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N-jettiness soft function at NNLO in QCD

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In this proceeding we present our derivation of a compact representation of the renormalized N-jettiness soft function free of infrared and collinear divergences through next-to-next-to-leading order (NNLO) in perturbative QCD. The number N of hard partons enters as a parameter in the formula of the finite remainder. We demonstrate analytically the cancellation of all infrared and collinear singularities between the bare soft function and its renormalization matrix in color space and we compare our results with the ones available in the literature.

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1. Introduction

Driven by the increasing precision of the experimental analyses at the Large Hadron Collider (LHC), the corresponding increasing accuracy in the theoretical description of high-multiplicity final states produced in hard processes led to many studies of the so-called subtraction and slicing schemes for perturbative QCD calculations. The role of these schemes is to enable cancellation of soft and collinear divergences between virtual and real contributions to cross sections without compromising the fully-differential nature of theoretical predictions.

The subtraction schemes are the most popular approach at NLO accuracy. In this case the idea is to construct a set of integrable counterterms for the real emission contributions that cancel the divergences in all IR limits. These counterterms can then be readily integrated in order to analytically remove the dimensionally regularized $1/\epsilon^n$ IR poles in the virtual contributions. Examples of this approach at NNLO are the Antenna subtraction [1], Projection-to-Born [2], CoLoRFul subtraction [3], Nested soft-collinear subtraction [4], Sector subtraction [5] and Local analytic sector subtraction [6], among many others.

The alternative popular approach is using phase space slicing methods. In these methods a global parameter is used to impose cuts that divide the phase space into two (or possibly more) regions. At NNLO, the idea is to separate the region below the cut, where the double unresolved emission takes place and a soft-collinear approximation can be used. In the other region, where there is at most one unresolved parton, standard NLO techniques can be used to calculate the cross section. The two most used slicing methods are q_t-slicing [7] and N-jettiness slicing [8, 9]. The q_t-slicing uses the transverse momentum q_t of the final state to split the phase space, and in the region of small q_t the cross section is calculated with the use of the Collins–Soper–Sterman factorization theorem [10].

In our work we focus in the other method, the N-jettiness slicing, which uses the N-jettiness $\mathcal{T}$ as the cutting variable. It is defined as

$$\mathcal{T}(\mathcal{R},\mathcal{U}) = \sum_{x\in\mathcal{U}} \min\left\{\frac{2p_x p_{h_1}}{P_{h_1}}, \frac{2p_x p_{h_2}}{P_{h_2}}, \frac{2p_x p_{h_3}}{P_{h_3}}, \ldots\right\}, \tag{1}$$

where we split all final-state partons in the process under consideration into resolved $\mathcal{R}$ and unresolved $\mathcal{U}$ ones, with $x \in \mathcal{U}$ and $h_i \in \mathcal{R}$, and P_{h_i} are arbitrary normalization constants. The cross section is then split as

$$\sigma = \int^{\mathcal{T}_0} d\mathcal{T}\frac{d\sigma}{d\mathcal{T}} + \int_{\mathcal{T}_0} d\mathcal{T}\frac{d\sigma}{d\mathcal{T}}, \tag{2}$$

and the first term can be calculated with the use of the factorization theorem from Soft-Collinear Effective Field theory (SCET) [11–15]

$$\int^{\mathcal{T}_0} d\mathcal{T}\frac{d\sigma}{d\mathcal{T}} = \int B\otimes B\otimes S\otimes H\otimes \prod_i^N J_i + O(\mathcal{T}_0), \tag{3}$$

where the beam function and jet functions (B and J_i) describe the initial- and final-state collinear radiation, the soft function S describes the soft radiation, and the (process dependent) hard function H encodes the virtual corrections. At NNLO, all these ingredients have been available for some time. In particular, the soft function was available for 0-, 1- and 2-jettiness, but only recently for generic N-jettiness [16, 17].

In this proceeding we present our recent calculation of the N-jettiness soft function at NNLO [17]. Previous NNLO calculations were mainly based on mapping the available phase space of soft-gluon emissions onto hemispheres and computing the required integrals numerically [16, 18, 19]. In our case, we use the well established subtraction methods to calculate this ingredient of the N-jettiness phase space slicing method, showing the *explicit analytical cancellation of divergences*. Also, in our calculation N is treated genuinely as a parameter in the formula. In this work we try to show how borrowing ideas from generic NNLO QCD subtraction schemes can be very beneficial for computing ingredients of modern slicing calculations.

