be fixed by direct evaluation of some integrals in certain kinematical limits. The calculation of boundary conditions for phase-space integrals faces a bottleneck in the latter step, since it is usually the case that interesting kinematical limits, where the integrals become simple, correspond to singular points. Moreover, a general and algorithmic strategy for the direct evaluation of phase space integrals does not exists, causing to resort to dedicated phase-space parametrizations depending on the process or on the type of integrals in consideration [9].

In this talk we present an algorithmic method for the evaluation of phase-space integrals in a given kinematical limit, based on the popular Auxiliary Mass Flow (AMFlow) [10–12] strategy, but yielding analytic results, therefore named Analytic Auxiliary Mass Flow (AAMFlow) [13]. We apply it to the real NNLO corrections to DIS phase-space integrals, a necessary ingredient for the calculation of NNLO initial-final antenna functions [14–17], that, thanks to this strategy, have been re-calculated and extended to higher order in the dimensional regulator.

2. Kinematics and notation

We consider the real emission NNLO corrections to DIS phase-space integrals. The kinematics is as follows

$$\begin{gathered} q_1 + q_2 \to p_1 + p_2 (+p_3), \\ q_1^2 = 0, \quad q_2^2 = -Q^2 < 0, \quad p_i^2 = 0 \quad (i = 1,2,3). \end{gathered} \tag{1}$$

We have two contributions, namely the double-real (RR) emission integrals (where p_3 is present) and the one loop real emission integrals, real virtual (RV). Thanks to reverse unitarity [18–20], we can map the phase-space integrals to cuts of loop integrals in forward kinematics. We obtain the following structure for RR integrals

$$I_{\mathrm{RR}} = (-i)^3 \int \prod_{i=1}^{3} \left(\frac{\mathrm{d}^d p_i}{(2\pi)^d} \frac{1}{\not{p}_i{}^2} \right) (2\pi)^d \, \delta\left(q_1 + q_2 - \sum_{i=1}^{3} p_i \right) \prod_{j \in \mathrm{uncut}} \frac{1}{D_j^{\alpha_j}}, \quad \alpha_j \in \mathbb{Z}, \tag{2}$$

where we indicate a cut propagator with the slashed notation $\not{p}_i{}^2$. RR integrals present 3 cut propagators, corresponding to the 3 final state particles of the starting phase-space integral. For the

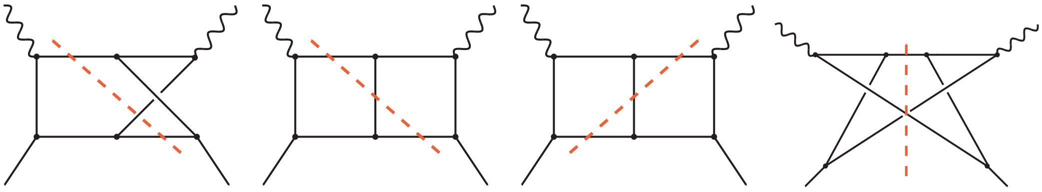

Figure 1: Top sectors of the RR integral families.

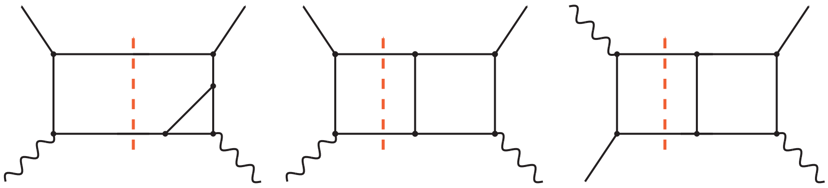

Figure 2: Top sectors of the RV integral families.

RV-integrals we have

$$I_{\text{RV}} = (-i)^2 \int \prod_{i=1}^{2} \left(\frac{\mathrm{d}^d p_i}{(2\pi)^d} \frac{1}{\not{p}_i^{\,2}} \right) (2\pi)^d \, \delta \left(q_1 + q_2 - \sum_{i=1}^{3} p_i \right) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \prod_{j \in \text{uncut}} \frac{1}{D_j^{\alpha_j}}, \; \alpha_j \in \mathbb{Z}, \quad (3)$$

with 2 cut propagators, corresponding to the 2 final state particles of the original phase-space integral, and one pure loop integration. For convenience, we set $Q^2 = 1$ and we express the integrals as functions of only one dimensionless variable z

$$z \equiv \frac{Q^2}{2q_1 \cdot q_2}. \quad (4)$$

3. Differential equation

We obtain the integral families by finding the physical 2−(for RV) and 3− (for RR) particle cuts of the inclusive forward DIS scattering amplitude at 2 loops [21]. The top sectors of the 4 RR families are depicted in Fig. 1, while the ones for the 3 RV families are presented in Fig. 2. We reduce the families to master integrals using `Kira` [22]. We set up the systems of differential equations in the variable z for the master integrals of RR and RV families. We put them in canonical form [6] using `Fuchsia` [23] and we find a generic solution in terms of Harmonic Polylogarithms [24].

