PROCEEDINGS OF SCIENCE

Causality and differential cross sections

Germán Rodrigo[a,*]

[a] *Instituto de Física Corpuscular, Universitat de València - Consejo Superior de Investigaciones Científicas, Parc Científic, E-46980 Paterna, Valencia, Spain*

E-mail: german.rodrigo@csic.es

The artificial separation between loop and tree-level contributions is at the origin of most of the technical difficulties in quantum field theory at high perturbative orders. The original motivation for the loop-tree duality (LTD), as explained in the seminal paper by Stefano Catani, was to circumvent this separation by opening the loops to tree-level objects in such a way that both contributions would be treated on the same footing. One of the unexpected properties of LTD is that the integrand of scattering amplitudes becomes manifestly causal. By exploiting this physically motivated property, we propose vacuum amplitudes in LTD as the optimal building blocks to assemble theoretical predictions for differential cross sections at high-energy colliders.

Loops and Legs in Quantum Field Theory (LL2024)
14-19, April, 2024
Wittenberg, Germany

*Speaker

PoS(LL2024)024

1. Causality in Feynman diagrams

Scattering amplitudes in the Feynman representation are constructed from Feynman propagators and interaction vertices. A Feynman propagator describes the propagation of a particle between two interaction vertices in either directions. Therefore, abusing of a quantum mechanical notation, a Feynman propagator can be formally written as

$$G_{\mathrm{F}}(q_i) = \frac{1}{q_i^2 - m_i^2 + \imath 0} \equiv \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) , \tag{1}$$

where $|0\rangle$ and $|1\rangle$ represent the two propagation states. When a Feynman diagram contains n propagators, it actually encodes the quantum superposition of 2^n states, with each state defining the directions in which the particles propagate. The interaction vertices are defined in the spacetime, so when a particle in a Feynman loop diagram departs from an interaction vertex and describes a closed cycle returning to the initial point, it necessarily travels back in time and thus breaks causality.

The directions of propagation are not usually specified in theoretical calculations because the Feynman representation provides a very compact mathematical description. However, this apparent simplicity comes at a price. It involves unphysical cyclic configurations which inevitably lead to unphysical singularities of the integrand. A solution is provided by the loop-tree duality (LTD) [1–18]. In LTD, one degree of freedom of each loop momenta is integrated out by using the Cauchy residue theorem, which is equivalent to set on shell certain internal particles. In other words, loop scattering amplitudes are open in LTD to a sum of tree-level objects. If the energy components of the loop momenta are integrated out, the integrand of the scattering amplitude gets support in the Euclidean space of the loop three-momenta, which has some advantages for asymptotic expansions [19, 20]. The most general form of an scattering amplitude in LTD in d spacetime dimensions is

$$\mathcal{A}^{(\Lambda)} = \int_{\boldsymbol{\ell}_1 \cdots \boldsymbol{\ell}_\Lambda} \mathcal{A}_{\mathrm{D}}^{(\Lambda)} , \qquad \int_{\boldsymbol{\ell}_j} = \mu^{4-d} \int \frac{d^{d-1} \boldsymbol{\ell}_j}{(2\pi)^{d-1}} , \tag{2}$$

where Λ is the number of independent loop momenta, and the integrand $\mathcal{A}_{\mathrm{D}}^{(\Lambda)}$ is the dual amplitude, which is a function of the loop three-momenta and of the external momenta. The scale μ is arbitrary.

Remarkably, the dual amplitude contains only configurations that respect causality [21–28]. These configurations are equivalent to directed acyclic graph (DAG) configurations in graph theory [29–31]. Another interesting interpretation sees causality as the concatenation of entangled causal thresholds. Each causal threshold divides the scattering amplitude into two subamplitudes, in such a way that all the particles involved are aligned in the same propagation direction. Different causal thresholds are compatible if the shared particles are aligned in the same direction. The causal thresholds cannot cross. In a more mathematical approach, each causal threshold is a bipartite partition of the set of propagators, and the causal configurations are the result of a recursive application of bipartite partitions [25].