The rest of the text is organized as follows: in Sec. 2 we first discuss how the renormalization of the N-jettiness soft function works, then we present our calculation of the NLO and NNLO soft function in Secs. 3 and 4, respectively. In Sec. 5 we discuss the implementation of our formulas and the comparison with other results available in the literature. Finally, we conclude in Sec. 6

2. Soft function renormalization

Before we discuss the NLO expression of the N-jettiness soft function, we first explain how its renormalization works. Since pure loop corrections do not contribute to the soft function, the infrared divergences turn into ultraviolet ones that require renormalization in order to cancel them. It is convenient to work in Laplace space, using the Laplace transform of the soft function

$$S(u) = \int_0^\infty d\mathcal{T}\, S_{\mathcal{T}}(\mathcal{T}) e^{-u\mathcal{T}}, \tag{4}$$

where $\mathcal{T}$ is the N-jettiness variable. When working in Laplace space, the renormalization works as a multiplicative matrix renormalization (in color space)

$$S = Z\tilde{S}Z^\dagger, \tag{5}$$

where S and $\tilde{S}$ are the bare and renormalized soft functions, respectively, and Z is a matrix in color space that can be found in Appendix A of Ref. [17]. It is beneficial to directly compute the combinations of Z and S that are actually needed for $\tilde{S}$. If we write the expansion of Z and S in powers of α_s

$$\begin{aligned} Z &= 1 + Z_1 + Z_2, \\ S &= 1 + S_1 + S_2, \\ \tilde{S} &= 1 + \tilde{S}_1 + \tilde{S}_2, \end{aligned} \tag{6}$$

and substitute them in Eq. (5) we get

$$\begin{aligned} \tilde{S}_1 &= S_1 - Z_1 - Z_1^\dagger, \\ \tilde{S}_2 &= S_2 - Z_2 - Z_2^\dagger + Z_1 Z_1 + Z_1^\dagger Z_1^\dagger - Z_1 S_1 - S_1 Z_1^\dagger + Z_1 Z_1^\dagger \\ &= \frac{1}{2}\tilde{S}_1\tilde{S}_1 + \frac{1}{2}[Z_1, Z_1^\dagger] + \frac{1}{2}\left[S_1, Z_1 - Z_1^\dagger\right] + S_{2,r} - Z_{2,r} - Z_{2,r}^\dagger. \end{aligned} \tag{7}$$

where, as it was done in Ref. [20], we separated iterations of $O(\alpha_s)$ soft, soft-collinear and virtual contributions to cross sections from the rest of NNLO contributions

$$
\begin{aligned}
S_2 &= \frac{1}{2} S_1 S_1 + S_{2,r}, \\
Z_2 &= \frac{1}{2} Z_1 Z_1 + Z_{2,r}.
\end{aligned}
\tag{8}
$$

This last step, as it will be shown later, is very helpful for the computation of the renormalized N-jettiness soft function.

3. N-jettiness soft function at NLO

If we take $P_{h_i} = E_i$ in Eq. (1) with an unresolved gluon m (where E_i is the energy of the parton or jet i), the N-jettiness is given by

$$
\mathcal{T}(m) = E_m\, \psi_m = E_m \min\{\rho_{1m}, \rho_{2m}, \rho_{3m}, ..., \rho_{Nm}\}, \tag{9}
$$

with $\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j$ and $\vec{n}_i$ is a unit vector pointing in the direction of the three-momentum of parton i. Then, the soft function at NLO is given by

$$
S(\tau) = -\sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j\, g_s^2 \int \frac{d\Omega_m^{(d-1)}}{2(2\pi)^{d-1}} \frac{dE_m}{E_m^{1+2\epsilon}} E_m^2\, \delta(\tau - E_m \psi_m) \left\langle S_{ij}(m) \right\rangle_m, \tag{10}
$$

where $\langle .. \rangle_m$ indicates integration over the directions of the vector $\vec{n}_m$, and

$$
S_{ij}(m) = \frac{p_i p_j}{(p_i p_m)(p_j p_m)} = \frac{1}{E_m^2} \frac{\rho_{ij}}{\rho_{im}\rho_{jm}}, \tag{11}
$$

is the soft eikonal function.

