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Refactorisation and Subtraction

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Infrared subtraction algorithms beyond next-to-leading order necessitate the analysis of multiple infrared limits of scattering amplitudes, where several particles sequentially become soft or collinear. In this contribution, we report on the study performed in Ref. [1], which investigates these limits from the perspective of infrared factorisation, offering general definitions for strongly-ordered soft and collinear kernels, expressed in terms of gauge-invariant operator matrix elements. These definitions facilitate the identification of local subtraction counterterms for strongly-ordered configurations, whose integrals are designed to cancel the IR poles of real-virtual counterterms. This framework is validated at tree level for multiple emissions, and at one loop for single and double emissions.

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1. Introduction

High-order calculations of collider observables are crucial for achieving the theoretical precision required by current and upcoming experiments, and for identifying potential new physics signals. Such calculations require, in turn, complete control over infrared (IR) divergences, which becomes problematic at high orders in perturbation theory (see Ref. [2] for a recent review on the topic). At next-to-leading order (NLO), general and efficient algorithms for the subtraction of IR divergences have been developed [3, 4], and are implemented in simulation codes heavily used by the experimental community. Extending subtraction algorithms to NNLO and beyond (see, for example, [5–12]) is a challenging but necessary step to upgrade the accuracy standard of theoretical predictions, relevant to collider phenomenology[1]. A significant obstacle in tackling this problem is the appearance, starting at NNLO, of *strongly-ordered* singular configurations, which arise when two or more partons become unresolved at a different rate. There is a subtle interplay between these configurations and the infrared poles of mixed real-virtual counterterms, which has not so far been systematically understood. The present work is a contribution in this direction.

In what follows, we focus on the construction of strongly-ordered subtraction counterterms from the perspective of IR factorisation. We demonstrate how matrix elements of fields and Wilson lines, which describe the factorised emission of soft and collinear particles, can be 'refactorised' in strongly-ordered configurations. This procedure provides formal expressions for strongly-ordered counterterms to all orders in perturbation theory, and identifies the pattern of cancellations between such counterterms and the singularities involving mixed real and virtual corrections. Our discussion is based on, but not limited to, the framework of *local analytic sector subtraction* [12, 14–19].

2. The architecture of infrared subtraction

To introduce our notation we start by considering the distribution of an IR-safe observable X, expanded in powers of the strong coupling as

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\rm LO}}{dX} + \frac{d\sigma_{\rm NLO}}{dX} + \frac{d\sigma_{\rm NNLO}}{dX} + \dots\,. \tag{1}$$

The complexity of the subtraction problem emerges at NNLO, where the cancellation of IR divergences entails considering double-virtual corrections VV_n, integrated over an n-body phase space, together with real-virtual contributions RV_{n+1}, integrated over an $(n+1)$-body phase space, and with double-real radiation, RR_{n+2}, integrated in an $(n+2)$-body phase space. The observable NNLO distribution is, in fact, given by

$$\frac{d\sigma_{\rm NNLO}}{dX} = \lim_{d\to 4}\left[\int d\Phi_n\, VV_n\, \delta_n(X) + \int d\Phi_{n+1}\, RV_{n+1}\, \delta_{n+1}(X) + \int d\Phi_{n+2}\, RR_{n+2}\, \delta_{n+2}(X)\right]. \tag{2}$$

The relevant squared matrix elements are

$$VV_n = \left|\mathcal{A}_n^{(1)}\right|^2 + 2\mathbf{Re}\left[\mathcal{A}_n^{(0)\dagger}\mathcal{A}_n^{(2)}\right], \quad RV_{n+1} = 2\mathbf{Re}\left[\mathcal{A}_{n+1}^{(0)\dagger}\,\mathcal{A}_{n+1}^{(1)}\right], \quad RR_{n+2} = \left|\mathcal{A}_{n+2}^{(0)}\right|^2, \tag{3}$$

[1]Recent developments in this field are discussed in Ref. [13] and in the references therein.