3.1 Boundary conditions

We employ two strategies to fix the boundary conditions. First, we minimize the ones to be actually calculated by exploiting the known behaviour of the integrals in their soft limit $z \to 1$. It corresponds to a singular point of the integral, regulating soft initial-state singularities, and in this limit the integrals can be expanded as follows

$$I_i^{\text{RR}} \sim (1-z)^{n_i - 2\epsilon} \sum_{j \geq 0} c_{ij}(\epsilon)(1-z)^j \,,$$

$$I_i^{\mathrm{RV}} \sim (1-z)^{m_i-\epsilon} \sum_{j\geq 0} d_{ij}(\epsilon)(1-z)^j + (1-z)^{l_i-2\epsilon} \sum_{j\geq 0} e_{ij}(\epsilon)(1-z)^j, \tag{5}$$

with $n_i, m_i, l_i \in \mathbb{Z}$. In here the boundary terms correspond to the first terms in the Tayor expansions, namely $c_{i0}(\epsilon), d_{i0}(\epsilon)$ and $e_{i0}(\epsilon)$. For RR integrals, we factor out each power of $(1-z)^{n_i-2\epsilon}$ and we impose the cancellation of terms proportional to $(1-z)^{-j}, \log^j(1-z), (j \geq 1)$ in the limit $z \to 1$. For the RV case, in each branch of the expansion, we factor out the corresponding factor of $(1-z)^{-\epsilon}$ or $(1-z)^{-2\epsilon}$ and we impose the vanishing of terms proportional to $\log^j(1-z), (j \geq 1)$. This produces a system of equations for the boundaries $c_{i0}(\epsilon), d_{i0}(\epsilon)$ and $e_{i0}(\epsilon)$, whose solution gives the independent boundaries to be calculated.

4. Auxiliary mass strategies

We calculate the independent boundary conditions by explicit evaluation of the integrals in the soft limit $z \to 1$. To do this, we employ a strategy built on the introduction of an auxiliary mass, a technique of which we give a brief account in the next paragraph.

4.1 AMFLow

The original formulation of the auxiliary mass flow technique can be found in [10–12]. It has recently been systematized and extended in [10, 25–28], thus becoming the known AMFlow method. Its main points be traced as follows

- `INPUT:` integral family that needs to be calculated, characterized by a set of physical master integrals, $\vec{M}$. All the kinematic invariants and the dimensional regulator are set to numerical values.

  1. Introduction of an auxiliary family obtained by modifying the original one by adding an auxiliary mass η to some propagators. The auxiliary family is characterized by a new, larger set of master integrals, $\vec{M}_\eta$.
  2. Derivation of a system of differential equations for the auxiliary master integrals with respect to η

     $$\frac{d}{d\eta}\vec{M}_\eta = A_\eta \cdot \vec{M}_\eta, \tag{6}$$

     that is then solved numerically.
  3. Fixing the constants of integration in the large-mass limit, $\eta \to \infty$ [29, 30].
  4. A flow to vanishing auxiliary mass $\eta \to 0$ to recover the physical solution is performed. It consists of the set up of consecutive series expansions from large values of η to $\eta \to 0$ that are matched in overlapping regions of convergence.

- `OUTPUT:` high-precision numerical evaluation of $\vec{M}$ at a given kinematical point.

The application of this method is limited to non-singular points. We develop an analytic strategy that can be used also near singular points and we employ it to the evaluation of RR and RV phase-space integrals in the singular soft limit.

4.2 Analytic AMFLow

For simplicity, we describe the application of AAMFlow to RR integrals in the soft limit. A more comprehensive and general treatment can be found in [13]. The list of RR integrals that are needed for the boundary conditions is our starting point. To some of their propagators, we add the auxiliary mass, thereby obtaining auxiliary RR families. We highlight that, to have a fully-analytic method, we introduce the auxiliary mass in a way that the two limits we take in the auxiliary mass ($\eta \to 0$ and $\eta \to \infty$) commute with the soft limit $z \to 1$. After reducing the auxiliary families to master integrals, we derive a system of differential equations with respect to the auxiliary mass η. We then expand the master integrals in their soft limit, as described in eq. (5), obtaining a differential equation for the coefficients $c_{ij}^{\text{aux}}(\eta, \epsilon)$ of the soft-limit expansion:

$$\frac{d}{d\eta}\vec{c}^{\,\text{aux}} = A(\eta, \epsilon) \cdot \vec{c}^{\,\text{aux}} \,, \tag{7}$$

that we can solve by decoupling. We then fix the constant of integration in the large-mass limit, noting that, in the case of RR integrals, the loop momenta can scale only soft with respect to the large auxiliary mass. This means that all the propagators containing an auxiliary mass are substituted, in this limit, by a factor $-1/\eta^2$. Thanks to this, the large-mass expansion of the RR auxiliary master integrals reduces to known integrals. After this step, we have a full solution for the needed auxiliary boundary terms, $\vec{c}^{\,\text{aux}}(\eta, \epsilon)$.