We then integrate over the energy of the gluon E_m and make use of the fact that we know the limit $\lim_{m||i} \psi_m = \rho_{im}$, so we can rewrite

$$
\psi_m^{2\epsilon} \frac{\rho_{ij}}{\rho_{im}\rho_{jm}} = \left(\frac{\psi_m \rho_{ij}}{\rho_{im}\rho_{jm}} \right)^{2\epsilon} \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}} = \left(1 + 2\epsilon g_{ij,m}^{(2)}\right) \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}}, \tag{12}
$$

where the function $g_{ij,m}^{(2)}$ contains the higher powers of ϵ terms in addition to the first power that it is shown explicitly. Integration of the first term in Eq. (12) can be done right away

$$
\left\langle \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}} \right\rangle_m = \frac{2\eta_{ij}^{\epsilon}}{\epsilon} K_{ij}^{(2)} = \frac{2\eta_{ij}^{\epsilon}}{\epsilon} \frac{\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)}\, {}_2F_1\left(\epsilon, \epsilon, 1-\epsilon, 1-\eta_{ij}\right), \tag{13}
$$

with $\eta_{ij} = \rho_{ij}/2$. Putting everything together, we find that in Laplace space (and defining $\bar{u} = u e^{\gamma_E}$)

$$
S_1 = a_s\, (\mu\bar{u})^{2\epsilon} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)e^{\epsilon\gamma_E}} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left[\frac{\eta_{ij}^{\epsilon}}{\epsilon^2} K_{ij}^{(2)} + \left\langle g_{ij,m}^{(2)} \frac{\rho_{ij}^{1-2\epsilon}}{\rho_{im}^{1-2\epsilon}\rho_{jm}^{1-2\epsilon}} \right\rangle_m \right]. \tag{14}
$$

It follows from this equation that all $1/\epsilon$ poles have been explicitly separated and that the remnant of the N-jettiness function appears only in the finite remainder, which can be computed numerically.

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By combining S_1 with the renormalization matrices Z_1 and $Z_1^\dagger$ and discarding all terms beyond $O(\epsilon^0)$, we finally obtain the renormalized NLO N-jettiness soft function

$$\tilde{S}_1 = a_s \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left[2L_{ij}^2 + \mathrm{Li}_2(1-\eta_{ij}) + \frac{\pi^2}{12} + \left\langle \ln\left(\frac{\psi_m \rho_{ij}}{\rho_{im}\rho_{jm}}\right) \frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right\rangle_m + O(\epsilon) \right], \tag{15}$$

where $L_{ij} = \ln\left(\mu\bar{u}\sqrt{\eta_{ij}}\right)$. This result can be easily evaluated numerically for an arbitrary number N of hard partons and/or jets. The logarithm $\ln(\psi_m \rho_{ij}/(\rho_{im}\rho_{jm}))$ in Eq. (15) provides an infrared regulator for an integral that would exhibit collinear divergences otherwise.

4. N-jettiness soft function at NNLO

We will now show how to extend the approach used in our NLO calculation to compute the NNLO contribution to the N-jettiness soft function. The NNLO contribution to the bare soft function is

$$S_2 = S_{2,RR} + S_{2,RV} - a_s \frac{\beta_0}{\epsilon} S_1, \tag{16}$$

where $S_{2,RR}$ is the double real-emission contribution, $S_{2,RV}$ is the real-virtual contribution and the last term is due to the renormalization of the strong coupling constant in the NLO soft function.

We further split the double-real contribution into correlated and uncorrelated pieces following the color structure

$$S_{2,RR,\tau} = S_{2,RR,T^4} + S_{2,RR,T^2} = \frac{1}{2} \sum_{(ij),(k,l)} \{\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l\}\, I_{T^4,ij,kl} - \frac{C_A}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j\, I_{T^2,ij}, \tag{17}$$

The real-virtual contribution reads

$$\begin{aligned} S_{2,RV,\tau} &= S_{RV,T^2} + S_{RV,\mathrm{tc}} \\ &= \frac{[\alpha_s]\, 2^{-\epsilon}}{\epsilon^2} C_A A_K(\epsilon) \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j\, I_{RV,ij} + [\alpha_s] \frac{4\pi N_\epsilon}{\epsilon} \sum_{(kij)} \kappa_{ij} F^{kij} I_{kij}, \end{aligned} \tag{18}$$

where $\kappa_{ij} = \lambda_{ij} - \lambda_{im} - \lambda_{jm}$, with $\lambda_{ij} = 1$ if both i and j refer to incoming/outgoing partons and zero otherwise. We have defined $F^{kij} = f_{abc} T_k^a T_i^b T_j^c$, while $A_K(\epsilon)$ and N_ϵ are normalization factors

$$A_K(\epsilon) = \frac{\Gamma^3(1+\epsilon)\Gamma^5(1-\epsilon)}{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}, \quad N_\epsilon = \frac{\Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(1-2\epsilon)}. \tag{19}$$