The LTD representation of a scattering amplitude is then the sum of all possible causal configurations where internal particles propagate in specific directions. Conversely, once the directions of propagation are known, they provide information to guide the bootstrapping of the corresponding LTD representation. Because of Eq. (1), identifying the causal / acyclic configurations of multiloop

Feynman diagrams marks a benchmark application to challenge quantum algorithms in particle physics [29–31].

2. Singularities of scattering amplitudes

Scattering amplitudes at high perturbative orders in quantum field theory are the central objects from which precise theoretical predictions for high-energy colliders are derived. They exhibit very interesting mathematical properties, which extend the interest in their study beyond particle physics, e.g. in gravitational physics [32]. Highly efficient and sophisticated methods and tools have been developed for their evaluation that are presented in this conference (e.g. [33–35]).

However, scattering amplitudes are defined for a fixed number of external particles, and different scattering amplitudes squared with different numbers of external particles must be combined to obtain theoretical predictions for collider observables. In theories such as Quantum Chromodynamics, where particles can be emitted at exactly zero energy or collinear with each other, fixing the number of external particles creates a mismatch between the mathematical description and the physics behind it, and makes scattering amplitudes tricky, specially in the four dimensions of the spacetime. In addition, loop configurations involve extreme quantum fluctuations at infinite energy (or exactly zero distance), where the theory is no longer strictly valid. The most common workaround is to evaluate loop and tree-level scattering amplitudes in arbitrary spacetime dimensions, e.g. in Dimensional Regularisation (DREG), where the ambiguities from infrared (IR) and ultraviolet (UV) configurations are translated into poles of the extra dimensions. Subtraction [36] methods are usually employed to get rid of these singularities.

3. Vacuum amplitudes in the loop-tree duality

The artificial separation between loop and tree-level contributions is at the origin of most of the technical difficulties encountered in theoretical calculations at high perturbative orders. To circumvent this separation, we recently proposed [37, 38] vacuum amplitudes, i.e. scattering amplitudes without external particles, in LTD as the optimal building blocks to assemble theoretical predictions for differential observables at colliders. We work in LTD because it provides a manifestly causal representation.

The Feynman propagators of a vacuum amplitude are substituted in LTD by causal propagators of the form [21]

$$\frac{1}{\lambda_{i_1 i_2 \cdots i_n}} = \frac{1}{\sum_{s=1}^{n} q_{i_s,0}^{(+)}} , \tag{3}$$

with $q_{i_s,0}^{(+)} = \sqrt{\mathbf{q}_{i_s}^2 + m_{i_s}^2 - \imath 0}$ the on-shell energies of the internal momenta, $\mathbf{q}_{i_s}$ the spatial components and m_{i_s} their masses. The factor $\imath 0$ stems from the original infinitesimal complex prescription of the Feynman propagators.

A causal propagator of the form in Eq. (3) represents a causal threshold involving a set of internal particles that divide the vacuum amplitude into two subamplitudes. And the dual amplitude is a sum of terms consisting of products of causally compatible causal propagators. A causal propagator would turn singular, $\lambda_{i_1 i_2 \cdots i_n} \to 0$, if all the particles involved were on shell. The difference with a conventional scattering amplitude is that, in the absence of external particles, a causal propagator cannot generate any IR or threshold singularity because the on-shell energies of the internal particles are, by definition, positive. Only UV singularities are allowed.