where $\mathcal{A}_m^{(k)}$ is the k-loop correction to the m-point scattering amplitude for the process under consideration. The double-virtual matrix element VV_n features up to quadruple poles in the dimensional regulator $\epsilon = (4-d)/2$, while the real-virtual correction RV_{n+1} displays up to double poles in ϵ, and up to two phase-space singularities. Finally, the double-real matrix element RR_{n+2} is finite in $d = 4$, but it is affected by up to four phase-space singularities. At this order, we need to consider all possible double- and single-unresolved limits of RR_{n+2}, and carefully account for their overlap; moreover, we have to properly subtract both the explicit poles and the single-unresolved limits of the real-virtual contribution. We collectively characterise single unresolved limits by the action of an operator $\overline{\mathbf{L}}^{(1)}$ on squared real-emission matrix elements, defined in such a way as to remove any double counting. Similarly, double-unresolved limits are given by an operator $\overline{\mathbf{L}}^{(2)}$, and their overlap by $\overline{\mathbf{L}}^{(12)}$. The action of these operators on double-real and real-virtual contributions defines the minimal set of counterterms required for an NNLO subtraction approach. They read

$$K_{n+2}^{(\mathbf{1})} = \overline{\mathbf{L}}^{(1)} RR_{n+2}\,, \qquad K_{n+2}^{(\mathbf{2})} = \overline{\mathbf{L}}^{(2)} RR_{n+2}\,, \tag{4}$$

$$K_{n+2}^{(\mathbf{12})} = \overline{\mathbf{L}}^{(12)} RR_{n+2}\,, \qquad K_{n+1}^{(\mathbf{RV})} = \overline{\mathbf{L}}^{(1)} RV_{n+1}\,. \tag{5}$$

Upon designing a set of suitable phase-space mappings, to factorise resolved and unresolved phase-space measures, we can proceed to define *integrated counterterms*, as

$$I_{n+1}^{(\mathbf{1})} = \int d\Phi_{\mathrm{rad},1}^{n+2}\, K_{n+2}^{(\mathbf{1})}\,, \qquad I_{n}^{(\mathbf{2})} = \int d\Phi_{\mathrm{rad},2}^{n+2}\, K_{n+2}^{(\mathbf{2})}\,, \tag{6}$$

$$I_{n+1}^{(\mathbf{12})} = \int d\Phi_{\mathrm{rad},1}^{n+2}\, K_{n+2}^{(\mathbf{12})}\,, \qquad I_{n}^{(\mathbf{RV})} = \int d\Phi_{\mathrm{rad},1}^{n+1}\, K_{n+1}^{(\mathbf{RV})}\,, \tag{7}$$

where the radiative phase spaces are defined by

$$d\Phi_{n+2} = \frac{\varsigma_{n+2}}{\varsigma_{n+1}}\, d\Phi_{n+1}\, d\Phi_{\mathrm{rad},1}^{n+2}\,, \quad d\Phi_{n+2} = \frac{\varsigma_{n+2}}{\varsigma_{n}}\, d\Phi_{n}\, d\Phi_{\mathrm{rad},2}^{n+2}\,, \quad d\Phi_{n+1} \equiv \frac{\varsigma_{n+1}}{\varsigma_{n}}\, d\Phi_{n}\, d\Phi_{\mathrm{rad},1}^{n+1}\,, \tag{8}$$

with ς_p denoting the appropriate symmetry factors. Putting together the ingredients assembled so far, we can now write a fully subtracted form of the generic NNLO distribution,

$$\begin{aligned} \frac{d\sigma_{\mathrm{NNLO}}}{dX} &= \int d\Phi_n \left[VV_n + I_n^{(\mathbf{2})} + I_n^{(\mathbf{RV})} \right] \delta_n(X) \\ &+ \int d\Phi_{n+1} \left[\left(RV_{n+1} + I_{n+1}^{(\mathbf{1})} \right) \delta_{n+1}(X) - \left(K_{n+1}^{(\mathbf{RV})} + I_{n+1}^{(\mathbf{12})} \right) \delta_n(X) \right] \\ &+ \int d\Phi_{n+2} \left[RR_{n+2}\, \delta_{n+2}(X) - K_{n+2}^{(\mathbf{1})}\, \delta_{n+1}(X) - \left(K_{n+2}^{(\mathbf{2})} - K_{n+2}^{(\mathbf{12})} \right) \delta_n(X) \right]. \end{aligned} \tag{9}$$