Our calculation of the renormalized soft function at NNLO is organized as follows

$$\tilde{S}_2 = \tilde{S}_2^{\mathrm{uncorr}} + \tilde{S}_2^{\mathrm{corr}} + \tilde{S}_2^{\mathrm{tc}},$$

where the different pieces are given by the following contributions

$$\begin{aligned} \tilde{S}_2^{\mathrm{uncorr}} &= \frac{1}{2} \tilde{S}_1 \tilde{S}_1, \\ \tilde{S}_2^{\mathrm{corr}} &= S_{2,RR,T^2} + S_{RV,T^2} - Z_{2,r} - Z_{2,r}^\dagger - \frac{a_s \beta_0}{\epsilon} S_1, \\ \tilde{S}_2^{\mathrm{tc}} &= \frac{1}{2}\left[Z_1, Z_1^\dagger\right] + \frac{1}{2}\left[S_1, Z_1 - Z_1^\dagger\right] + S_{RV,\mathrm{tc}}. \end{aligned} \tag{20}$$

In the following subsections, we will discuss how each of these contributions is treated.

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4.1 Uncorrelated emission

The most straightforward contribution comes from the iterated contribution of the NLO soft function S_1 that is contained in S_2. The integral that appears in this contribution is

$$I_{T^4,ij,kl} = \frac{[\alpha_s]^2}{2} \left\langle \int_0^\infty \frac{dE_m}{E_m^{1+2\epsilon}} \frac{dE_n}{E_n^{1+2\epsilon}} \, \delta(\tau - E_m\psi_m - E_n\psi_n) \frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \frac{\rho_{kl}}{\rho_{kn}\rho_{ln}} \right\rangle_{mn}. \tag{21}$$

If we integrate over both energies

$$\int_0^\infty \frac{dE_m}{E_m^{1+2\epsilon}} \frac{dE_n}{E_n^{1+2\epsilon}} \, \delta(\tau - E_m\psi_m - E_n\psi_n) = \frac{\tau^{-1-4\epsilon}}{\Gamma(-4\epsilon)} \frac{\psi_m^{2\epsilon}\Gamma(1-2\epsilon)}{2\epsilon} \frac{\psi_n^{2\epsilon}\Gamma(1-2\epsilon)}{2\epsilon}, \tag{22}$$

and apply the Laplace transformation, we can then properly identify the iteration of the NLO result

$$\begin{aligned} S_{2,RR,T^4} &= \frac{[\alpha_s]^2}{4} \sum_{(ij),(kl)} \{\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l\} \left(\frac{u^{2\epsilon}\Gamma(1-2\epsilon)}{2\epsilon}\right)^2 \left\langle \psi_m^{2\epsilon} \frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right\rangle_m \left\langle \psi_n^{2\epsilon} \frac{\rho_{kl}}{\rho_{kn}\rho_{ln}} \right\rangle_n \\ &= \frac{1}{2} \, S_1 S_1, \end{aligned} \tag{23}$$

where in the last step we used the symmetry between the (ij) and (kl) summation indices. It is straightforward to combine the above result with the iterated contribution of the renormalization constant Z_1 to arrive at the relevant contribution to the renormalized soft function.

4.2 Terms with three color charges

This contribution depends on triple products of color charges. In processes with only three hard partons this contribution vanishes due to color conservation, but in the case of four or more partons it can lead to relevant corrections. It originates from the following terms in $\tilde{S}_2$

$$\tilde{S}_2^{\rm tc} = \frac{1}{2}\left[Z_1, Z_1^\dagger\right] + \frac{1}{2}\left[S_1, Z_1 - Z_1^\dagger\right] + S_{RV,\rm tc}.$$

The commutators appearing in this expression can be computed as shown in [20]

$$\begin{aligned} \frac{1}{2}[Z_1, Z_1^\dagger] &= -\frac{2\pi a_s^2}{\epsilon^2} \sum_{(kij)} \lambda_{kj} L_{ij} F^{kij} = -\frac{\pi a_s^2}{\epsilon^2} \sum_{(kij)} \lambda_{kj} \ln\eta_{ij} \, F^{kij}, \\ \frac{1}{2}[S_1, Z_1 - Z_1^\dagger] &= -\frac{a_s^2\pi(\mu u)^{2\epsilon}}{\epsilon^2} \frac{e^{\gamma_E\epsilon}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \sum_{(kij)} \kappa_{kj} \left\langle \psi_m^{2\epsilon} \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \right\rangle_m F^{kij}. \end{aligned} \tag{24}$$