The *central hypothesis* of our proposal [37, 38] is that by analytically continuing the on-shell energies of the particles that will be identified as incoming to negative values, and by considering different residues on the causal propagators, $1/\lambda_{i_1 i_2 \cdots i_n}$, dubbed phase-space residues, all the amplitude interferences with different numbers of external particles contributing to a differential cross section or a decay rate are generated at once, with the dual vacuum amplitude acting as a kernel for all contributions. Since IR and threshold singularities are absent in the vacuum amplitude, and since all contributions are generated from the same vacuum amplitude, collinear, soft and threshold singularities locally match among the different phase-space residues, with the sole exception of initial-state collinear singularities whose local cancellation is limited by kinematics. Another advantage is that the vacuum amplitude includes selfenergy insertions. Therefore, the wave-function renormalisation of the external legs is naturally incorporated by the phase-space residues. These contributions are essential for the local cancellation of IR singularities.

4. Differential observables from vacuum amplitudes: LTD causal unitary

The fundamental building blocks in LTD causal unitary [37, 38] for theoretical predictions of differential observables at colliders are the phase-space residues defined as

$$\mathcal{A}_{\mathrm{D}}^{(\Lambda,\mathrm{R})}(i_1 \cdots i_n ab) \quad = \quad \mathrm{Res}\left(\frac{x_{ab}}{2}\,\mathcal{A}_{\mathrm{D}}^{(\Lambda)}, \lambda_{i_1 \cdots i_n ab}\right) - \mathcal{A}_{\mathrm{UV/C}}^{(\Lambda)}(i_1 \cdots i_n ab)\,, \tag{4}$$

where the first term on the r.h.s of Eq. (4) is the residue of the kernel vacuum amplitude $\mathcal{A}_{\mathrm{D}}^{(\Lambda)}$ in LTD at $\lambda_{i_1 \cdots i_n ab} = 0$. The number of loops of $\mathcal{A}_{\mathrm{D}}^{(\Lambda)}$ is $\Lambda = L + N - 1$, where N is the total number of external particles in leading order (LO) kinematics, and L is the maximum number of loops that contribute at N^{k}LO. Typically, $L = 2$ at next-to-next-to-leading order (NNLO). Each residue on the kernel vacuum amplitude implements a different n-particle final state. The indices of the initial-state particles are a and b, and their on-shell energies are analytically continued to negative values. We defined $x_{ab} = 4 q_{a,0}^{(+)} q_{b,0}^{(+)}$. The counterterm $\mathcal{A}_{\mathrm{UV/C}}^{(\Lambda)}$ implements a local UV renormalisation, and a local subtraction of initial-state collinear singularities.

Then, the differential representation of the N^{k}LO contribution to a physical observable is

$$d\sigma_{\mathrm{N}^k\mathrm{LO}} = \frac{d\Lambda}{2s} \sum_{(i_1 \cdots i_n ab) \in \Sigma} \mathcal{A}_{\mathrm{D}}^{(\Lambda,\mathrm{R})}(i_1 \cdots i_n ab)\, O_{i_1 \cdots i_n}\, \widetilde{\Delta}_{i_1 \cdots i_n \bar{a} \bar{b}}\,, \tag{5}$$

For a scattering process, the integration measure,

$$d\Lambda = \prod_{j=1}^{\Lambda-2} d\Phi_{\boldsymbol{\ell}_j} = \prod_{j=1}^{\Lambda-2} \mu^{4-d} \frac{d^{d-1}\boldsymbol{\ell}_j}{(2\pi)^{d-1}}\,, \tag{6}$$

is written in terms of the spatial components of $\Lambda - 2$ primitive loop momenta because two of the loop three-momenta are fixed by the initial-state. The Dirac-delta function in the spatial components

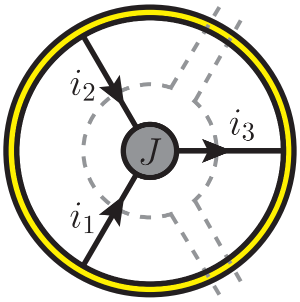

Figure 1: Diagrammatic interpretation of the local cancellation of final-state double-collinear singularities, including quasicollinear configurations. Causal propagators represented by dashed lines. The blob J stands for a multiloop subdiagram. Each phase-space residue is identified with the interference of scattering amplitudes with different numbers of external particles, namely different numbers of loops.