We note that the third line is integrable in Φ_{n+2} by construction, since all singular regions have been subtracted with no double counting. In the second line, the integral $I_{n+1}^{(\mathbf{1})}$ cancels the ϵ poles of RV_{n+1}, but their combination is still affected by phase-space singularities. Those affecting $I_{n+1}^{(\mathbf{1})}$ and RV_{n+1} are cured by $I_{n+1}^{(\mathbf{12})}$ and $K_{n+1}^{(\mathbf{RV})}$, respectively: we conclude that the second line in eq. (9) is free from phase-space singularities. On the other hand, there is in principle no guarantee that the ϵ poles of $K_{n+1}^{(\mathbf{RV})}$ will cancel those of $I_{n+1}^{(\mathbf{12})}$, given the considerable degree of arbitrariness in constructing radiative counterterms. The goal of this note is to provide precise definitions of the counterterms that are geared towards making this cancellation automatic. A more detailed discussion is presented in Sec. 2.2 of [1]. Having established the finiteness and integrability of both the second and the third line of eq. (9), the cancellation of poles in the first line directly follows from the KLN theorem.

The arguments presented in this section can be formally generalised to N^{3}LO, and beyond. This requires organising a larger number of singular configurations and related overlaps. To illustrate this, we can consider triple-real corrections arising at N^{3}LO: in that case, one needs to define unresolved limits and integrated counterterms as

$$K_{n+3}^{(\mathbf{h})} = \overline{\mathbf{L}}^{(\mathbf{h})} RRR_{n+3} , \qquad I_{n+3-q}^{(\mathbf{h})} = \int d\Phi_{\mathrm{rad},q}^{n+3} K_{n+3}^{(\mathbf{h})} , \qquad \mathbf{h} \in \{\mathbf{3}, \mathbf{13}, \mathbf{23}, \mathbf{123}\} , \tag{10}$$

where q is the number of particles going unresolved at the highest rate. For instance, $\overline{\mathbf{L}}^{(\mathbf{3})}$ collects the limits where three particles become unresolved at the same rate; $\overline{\mathbf{L}}^{(\mathbf{13})}$ corresponds to the strongly-ordered case where one particle becomes unresolved faster than the other two; $\overline{\mathbf{L}}^{(\mathbf{23})}$ describes the configurations where two particles become unresolved at the same rate, but significantly faster than the third particle; finally, $\overline{\mathbf{L}}^{(\mathbf{123})}$ describes the strongly-ordered scenario where each particle becomes unresolved at a different rate. Including mixed real-virtual corrections at one and two loops, one finds that N^{3}LO subtraction would require a total of 11 local counterterm functions, 5 of which involving strong ordering, with the remaining 6 corresponding to uniform limits. Generalising further to N^{k}LO, the number of required counterterms turns out to be given by $c(k) = 2^{k+1} - 2 - k$, of which only $k(k+1)/2$ corresponding to uniform limits: clearly, the problem of handling the cancellations between integrated strongly-ordered and real-virtual counterterms becomes increasingly significant at higher perturbative orders.

3. Democratic counterterms to any order

In this section we exploit the factorisation properties of gauge amplitudes to find explicit expressions for the *democratic* counterterms encoding uniform soft and collinear limits, following the discussion in Ref. [14]. This approach relies on the knowledge of the IR structure of *virtual* corrections, which is used as a starting point to infer suitable soft and collinear approximants of *real* corrections. We refer to such a method as a *top-down* approach, since we first analyse the first line of eq. (9), identify the required integrated counterterms, and then the corresponding integrands, which enter the second and the third lines.