The real-virtual triple-color correlated contribution reads

$$S_{RV,\rm tc} = \frac{a_s^2\pi(\mu\,\bar{u})^{4\epsilon} N_\epsilon 2^{-\epsilon}}{2\epsilon^2} \frac{\Gamma(1-4\epsilon)}{\Gamma^2(1-\epsilon)e^{2\gamma_E\epsilon}} \sum_{(kij)} \kappa_{kj} \left\langle \psi_m^{4\epsilon} \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \left(\frac{\rho_{kj}}{\rho_{km}\rho_{jm}}\right)^\epsilon \right\rangle_m F^{kij}. \tag{25}$$

Following the NLO case, we isolate the jettiness-dependent part of the integrals

$$\begin{aligned} \left\langle \psi_m^{2\epsilon} \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \right\rangle_m &= \left\langle (1 + 2\epsilon g_{ki,m}^{(2)}) \frac{\rho_{ki}^{1-2\epsilon}}{\rho_{km}^{1-2\epsilon}\rho_{im}^{1-2\epsilon}} \right\rangle_m, \\ \left\langle \psi_m^{4\epsilon} \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \left(\frac{\rho_{kj}}{\rho_{km}\rho_{jm}}\right)^\epsilon \right\rangle_m &= \left\langle \left(1 + 4\epsilon g_{ki,m}^{(4)}\right) \frac{\rho_{ki}^{1-4\epsilon}}{\rho_{km}^{1-4\epsilon}\rho_{im}^{1-4\epsilon}} \left(\frac{\rho_{kj}}{\rho_{km}\rho_{jm}}\right)^\epsilon \right\rangle_m. \end{aligned} \tag{26}$$

It is easy to show that the poles depending on the N-jettiness function cancel

$$\tilde{S}_2^{\text{tc}} \to -\frac{2a_s^2\pi}{\epsilon} \sum_{(kij)} \kappa_{kj} \left\langle \frac{\rho_{ik}}{\rho_{im}\rho_{km}} \left(g^{(2)}_{ki,m} - g^{(4)}_{ki,m} \right) \right\rangle_m F^{kij} = O(\epsilon^0), \tag{27}$$

since $g^{(2)} = g^{(4)}$ through order $O(\epsilon^0)$.

The N-jettiness dependent finite reminder is in this case given by

$$\tilde{S}_2^{\text{tc}} \to 2a_s^2\pi \sum_{(kij)} \kappa_{kj} \left\langle \frac{\rho_{ki}}{\rho_{im}\rho_{km}} \ln\left(\frac{\psi_m \rho_{ki}}{\rho_{km}\rho_{im}} \right) \ln\left(\frac{(\bar{u}\mu)^2 \psi_m \rho_{im}\rho_{kj}}{2\rho_{jm}\rho_{ki}} \right) \right\rangle_m F^{kij}. \tag{28}$$

For the rest of the finite part, the idea is to use the results of Ref. [20], where the analytic expression for the following integral

$$\left\langle \frac{\rho_{ki}}{\rho_{km}\rho_{im}} \left(\frac{\rho_{kj}}{\rho_{km}\rho_{jm}} \right)^{\epsilon} \right\rangle_m, \tag{29}$$

was calculated. The idea is to take the difference of the two results. By doing that, one finds that the complicated finite integral appears at order $O(\epsilon^2)$ only. This integral can then be numerically computed alongside with N-jettiness dependent contributions.

4.3 Correlated emission

The only contributions left to the NNLO soft function are those that are not iterations of NLO terms, and do not contain triple-color correlators. The calculation of these correlated terms are the main bulk of the calculation of the NNLO N-jettiness soft function

$$\tilde{S}_2^{\text{corr}} = S_{2,RR,T^2} + S_{RV,T^2} - Z_{2,r} - Z_{2,r}^{\dagger} - \frac{a_s\beta_0}{\epsilon} S_1. \tag{30}$$

Since the full calculation is quite technical, here we will only sketch the idea of how the calculation in performed; the full detailed computation can be found in Ref. [17]. The last three renormalization terms in Eq. (30) do not require any integration, and the real-virtual one is simply given by

$$S_{RV,T^2} = -\frac{[\alpha_s]^2 (\bar{u})^{4\epsilon} e^{-4\gamma_E \epsilon}\Gamma(1-4\epsilon)}{2^{2+\epsilon}\ \epsilon^3} C_A A_K(\epsilon) \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left\langle \psi_m^{4\epsilon} \left(\frac{\rho_{ij}}{\rho_{im}\rho_{jm}} \right)^{1+\epsilon} \right\rangle_m, \tag{31}$$

which can be computed with the standard NLO techniques.