of the external momenta is absent because momentum conservation in the spacial components is self-satisfied in the vacuum amplitude. Energy conservation is imposed by the Dirac-delta function

$$\widetilde{\Delta}_{i_1\cdots i_n\bar{a}\bar{b}} = 2\pi\, \delta(\lambda_{i_1\cdots i_n\bar{a}\bar{b}})\,, \tag{7}$$

where $\lambda_{i_1\cdots i_n\bar{a}\bar{b}} = \sum_{s=1}^{n} q_{i_s,0}^{(+)} - q_{a,0}^{(+)} - q_{b,0}^{(+)}$. The bar over a and b indicates that the corresponding on-shell energies bear a minus sign. Finally, the function $O_{i_1\cdots i_n}$ encodes the observable under consideration. The default choice $O_{i_1\cdots i_n} = 1$ gives the total cross section or decay rate after integration.

5. Collinear, soft and threshold singularities

The kernel vacuum amplitude $\mathcal{A}_{\rm D}^{(\Lambda)}$ is free of IR and threshold singularities because the on-shell energies are positive. However, the phase-space residues in Eq. (4) exhibit the expected singularities because some of the on-shell energies are promoted to negative values. These singularities locally match in the master differential representation in Eq. (5), and therefore Eq. (5) is well defined directly in the four physical dimensions of the spacetime. Detailed proofs of the local matching of singularities between phase-space residues have been presented in Ref. [37]. Here, we comment on the simplest configuration of double-collinear singularities in final states.

We consider the insertion of a multiloop interaction involving three particles labelled i_1, i_2 and i_3, as shown in Fig. 1. The dual vacuum amplitude contains a term proportional to

$$\mathcal{A}_{\rm D}^{(\Lambda)} \sim \frac{1}{\lambda_{i_1 i_2\cdots ab}\lambda_{i_3\cdots ab}}\,, \tag{8}$$

where the dots represent other particles, acting as spectators. Here, a and b are identified with the initial-state particles and their on-shell energies are promoted to negative values. The two phase-space residues

$$\mathcal{A}_{\rm D}^{(\Lambda)}(i_3\cdots ab) = {\rm Res}\left(\frac{x_{ab}}{2}\mathcal{A}_{\rm D}^{(\Lambda)}, \lambda_{i_3\cdots ab}\right) \sim \frac{1}{\lambda_{i_1 i_2 \bar{i}_3}}\,, \tag{9}$$

$$\mathcal{A}_{\rm D}^{(\Lambda)}(i_1 i_2\cdots ab) = {\rm Res}\left(\frac{x_{ab}}{2}\mathcal{A}_{\rm D}^{(\Lambda)}, \lambda_{i_1 i_2\cdots ab}\right) \sim -\frac{1}{\lambda_{i_1 i_2 \bar{i}_3}}\,, \tag{10}$$

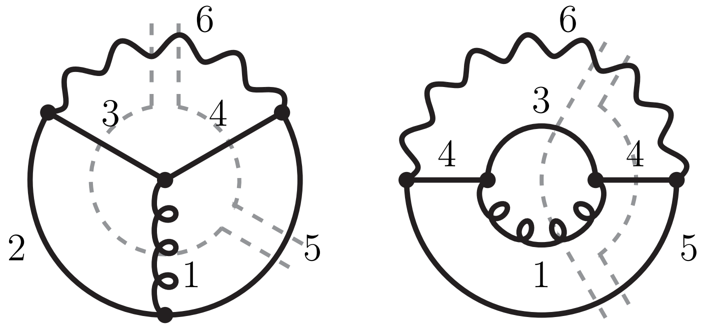

Figure 2: Three-loop vacuum diagrams contributing to the decay $\gamma^* \to q\bar{q}(g)$ at NLO. The gray dashed lines represent phase-space residues, i.e. different final states.

represent, respectively, a configuration with i_3 as an on-shell particle in the final state, and another configuration with one loop less, where i_1 and i_2 are on-shell and final.