The infrared factorisation formula for massless gauge-theory amplitudes reads [20–23]

$$\mathcal{A}_n(\{p_i\}) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i(p_i, n_i)}{\mathcal{J}_{\mathrm{E}_i}(\beta_i, n_i)} \right] \mathcal{S}_n(\{\beta_i\}) \, \mathcal{H}_n(\{p_i\}, \{n_i\}) \, . \tag{11}$$

The soft, jet, and eikonal jet functions $\mathcal{S}_n$, $\mathcal{J}_i$ and $\mathcal{J}_{\mathrm{E}_i}$ appearing in eq. (11) have explicit definitions (see for example Ref. [2]), in terms of operator matrix elements involving semi-infinite Wilson lines aligned with the external-particle velocities β_i,

$$\Phi_{\beta_i}(\infty, 0) \equiv P \exp \left\{ \mathrm{i} g_s \mathbf{T}^a \int_0^\infty dz \, \beta_i \cdot A_a(z) \right\} , \tag{12}$$

as well as auxiliary Wilson lines along the directions n_i ($n_i^2 \neq 0$), and quantum fields. In eq. (12), P denotes path ordering, A_a represents the gluon field, and g_s is the strong coupling constant. In

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the subtraction context, $\mathcal{S}_n$, $\mathcal{J}_i$ and $\mathcal{J}_{\mathrm{E}_i}$ can be considered special cases of more general functions, which can be used to model soft and collinear real radiation. For instance, the *eikonal form factor*

$$\mathcal{S}_{n,f_1\ldots f_m}\big(\{\beta_i\};\{k_j,\lambda_j\}\big) \;\equiv\; \langle\{k_j,\lambda_j\}|\,T\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)|0\rangle\;, \tag{13}$$

describes the radiation of m soft particles of flavours f_j, momenta k_j and spin polarisations λ_j ($j=0,\ldots,m$), from n Wilson lines representing hard particles, including virtual corrections in the soft approximation. Analogously, collinear radiation from an external particle i can be modelled via *collinear form factors* such as

$$\mathcal{J}^{\alpha}_{q,f_1\ldots f_m}\big(x;n;\{k_j,\lambda_j\}\big) \;\equiv\; \langle\{k_j,\lambda_j\}|\,T\left[\bar{\psi}^{\alpha}(x)\Phi_n(x,\infty)\right]|0\rangle\;, \tag{14}$$

where we picked as an example a quark jet, $j=1,\ldots,m$, and the case $m=1$ represents the purely virtual contribution, $\mathcal{J}_{f_i,f_i}=\mathcal{J}_i$. Similarly, the soft function in eq. (11) corresponds to the $m=0$ case of eq. (13). Eikonal jets are obtained by replacing the quark field in eq. (14) with an appropriate Wilson line, aligned with the classical quark trajectory.

At cross-section level, eikonal and collinear form factors must be squared, building up radiative soft and jet functions, which are fully local in the degrees of freedom of (multiple) soft and collinear real radiation. In the case of soft functions, the definition is straightforward [14], while collinear emissions require more care. Indeed, radiative jet functions must account for hard-collinear emissions to have non-zero momentum: therefore, at cross-section level, one of the two collinear form factors is evaluated at a shifted position x, which is Fourier-conjugate to the total momentum ℓ carried by final-state particles. We define then

$$J^{\alpha\beta}_{f,f_1\ldots f_m}\big(\ell;n;\{k_j\}\big) = \sum_{\{\lambda_j\}}\int d^dx\,\mathrm{e}^{\mathrm{i}\ell\cdot x}\,\mathcal{J}^{\alpha,\dagger}_{f,f_1\ldots f_m}\big(0;n;\{k_j,\lambda_j\}\big)\mathcal{J}^{\beta}_{f,f_1\ldots f_m}\big(x;n;\{k_j,\lambda_j\}\big)\,. \tag{15}$$

In eq. (15), f denotes the flavour of the parent particle, carrying the open spin indices α and β. Performing the x integral will fix $\ell=\sum_j k_j$. At tree level, for $m=2$, it is straightforward to show that $J^{(0)}_{f,f_1f_2}$ reproduces the tree-level splitting kernel for $f\to f_1+f_2$. Integrating eq. (15) over the radiative m-particle phase space, and summing over m, one finds

$$\sum_{m=1}^{\infty}\sum_{\{f_i\}}\int d\Phi_m\,J^{\alpha\beta}_{q,f_1\ldots f_m} = \mathrm{Disc}\left\{\int d^dx\,\mathrm{e}^{\mathrm{i}\ell\cdot x}\,\langle 0|\,T\Big[\Phi_n(\infty,x)\psi^{\beta}(x)\bar{\psi}^{\alpha}(0)\Phi_n(0,\infty)\Big]|0\rangle\right\}. \tag{16}$$