The first term in Eq. (30), that involves the correlated emission eikonal term $S_{ij}^{gg}(m,n)$, is the one that requires attention

$$\begin{aligned} S_{2,RR,T^2,\tau} &= -\frac{C_A}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j\ I_{ij,\tau} \\ &= -\frac{C_A}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j\ \frac{g_s^4}{2} \int [dp_m][dp_n]\ \delta\left(\tau - E_m\psi_m - E_n\psi_n\right) \tilde{S}_{ij}^{gg}(m,n), \end{aligned} \tag{32}$$

where $\tilde{S}_{ij}^{gg}(m,n)$ is the correlated-emission eikonal function defined in Ref. [21].

To calculate this last term, the idea is to perform a nested subtraction of the soft and collinear divergent limits, following the work done in Refs. [4, 20]. We begin by subtracting the "strongly ordered" limit, which corresponds to the double soft limit with energy ordering

$$S_{2,RR,T^2} = -\frac{C_A}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \left[\bar{S}_\omega I_{ij} + S_\omega I_{ij} \right], \tag{33}$$

with S_ω being the operator that enforces the strongly-ordered limit. The divergent terms of $S_\omega I_{ij}$ can be calculated analytically by performing subtractions. The N-jettiness dependent poles cancel against those of the RV contribution.

After the calculating the strongly ordered limit, there are only collinear divergences remaining in $\bar{S}_\omega I_{ij} = (1 - S_\omega)\, I_{ij}$. The idea now is to introduce partitions functions [20] to separate double-collinear and triple-collinear singularities, allowing us to split the integral

$$\bar{S}_\omega [I_{ij}] = \bar{S}_\omega [I_{ij}^{dc}] + \bar{S}_\omega [I_{ij}^{tc}]. \tag{34}$$

The double-collinear contribution in Eq. (34)

$$\bar{S}_\omega [I_{ij}^{dc}] = \frac{N_u}{\epsilon} \int_0^1 \frac{d\omega}{\omega^{1+2\epsilon}} \left\langle \psi_{mn}^{4\epsilon} \left(w^{mi,nj} + w^{ni,mj} \right) \bar{S}_\omega \left[\omega^2 \tilde{S}_{ij}^{gg} \right] \right\rangle_{mn} \tag{35}$$

has ϵ poles independent of the N-jettiness, which can be obtained from the calculation done in [22] (which is exactly the same, but without the N-jettiness delta function). The jettiness-dependent part can be safely calculated numerically.

The triple-collinear contribution in Eq. (34) is

$$\bar{S}_\omega [I_{ij}^{tc}] = \frac{N_u}{\epsilon} \int_0^1 \frac{d\omega}{\omega^{1+2\epsilon}} \left\langle \psi_{mn}^{4\epsilon}\, w^{tc}\, \bar{S}_\omega \left[\omega^2 \tilde{S}_{ij}^{gg} \right] \right\rangle_{mn}, \tag{36}$$

where the partition function $w^{tc} = w^{mi,ni} + w^{mj,nj}$ combines both triple-collinear divergent contributions (the one collinear to parton i and to j). In this case we also need to introduce a further split of the partitions into sectors [20] in order to deal with the collinear singularity between the two unresolved partons, that is, when $m||n$.

Once again, the strategy is to identify the terms that correspond to the calculation without the N-jettiness constraint. This allows us to avoid calculating complicated finite terms, leaving only the jettiness-dependent ones for numeric evaluation.

4.4 Soft quark contribution

The NNLO N-jettiness soft function also includes a contribution of a soft $q\bar{q}$ pair. The entire contribution comes from correlated emissions

$$S^{q\bar{q}}_{2,RR,T^2,\tau} = \frac{n_f T_R}{2} \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j \; I^{q\bar{q}}_{ij,\tau}, \tag{37}$$

where

$$I^{q\bar{q}}_{ij,\tau} = g_s^4 \int [\mathrm{d}p]_m [\mathrm{d}p]_n \delta(\tau - E_m \psi_m - E_n \psi_n)\; \tilde{S}^{q\bar{q}}_{ij}(m,n), \tag{38}$$

and the quark eikonal function $\tilde{S}_{ij}^{q\bar{q}}$ is defined in Ref. [21]. For this contribution we proceed in exactly the same way as with the gluon contributions. This case is significantly easier since the $q\bar{q}$ eikonal function does not possess a strongly-ordered singular limit, and some of the previous results can be reused. Since the computation of the quark contribution is analogous to what has been discussed in the two-gluon case we do not discuss it further.