For massless particles and in the limit $\lambda_{i_1 i_2 \bar{i}_3} = q_{i_1,0}^{(+)} + q_{i_2,0}^{(+)} - q_{i_3,0}^{(+)} \to 0$, each of these phase-space residues develops a collinear singularity, but the sum of both contributions is finite:

$$\lim_{\lambda_{i_1 i_2 \bar{i}_3} \to 0} \left(\mathcal{A}_{\mathrm{D}}^{(\Lambda)}(i_1 i_2 \cdots ab) \, \widetilde{\Delta}_{i_1 i_2 \cdots \bar{a}\bar{b}} + \mathcal{A}_{\mathrm{D}}^{(\Lambda)}(i_3 \cdots ab) \, \widetilde{\Delta}_{i_3 \cdots \bar{a}\bar{b}} \right) = O(\lambda^0_{i_1 i_2 \bar{i}_3}) \,, \tag{11}$$

because of the sign change from Eq. (9) to Eq. (10). The local cancellation of this collinear singularity is compatible with energy conservation: $\lim_{\lambda_{i_1 i_2 \bar{i}_3} \to 0} \widetilde{\Delta}_{i_1 i_2 \cdots \bar{a}\bar{b}} = \widetilde{\Delta}_{i_3 \cdots \bar{a}\bar{b}}$. It is interesting to note that for massive particles quasicollinear configurations in the form of large massive logarithms also match since particle masses appear implicitly in the causal propagators through the on-shell energies. This fact provides a seamless transition between a massive and a massless calculation, allowing for an almost identical implementation [8].

The local cancellation of collinear singularities in the initial state is limited by the phase space. However, LTD predicts that the functional form of the collinear singularity is the same for the loop and tree-level phase-space residues, and hence the unintegrated contributions to the Altarelli-Parisi splitting functions [39]. For example, the collinear splitting $q \to qg$ is described by the local splitting function

$$\mathcal{P}_{qq}^{(0)}(z_{\mathrm{V}}, z; \epsilon) = \mathcal{P}_{qq}^{(0)}(z_{\mathrm{V}}; \epsilon) \left(\delta(z_{\mathrm{V}} - z) - \delta(1 - z) \right) \,, \tag{12}$$

where

$$\mathcal{P}_{qq}^{(0)}(z; \epsilon) = \frac{1 + z^2}{1 - z} - \epsilon(1 - z) \tag{13}$$

is the customary bare splitting function, with $1 - z$ the longitudinal momentum fraction of the gluon when emitted as real, and $1 - z_{\mathrm{V}}$ the longitudinal momentum fraction of the same gluon in the loop contribution. Integrating over z_{V}, with $z_{\mathrm{V}} \in [0, 1]$, we obtain the expected loop factor $3/2 \, \delta(1 - z)$.

6. Proof of concept at NLO and NNLO

As benchmark decay rates at NLO, the decay of a heavy scalar into lighter scalars, and the decay of a Higgs boson or an off-shell photon into a pair of massive quarks and antiquarks were