The r.h.s. of eq. (16) is the discontinuity of a two-point function in the presence of Wilson lines, which can be shown to be IR finite order by order. Such a *finiteness* condition is crucial for applying factorisation arguments to the construction of local IR counterterms at any order in perturbation theory [14]: similar conditions apply for soft functions and eikonal jets. Indeed, by expanding the l.h.s. of eq. (16), and of its analogue for the soft function, we find order-by-order finiteness conditions that embody the KLN cancellations. At NLO, for example

$$S^{(1)}_n\big(\{\beta_i\}\big)+\int d\Phi(k)\,S^{(0)}_{n,g}\big(\{\beta_i\};k\big) \;=\; \text{finite}\,, \tag{17}$$

$$\sum_{f_1}\int d\Phi(k_1)J^{(1)\alpha\beta}_{f,f_1}(\ell;k_1)+\sum_{f_1,f_2}\varsigma_{f_1f_2}\int d\Phi(k_1)d\Phi(k_2)\,J^{(0)\,\alpha\beta}_{f,f_1f_2}(\ell;k_1,k_2) \;=\; \text{finite}\,, \tag{18}$$

where $\varsigma_{f_1 f_2}$ is a phase-space symmetry factor. The conditions in eqs. (17-18) immediately suggest that the integrands of the real-radiation contributions can serve as candidate soft and collinear local NLO counterterms. At NNLO, the analogues of eqs. (17-18) involve a double-virtual correction, a real-virtual term and a double-radiative function, as expected from general IR-cancellation theorems. It is important to notice that counterterms identified via finiteness relations directly reproduce only uniform infrared limits. The construction of strongly-ordered counterterms from factorisation is discussed in the next section.

4. Strongly-ordered counterterms to any order

In this section we aim to express strongly-ordered limits in terms of universal operator matrix elements, in the spirit of factorisation. To deduce their form, we start by considering uniform double-unresolved limits, and applying strong-ordering conditions. We analyse the soft limit first. The tree-level double-soft current for gluons 1 and 2 with momenta k_1 and k_2 [24] simplifies considerably in the strongly-ordered limit in which $k_2 \ll k_1 \ll \mu$, with μ the hard scale of the process. The corresponding form factor is given by an interesting 'refactorisation' of the double-radiative soft function:

$$\begin{aligned}
\left[\mathcal{S}^{(0)}_{n;\,g,g}\right]^{a_1 a_2}_{\{d_i e_i\}}(\{\beta_i\}; k_1, k_2) \;\; &\equiv \;\; \langle k_2, a_2 |\, T\Big[\Phi_{\beta_{k_1}}^{\;a_1 b}(0,\infty) \prod_{i=1}^{n} \Phi_{\beta_i,\, d_i}^{\;\;c_i}(\infty, 0)\Big]\, |0\rangle \\
&\qquad \times \langle k_1, b |\, T\Big[\prod_{i=1}^{n} \Phi_{\beta_i,\, c_i e_i}(\infty,0)\Big]\, |0\rangle \Big|_{\rm tree} \\
&= \;\; \left[\mathcal{S}^{(0)}_{n+1,\,g}\right]^{a_2,\,a_1 b}_{\{d_i c_i\}}\big(\beta_{k_1}, \{\beta_i\}; k_2\big)\, \left[\mathcal{S}^{(0)}_{n,\,g}\right]_{b,\,\{c_i e_i\}}(\{\beta_i\}; k_1)\,.
\end{aligned} \tag{19}$$

Eq. (19) can be interpreted as follows: gluon 1 is soft compared to the n hard Born partons, but appears as hard when probed by gluon 2, with $k_2 \ll k_1$. The original system of n Wilson lines thus radiates the harder gluon 1, which then '*Wilsonises*': indeed, at this stage, the new system of $(n + 1)$ Wilson lines radiates the softer gluon 2. This is described by a factorised matrix element, where gluon 2 remains a final-state parton, while gluon 1 plays the double role of final-state parton (when radiated by one of the n original hard legs) and of Wilson line in the adjoint representation (when radiating gluon 2). Tree-level soft refactorisation has been tested against the expressions in Ref. [25] up to 3 gluons, and it is natural to conjecture its validity for any number of gluons.