4.5 The final result

Having discussed all the relevant contributions, we can now present the renormalized N-jettiness soft function at NNLO in QCD. The NNLO result is constructed from different pieces. We write

$$\tilde{S}_2 = \frac{1}{2}\tilde{S}_1^2 + a_s^2 C_A \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j\, G_{ij} + a_s^2\, n_f\, T_R \sum_{(ij)} \mathbf{T}_i \cdot \mathbf{T}_j\, Q_{ij} + a_s^2 \pi \sum_{(kij)} F^{kij}\, \kappa_{kj} G_{kij}^{\text{triple}}, \tag{39}$$

where G_{ij}, Q_{ij} and G_{kij}^{triple} are finite functions with analytical terms along with a *reduced number of numerical integrations* over one- and two-particle phase space in four-dimensions. Their explicit expressions are somewhat lengthy, so we refer the reader to Ref. [17]. The first term in the right-hand side of Eq. (39) corresponds to the iterated contribution of the renormalized NLO soft function of Eq. (15). The second and third terms come from the correlated gluon and quark emission contributions, respectively, while the last one is the one with the one that depends on triple products of color charges. This last term can only contribute to 2- or higher jettiness processes in hadron colliders, or to 4 or higher-jet production at $e^+ + e^-$-colliders.

5. Numerical implementation and checks

In order to corroborate our approach, we compared our results for the N-jettiness soft function with the recent ones in Ref. [16]. We implemented our formulas into a Fortran code. For the computation of the terms that require integration over directions of the unresolved particles' momenta, we use the phase space parametrization described in Refs. [4, 5, 23].

Here, we focus in the 3-jettiness case, that was calculated for the first time in Ref. [16] as a benchmark. A more detailed analysis can be found in Ref. [17]. We consider the configuration with two back-to-back beams and three final-state jets. The five relevant directions are given by

$$\begin{gathered} n_1 = (0,0,1), \quad n_2 = (0,0,-1), \quad n_3 = (\sin\theta_{13}, 0, \cos\theta_{13}), \\ n_4 = (\sin\theta_{14}\cos\phi_4, \sin\theta_{14}\sin\phi_4, \cos\theta_{14}), \quad n_5 = (\sin\theta_{15}\cos\phi_5, \sin\theta_{15}\sin\phi_5, \cos\theta_{15}). \end{gathered} \tag{40}$$

We choose in the (arbitrary) phase space point with the angles

$$\theta_{13} = \frac{3\pi}{10}, \quad \theta_{14} = \frac{6\pi}{10}, \quad \theta_{15} = \frac{9\pi}{10}, \quad \phi_4 = \frac{3\pi}{5}, \quad \phi_5 = \frac{6\pi}{5}. \tag{41}$$

For this comparison, we define two functions G_{ij}^{nl} and Q_{ij}^{nl} that are obtained from G_{ij} and Q_{ij} by setting all terms with L_{ij} to zero

$$\begin{aligned} G_{ij} =& \frac{22}{9}L_{ij}^3 + \left(\frac{67}{9} - \frac{\pi^2}{3}\right)L_{ij}^2 + L_{ij}\Big(\frac{11}{3}\left\langle L_{ij,m}^{\psi}\frac{\rho_{ij}}{\rho_{im}\rho_{jm}}\right\rangle_m + \frac{11}{3}\mathrm{Li}_2(1-\eta_{ij}) \\ &+ \frac{202}{27} - 7\zeta_3\Big) + G_{ij}^{nl} \end{aligned}, \tag{42}$$

$$Q_{ij} = -\frac{8}{9}L_{ij}^3 - \frac{20}{9}L_{ij}^2 - L_{ij}\left(\frac{4}{3}\left\langle L_{ij,m}^{\psi}\frac{\rho_{ij}}{\rho_{im}\rho_{jm}}\right\rangle_m + \frac{4}{3}\,\mathrm{Li}_2(1-\eta_{ij}) + \frac{56}{27}\right) + Q_{ij}^{nl}. \tag{43}$$

In Table 1 we compare our 3-jettiness dipole contributions to the benchmark. In all the dipole configurations we find excellent agreement, specially if we take into account that the error shown are from the Vegas integration which are likely to underestimate the true uncertainties.