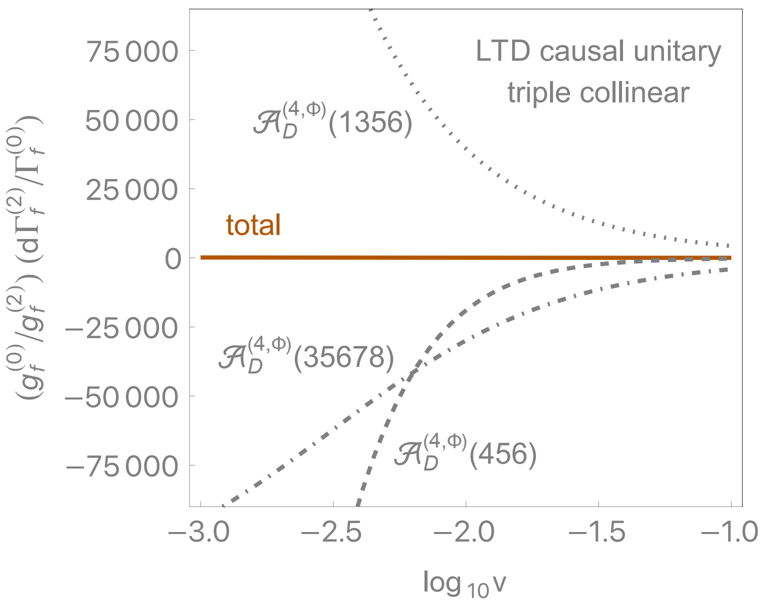

Figure 3: Local cancellation of tripple collinear singularities at NNLO. Details are provided in Ref. [38].

implemented as a proof of concept of LTD causal unitary in Ref. [38], where classical integration methods were used to predict the total decay rates. The good behaviour of the integrand also makes it suitable for a quantum integrator implementation [40, 41]. We refer to Ref. [38] for a detailed presentation of the expressions used in the numerical implementation. The vacuum diagrams that contribute to the decay $\gamma^* \to q\bar{q}(g)$ are shown in Fig. 2. Similar vacuum diagrams describe the other two decay processes considered. At NNLO, the decay of a heavy scalar into lighter scalars have been analysed. The local cancellation of a tripple-collinear singularity is illustrated in Fig. 3.

7. Conclusions

We have presented a novel representation of differential observables at high-energy colliders, where all final states contributing to a scattering or a decay process are coherently generated from a multiloop vacuum amplitude in LTD. This representation is well defined directly in the four physical dimensions of the spacetime. Exploiting the manifestly causal properties of vacuum amplitudes in LTD provides a consistent theoretical framework in which certain mathematical artefacts are absent and many technical difficulties are solved in one go. For example, our formalism deals directly with the actual momenta of the external particles, which is more convenient for predicting differential observables. The absence of collinear, soft and threshold singularities in the vacuum amplitude leads to a local matching of these singularities in the sum over all phase-space residues to all perturbative orders. Large logarithmic terms from quasicollinear configurations of massive particles are matched. Furthermore, the vacuum amplitude consistently incorporates gauge invariance and the wave function renormalisation of external particles, which is crucial for the IR local cancellation with tree-level contributions. The methodology has successfully been tested with proof-of-concept implementations at LO, NLO and NNLO.

Acknowledgements

This work is supported by the Spanish Government - Agencia Estatal de Investigación (MCIN/AEI/10.13039/501100011033) Grant No. PID2020-114473GB-I00, and Generalitat Valenciana Grants No. PROMETEO/2021/071 and ASFAE/2022/009 (Planes Complementarios de I+D+i, NextGenerationEU).

References

[1] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J.-C. Winter, *From loops to trees by-passing Feynman's theorem*, *JHEP* **09** (2008) 065, [0804.3170].

[2] I. Bierenbaum, S. Catani, P. Draggiotis and G. Rodrigo, *A Tree-Loop Duality Relation at Two Loops and Beyond*, *JHEP* **10** (2010) 073, [1007.0194].

[3] I. Bierenbaum, S. Buchta, P. Draggiotis, I. Malamos and G. Rodrigo, *Tree-Loop Duality Relation beyond simple poles*, *JHEP* **03** (2013) 025, [1211.5048].

[4] S. Buchta, G. Chachamis, P. Draggiotis, I. Malamos and G. Rodrigo, *On the singular behaviour of scattering amplitudes in quantum field theory*, *JHEP* **11** (2014) 014, [1405.7850].

[5] S. Buchta, G. Chachamis, P. Draggiotis and G. Rodrigo, *Numerical implementation of the loop–tree duality method*, *Eur. Phys. J. C* **77** (2017) 274, [1510.00187].