In the case of multiple collinear emissions, the situation is more involved due to spin correlations, but a refactorised form can still be identified. For instance, the NNLO strongly-ordered collinear configuration for a $q \to q'_1\, \bar{q}'_2\, q_3$ branching is given by

$$\begin{aligned}
\lim_{\theta_{12} \ll \theta_{13} \to 0} RR_{n+2} \;\; &= \;\; \frac{(8\pi\alpha_s)^2}{s_{12}\, s_{[12]3}}\, P^{\rho\sigma}_{q\to gq}\big(z_{[12]}, q_\perp\big)\, d_{\rho\mu}\big(k_{[12]}\big) \\
&\qquad \times P^{\mu\nu}_{g\to q\bar{q}}\big(z_1/z_{[12]}, k_\perp\big)\; d_{\sigma\nu}\big(k_{[12]}\big)\, B_n\,,
\end{aligned} \tag{20}$$

where the intermediate-particle momentum is $k_{[12]} \equiv k_1 + k_2$, its collinear energy fraction is $z_{[12]} \equiv z_1 + z_2 = 1 - z_3$, and $s_{[12]3} = 2\, k_{[12]} \cdot k_3$. Finally, $d_{\mu\nu}(k) = -g_{\mu\nu} + (k_\mu n_\nu + k_\nu n_\mu)/(k \cdot n)$

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is the intermediate gluon polarisation sum. As expected, the same result can be written in terms of an appropriate convolution of radiative jet functions:

$$\int \frac{d^d \ell}{(2\pi)^d} \left[\lim_{\theta_{12} \ll \theta_{13} \to 0} J^{(0)}_{q,qq'\bar{q}'}(\ell; k_1, k_2, k_3) \right] \equiv \int \frac{d^d \ell}{(2\pi)^d} \, J^{(0)}_{q,gq;g,q'\bar{q}'}(\ell; k_1, k_2, k_3)$$
$$= \int \frac{d^d \ell}{(2\pi)^d} \, J^{;\rho\sigma\,(0)}_{q,gq}(\ell; k_{[12]}, k_3) \int \frac{d^d \ell'}{(2\pi)^d} \, J^{\rho\sigma\,(0)}_{g,q'\bar{q}'}(\ell'; k_1, k_2) \,, \qquad (21)$$

where the first line introduces the notation for a strongly-ordered jet function, outlining the sequential splittings involved. The integrals over $d^d\ell$ and $d^d\ell'$ address the delta-function constraints in the corresponding jet-function definitions, which ensure that the parent parton momentum equals the total momentum of its decay products. Spin indices of the parent quark are not explicitly shown, and are replaced by the semicolon. The Lorentz spin indices after the semicolon correspond to the daughter gluon produced by the splitting. Analogous versions eq. (21) hold for splittings of different flavours, and generalise to larger numbers of emitted particles [1].

As mentioned in Section 1, strongly-ordered counterterms have to combine, upon integration over the most unresolved parton, with real-virtual counterterms, cancelling their poles. In order to make such an interplay manifest, it is useful to exploit once more the idea of refactorisation. We need to consider one-loop radiative soft and jet functions: for the sake of illustration, we focus on the former. We note that, contrary to their virtual counterpart, they do not reduce to pure counterterms, and contain both IR poles and finite contributions. For our purposes, they can be considered as scattering amplitudes with Wilson-line sources, which points to a natural factorisation of their virtual IR poles. Indeed, for example, applying the standard soft-jet-hard factorisation to the single-radiative soft function leads to