The last step consists of the analytic flow to vanishing auxiliary mass, to recover the physical boundary conditions. This limit is described by an asymptotic expansion, namely

$$c_{ij}^{\text{aux}}(\eta, \epsilon) = c_{ij}^{\text{phys}}(\epsilon) + \eta^{-\epsilon} d_{ij,1}(\epsilon) + \eta^{-2\epsilon} d_{ij,2}(\epsilon) + O(\eta) \tag{8}$$

that is due to the scaling of the loop momenta with respect to the small auxiliary mass. The physical, or hard, region corresponds to $c_{ij}^{\text{phys}}(\epsilon)$. To recover the physical boundary conditions we have to fix all the needed coefficients of its $\epsilon-$expansion, $c_{ij}^{\text{phys}}(\epsilon) = \sum c_{ij}^{\text{phys},k}\,\epsilon^k$. The ϵ-expansion of this limit up to order $O(\epsilon^3)$ gives

$$\begin{aligned} c_{ij}^{\text{aux}} &= c_{ij}^{\text{phys},0} + d_{ij,1}^0 + d_{ij,2}^0 \\ &+ \epsilon \left(c_{ij}^{\text{phys},1} + \left(-d_{ij,1}^0 - 2d_{ij,2}^0\right)\log(\eta) + d_{ij,1}^1 + d_{ij,2}^1 \right) \\ &+ \epsilon^2 \left(c_{ij}^{\text{phys},2} + d_{ij,1}^2 + d_{ij,2}^2 + \frac{1}{2}\left(d_{ij,1}^0 + 4d_{ij,2}^0\right)\log^2(\eta) + \left(-d_{ij,1}^1 - 2d_{ij,2}^1\right)\log(\eta) \right) \\ &+ O(\epsilon^3). \end{aligned} \tag{9}$$

(where we assumed that the starting point of the expansion was at order $O(\epsilon^0)$). We can compare this limit with the naive expansion around $\eta \to 0$ of the known solution of the auxiliary RR boundary conditions, which reads

$$c_{ij}^{\text{aux}}(\eta, \epsilon) = \sum_{k=\min}^{\infty} \epsilon^k \left[r_{k,0} + \sum_{m=1}^{k} r_{k,m} \log^m(\eta) \right] , \tag{10}$$

leaving implicit the dependence of $r_{k,m}$ ($m \geq 0$) on i and j. In this form, all the terms in this expansion are known. We can set up a linear system of equations to relate the coefficients of the

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same order in ϵ and with the same log power. In this way we can, for example, fix the order $O(\epsilon^0)$ hard-region coefficient $c_{ij}^{\mathrm{phys},0}$ as

$$\begin{cases} -d_{ij,1}^{0} - 2d_{ij,2}^{0} = r_{1,1} \,, \\ d_{ij,1}^{0}/2 + 2d_{ij,2}^{0} = r_{2,2} \,, \\ c_{ij}^{\mathrm{phys},0} + d_{ij,1}^{0} + d_{ij,2}^{0} = r_{0,0} \,. \end{cases} \tag{11}$$

and we can repeat the procedure to fix all the ϵ-orders that we need, $c_{ij}^{\mathrm{phys},k}$. The same procedure, albeit a bit more involved, can be applied to RV integrals. The analytic results for RR and RV master integrals, as well for the integrated initial-final antenna functions are available in [13].

5. Conclusions

In this talk I presented an analytic variation on the AMFlow method, particularly suited for the evaluation of few-scale loop and phase-space integrals in some kinematical limits. It is based on the introduction of an auxiliary mass to some properly chosen propagators, on the construction and subsequent solution of a DE system with respect to the auxiliary mass and an analytic flow to vanishing auxiliary mass for recovering the physical solution. The strategy has been employed to fix the boundary conditions of the RR and RV phase-space integrals relevant to NNLO DIS, thereby obtaining a full analytic solution. The results have been applied to the derivation of initial-final antenna functions to higher order in ϵ, in view of a future N^{3}LO application.

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