[6] R. J. Hernandez-Pinto, G. F. R. Sborlini and G. Rodrigo, *Towards gauge theories in four dimensions*, *JHEP* **02** (2016) 044, [1506.04617].

[7] G. F. R. Sborlini, F. Driencourt-Mangin, R. Hernandez-Pinto and G. Rodrigo, *Four-dimensional unsubtraction from the loop-tree duality*, *JHEP* **08** (2016) 160, [1604.06699].

[8] G. F. R. Sborlini, F. Driencourt-Mangin and G. Rodrigo, *Four-dimensional unsubtraction with massive particles*, *JHEP* **10** (2016) 162, [1608.01584].

[9] E. T. Tomboulis, *Causality and Unitarity via the Tree-Loop Duality Relation*, *JHEP* **05** (2017) 148, [1701.07052].

[10] J. L. Jurado, G. Rodrigo and W. J. Torres Bobadilla, *From Jacobi off-shell currents to integral relations*, *JHEP* **12** (2017) 122, [1710.11010].

[11] F. Driencourt-Mangin, G. Rodrigo, G. F. R. Sborlini and W. J. Torres Bobadilla, *Universal four-dimensional representation of $H \to \gamma\gamma$ at two loops through the Loop-Tree Duality*, *JHEP* **02** (2019) 143, [1901.09853].

[12] R. Runkel, Z. Szőr, J. P. Vesga and S. Weinzierl, *Causality and loop-tree duality at higher loops*, *Phys. Rev. Lett.* **122** (2019) 111603, [1902.02135].

[13] J. J. Aguilera-Verdugo, F. Driencourt-Mangin, J. Plenter, S. Ramírez-Uribe, G. Rodrigo, G. F. R. Sborlini et al., *Causality, unitarity thresholds, anomalous thresholds and infrared singularities from the loop-tree duality at higher orders*, *JHEP* **12** (2019) 163, [1904.08389].

[14] R. Runkel, Z. Szőr, J. P. Vesga and S. Weinzierl, *Integrands of loop amplitudes within loop-tree duality*, *Phys. Rev. D* **101** (2020) 116014, [1906.02218].

[15] Z. Capatti, V. Hirschi, D. Kermanschah and B. Ruijl, *Loop-Tree Duality for Multiloop Numerical Integration*, *Phys. Rev. Lett.* **123** (2019) 151602, [1906.06138].

[16] F. Driencourt-Mangin, G. Rodrigo, G. F. R. Sborlini and W. J. Torres Bobadilla, *Interplay between the loop-tree duality and helicity amplitudes*, *Phys. Rev. D* **105** (2022) 016012, [1911.11125].

[17] Z. Capatti, V. Hirschi, D. Kermanschah, A. Pelloni and B. Ruijl, *Numerical Loop-Tree Duality: contour deformation and subtraction*, *JHEP* **04** (2020) 096, [1912.09291].

[18] R. M. Prisco and F. Tramontano, *Dual subtractions*, *JHEP* **06** (2021) 089, [2012.05012].

[19] F. Driencourt-Mangin, G. Rodrigo and G. F. R. Sborlini, *Universal dual amplitudes and asymptotic expansions for $gg \to H$ and $H \to \gamma\gamma$ in four dimensions*, *Eur. Phys. J. C* **78** (2018) 231, [1702.07581].

[20] J. Plenter and G. Rodrigo, *Asymptotic expansions through the loop-tree duality*, *Eur. Phys. J. C* **81** (2021) 320, [2005.02119].

[21] J. J. Aguilera-Verdugo, F. Driencourt-Mangin, R. J. Hernández-Pinto, J. Plenter, S. Ramirez-Uribe, A. E. Renteria Olivo et al., *Open Loop Amplitudes and Causality to All Orders and Powers from the Loop-Tree Duality*, *Phys. Rev. Lett.* **124** (2020) 211602, [2001.03564].