$$\mathcal{S}_{n,g}\big(\{\beta_i\}; k\big) \;=\; \mathcal{S}_{n+1}\big(\{\beta_i\}, \beta_k\big) \, \frac{\mathcal{J}^{\mu}_{g,g}(0; k)}{\mathcal{J}_{\mathrm{E_g}}(\beta_k)} \, \mathcal{S}^{\mathcal{H},\mu}_{n,g}\big(\{\beta_i\}; k\big) \,, \qquad (22)$$

where the factor $\mathcal{S}^{\mathcal{H},\mu}_{n,g}$ is finite in $d = 4$. Expanding to one-loop order, the terms containing IR poles are thus

$$\mathcal{S}^{(1)}_{n,g}\big(\{\beta_i\}; k\big) \;=\; \Big[\mathcal{S}^{(1)}_{n+1}\big(\{\beta_i\}, \beta_k\big) - \mathcal{J}^{(1)}_{\mathrm{E_g}}(\beta_k) \Big] \, \mathcal{S}^{(0)}_{n,g}\big(\{\beta_i\}; k\big)$$
$$+ \, \mathcal{J}^{(1)\mu}_{g,g}(0; k) \, \mathcal{S}^{(0)\mu}_{n,g}\big(\{\beta_i\}; k\big) \,. \qquad (23)$$

We will show in the next section how to reconstruct from eq. (23) (upon squaring) the soft contribution to $K^{(\mathbf{RV})}_{n+1}$, plus hard-collinear corrections, which will need to be subtracted. The remaining soft poles will naturally cancel against those arising in the integrated strongly-ordered soft counterterm.

5. Engineering cancellations

In this section we present a construction of strongly-ordered counterterms, starting from the expression of the real-virtual counterterm $K^{(\mathbf{RV})}_{n+1}$, such that the combination $K^{(\mathbf{RV})}_{n+1} + I^{(\mathbf{12})}_{n+1}$ is free of IR poles. For the sake of illustration, we focus on the soft component. We begin by constructing the cross-section-level radiative soft function from factorisation, using eq. (23). We find

$$S^{(1)}_{n,g}(\{\beta_i\};k) = \mathcal{S}^{(0)\dagger}_{n,g}(\{\beta_i\};k)\left[S^{(1)}_{n+1}(\{\beta_i\},\beta_k) - J^{(1)}_{\mathrm{E}_g}(\beta_k)\right]\mathcal{S}^{(0)}_{n,g}(\{\beta_i\};k) + \int \frac{d^d\ell}{(2\pi)^d}\left(\mathcal{S}^{(0)\mu}_{n,g}(\{\beta_i\};\ell)\right)^\dagger J^{(1)\mu\nu}_{g,g}(\ell;k)\;\mathcal{S}^{(0)\nu}_{n,g}(\{\beta_i\};\ell)\,. \tag{24}$$

We now use the finiteness conditions, suitably replacing one-loop factors with tree-level radiative factors. This leads to

$$S^{(1)}_{n,g}(\{\beta_i\};k_1) + \int d\Phi(k_2)\Bigg\{S^{(0)}_{n;g,g}(\{\beta_i\};k_{[12]};k_2) - \left(\mathcal{S}^{(0)}_{n,g}(\{\beta_i\};k_{[12]})\right)^\dagger J^{(0)}_{\mathrm{E}_g,g}(\beta_{k_1};k_2)\,\mathcal{S}^{(0)}_{n,g}(\{\beta_i\};k_{[12]}) + \int\frac{d^d\ell}{(2\pi)^d}\left(\mathcal{S}^{(0)\,\mu}_{n,g}(\{\beta_i\};\ell)\right)^\dagger\sum_{f_1,f_2} J^{(0)\,\mu\nu}_{g,f_1f_2}(\ell;k_1,k_2)\;\mathcal{S}^{(0)\,\nu}_{n,g}(\{\beta_i\};\ell)\Bigg\} = \text{finite}\,. \tag{25}$$