Dipoles	Gluons		Quarks	
	G_{ij}^{nl}	Ref. [16]	Q_{ij}^{nl}	Ref. [16]
12	116.20 ± 0.01	116.20 ± 0.16	-36.249 ± 0.001	-36.244 ± 0.009
13	38.13 ± 0.03	37.63 ± 0.03	-21.717 ± 0.007	-21.732 ± 0.005
14	63.63 ± 0.01	63.66 ± 0.06	-25.189 ± 0.003	-25.192 ± 0.006
15	107.17 ± 0.01	106.99 ± 0.12	-35.268 ± 0.001	-35.256 ± 0.009
23	97.11 ± 0.01	96.97 ± 0.10	-32.875 ± 0.002	-32.872 ± 0.008
24	67.36 ± 0.02	67.51 ± 0.08	-26.821 ± 0.003	-26.815 ± 0.007
25	30.87 ± 0.03	30.73 ± 0.04	-21.561 ± 0.009	-21.561 ± 0.005
34	69.43 ± 0.01	69.24 ± 0.07	-25.854 ± 0.002	-25.861 ± 0.006
35	106.13 ± 0.02	105.97 ± 0.13	-34.799 ± 0.002	-34.796 ± 0.008
45	74.45 ± 0.02	74.36 ± 0.09	-28.247 ± 0.004	-28.251 ± 0.007

Table 1: Comparison of the selected results for the 3-jettiness soft function for the kinematic point in Eq. (41). When quoting results for functions G_{ij}^{nl}, Q_{ij}^{nl}, we show Vegas integration errors which, most likely, underestimate the true uncertainties of the result.

We note that to obtain the high-precision numbers shown in Table 1, we used a large number of sample points, which results in runtimes as large as a few minutes per dipole. However, to obtain the same numbers with a better-than-percent precision, we need a relatively smaller number of sampling points, resulting in runtimes of an order of a few seconds per dipole. Runtimes for individual dipoles are largely *independent* of N. Since $(N+2)(N+1)/2$ dipoles need to be calculated to get the full N-jettiness soft function at a given phase space point, $O(1-2)$ minutes would be required to do that in case of three- or four-jet production at hadron colliders, or six-jet production at an e^+e^- collider.

For the tripole contributions (the terms with three color charges) in this configuration, there are four independent color structures. In Table 2 we present our results for each of those four color structures, as defined in [16]. Once again, we find excellent agreement between the two results.

	$\tilde{c}^{(2,124)}_{\text{tripoles}}$	$\tilde{c}^{(2,125)}_{\text{tripoles}}$	$\tilde{c}^{(2,145)}_{\text{tripoles}}$	$\tilde{c}^{(2,245)}_{\text{tripoles}}$
$\tilde{c}_{\text{tripoles}}$	-683.25 ± 0.01	-2203.3 ± 0.2	-6.324 ± 0.004	-0.837 ± 0.008
Ref. [16]	-683.23 ± 0.04	-2203.5 ± 0.1	-6.325 ± 0.04	-0.830 ± 0.039

Table 2: Same as in Table 1 for the four independent triple-color correlated contributions.

The performance situation with tripoles is similar to the dipole one. Since in this case we only require the integration over the direction of a single gluon, the integration converges rather fast. In general, for an N-jettiness case, the evaluation of $(N+2)(N+1)N$ independent functions

G^{triple}_{kij} is required from which a smaller number of independent color-correlated contributions can be constructed. We need about twenty seconds to compute all the required G^{triple}_{kij} functions for $N = 2$ and, therefore, we would need about a minute to compute them for $N = 3$, and about two minutes for $N = 4$.

6. Conclusions

We have presented our calculation of the N-jettiness soft function at NNLO in QCD. While keeping N as a parameter, we analytically demonstrated the cancellation of all $1/\epsilon$ poles against the soft-function renormalization matrix. We arrived at a simple representation for the finite, jettiness-dependent remainder valid for an arbitrary number of hard partons N, allowing for faster implementations of the calculation of the soft function.

We found excellent agreement between our numerical results for $N = 1, 2$ and $N = 3$ and the ones recently presented in Ref. [16]. We also found that the representation of the finite remainder that is derived in this paper leads to fast and rapidly convergent integration, which is relevant for the use of the N-jettiness soft function in the computation of higher-order corrections for multi-jet production at hadron colliders.

We have also shown that, at least for the N-jettiness observable, the benefits of applying subtraction-inspired methods to derive representations for the building blocks of modern slicing methods are significant. This application of the subtraction-inspired methods for computing the N-jettiness soft function becomes possible when one departs from the (by now) standard approach [18, 19, 24] to such computations where hemisphere soft functions are used as elementary building blocks and, instead, interprets N-jettiness as one of the many infrared safe observables that can be studied using available subtraction schemes.

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