We see that the refactorisation of strongly-ordered soft radiation suggests an expression for the soft component of the strongly-ordered counterterm $K^{(\mathbf{12})}_{n+2}$. Indeed, we can take the integrand appearing in eq. (25) as a definition of the local counterterm. With simple steps, one gets

$$K^{(\mathbf{12},\,\mathrm{s})}_{n+2} = \mathcal{H}^{(0)\dagger}_n\sum_{f_1,f_2}\left[S^{(0)}_{n;f_{[12]},f_2}\big(k_{[12]},k_2\big) + S^{(0)}_{n,f_{[12]}}\left(J^{(0)}_{f_{[12]},f_1f_2} - J^{(0)}_{\mathrm{E}_{[12]},f_2}\right)\right]\mathcal{H}^{(0)}_n\,. \tag{26}$$

In the soft sector, $K^{(\mathbf{12},\,\mathrm{s})}_{n+2}$ cancels all poles of $K^{(\mathbf{RV},\,\mathrm{s})}_{n+1}$ by construction. In fact, the explicit poles of the soft component $K^{(\mathbf{RV})}_{n+1}$ are encoded in the radiative, one-loop soft function, so that we have

$$K^{(\mathbf{RV},\,\mathrm{s})}_{n+1} = \mathcal{H}^{(0)\dagger}_n\;S^{(1)}_{n,g}\,\mathcal{H}^{(0)}_n + \text{finite}\,. \tag{27}$$

It is then straightforward to verify the cancellation occurring between $K^{(\mathbf{RV},\,\mathrm{s})}_{n+1}$ and $K^{(\mathbf{12},\,\mathrm{s})}_{n+2}$, upon integrating the latter over $d\Phi(k_2)$. The same steps apply to the collinear case, where the hard-collinear component of the real-virtual counterterm can be written as

$$K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,i} = \mathcal{H}^{(0)\dagger}_n\sum_{f_1,f_2} J^{(0),\,\mathrm{hc}}_{f_i,f_1f_2}\left[S^{(1)}_3 - J^{(1)}_{\mathrm{E}_i} + \sum_{k=1}^{2} J^{(1),\,\mathrm{hc}}_{f_k,f_k}\right]\mathcal{H}^{(0)}_n\,, \tag{28}$$

and the corresponding strongly-ordered counterterm reads

$$K^{(\mathbf{12},\,\mathrm{hc})}_{n+2,i} = \mathcal{H}^{(0)\dagger}_n\sum_{f_1,f_2,f_3}\Bigg[J^{(0),\,\mathrm{hc}}_{f_i,f_1f_2}(\bar k_1,\bar k_2)\,S^{(0)}_{3,f_3} - J^{(0),\,\mathrm{hc}}_{f_i,f_1f_2}(k_1,k_2)\,J^{(0)}_{\mathrm{E}_i,f_3} + \sum_{kl=\{12,21\}} J^{(0),\,\mathrm{hc}}_{f_i,f_{[k3]}f_l}\left(J^{(0)}_{f_{[k3]},f_kf_3} - J^{(0)}_{\mathrm{E}_k,f_3}\right)\Bigg]\mathcal{H}^{(0)}_n\,. \tag{29}$$

Again, the pole cancellation between eq. (28) and the integral of eq. (29) can be easily proven by exploiting finiteness relations involving one-loop and radiative soft and jet functions.

6. Outlook

In this contribution we have outlined a procedure to identify local subtraction counterterms describing the singular behaviour of real-radiation in soft and collinear limits, paying special attention to strongly-ordered configurations. We have presented the general structure of subtraction counterterms at NNLO, and we have sketched it at N^3LO, displaying a pattern that facilitates the generalisation to higher orders. Our procedure for constructing counterterms starts from the all-order factorisation of virtual amplitudes, and uses finiteness relations to deduce the form of infrared counterterms for real radiation, reversing the standard approach to infrared subtraction[2]. All counterterms are expressed in terms of gauge-invariant matrix elements of fields and Wilson lines. We have focused on the main challenge associated with strongly-ordered configurations, namely ensuring that the integrals of the corresponding counterterms cancel the poles of mixed real-virtual contributions. Even if our approach does not immediately translate into a concrete subtraction algorithm, it provides crucial insights on the architecture of infrared subtraction to all orders. Further work is underway to build an algorithmic implementation of these ideas.

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