[22] J. J. Aguilera-Verdugo, R. J. Hernandez-Pinto, G. Rodrigo, G. F. R. Sborlini and W. J. Torres Bobadilla, *Causal representation of multi-loop Feynman integrands within the loop-tree duality*, *JHEP* **01** (2021) 069, [2006.11217].

[23] J. Jesús Aguilera-Verdugo, R. J. Hernández-Pinto, G. Rodrigo, G. F. R. Sborlini and W. J. Torres Bobadilla, *Mathematical properties of nested residues and their application to multi-loop scattering amplitudes*, *JHEP* **02** (2021) 112, [2010.12971].

[24] S. Ramírez-Uribe, R. J. Hernández-Pinto, G. Rodrigo, G. F. R. Sborlini and W. J. Torres Bobadilla, *Universal opening of four-loop scattering amplitudes to trees*, *JHEP* **04** (2021) 129, [2006.13818].

[25] G. F. R. Sborlini, *Geometrical approach to causality in multiloop amplitudes*, *Phys. Rev. D* **104** (2021) 036014, [2102.05062].

[26] W. J. Torres Bobadilla, *Loop-tree duality from vertices and edges*, *JHEP* **04** (2021) 183, [2102.05048].

[27] W. J. T. Bobadilla, *Lotty – The loop-tree duality automation*, Eur. Phys. J. C **81** (2021) 514, [2103.09237].

[28] J. Rios-Sanchez and G. Sborlini, *Toward multiloop local renormalization within causal loop-tree duality*, Phys. Rev. D **109** (2024) 125004, [2402.13995].

[29] S. Ramírez-Uribe, A. E. Rentería-Olivo, G. Rodrigo, G. F. R. Sborlini and L. Vale Silva, *Quantum algorithm for Feynman loop integrals*, JHEP **05** (2022) 100, [2105.08703].

[30] G. Clemente, A. Crippa, K. Jansen, S. Ramírez-Uribe, A. E. Rentería-Olivo, G. Rodrigo et al., *Variational quantum eigensolver for causal loop Feynman diagrams and directed acyclic graphs*, Phys. Rev. D **108** (2023) 096035, [2210.13240].

[31] S. Ramírez-Uribe, A. E. Rentería-Olivo and G. Rodrigo, *Quantum querying based on multicontrolled Toffoli gates for causal Feynman loop configurations and directed acyclic graphs*, 2404.03544.

[32] C. Cheung, I. Z. Rothstein and M. P. Solon, *From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion*, Phys. Rev. Lett. **121** (2018) 251101, [1808.02489].

[33] J. Davies, K. Schönwald, M. Steinhauser and H. Zhang, *Electroweak corrections to $gg \to HH$: Factorizable contributions*, 2407.05787.

[34] F. Lange, J. Usovitsch and Z. Wu, *Towards the next Kira release*, 2407.01395.

[35] G. De Laurentis, *Non-Planar Two-Loop Amplitudes for Five-Parton Scattering*, 2406.18374.

[36] L. Magnea, C. Milloy, C. Signorile-Signorile and P. Torrielli, *Refactorisation and Subtraction*, 2407.07512.

[37] S. Ramírez-Uribe, P. K. Dhani, G. F. R. Sborlini and G. Rodrigo, *Rewording theoretical predictions at colliders with vacuum amplitudes*, 2404.05491.

[38] LTD collaboration, S. Ramírez-Uribe, A. E. Rentería-Olivo, D. F. Rentería-Estrada, J. J. M. de Lejarza, P. K. Dhani, L. Cieri et al., *Vacuum amplitudes and time-like causal unitary in the loop-tree duality*, 2404.05492.

[39] G. Altarelli and G. Parisi, *Asymptotic Freedom in Parton Language*, Nucl. Phys. B **126** (1977) 298–318.

[40] J. J. Martínez de Lejarza, L. Cieri, M. Grossi, S. Vallecorsa and G. Rodrigo, *Loop Feynman integration on a quantum computer*, 2401.03023.

[41] J. J. Martínez de Lejarza, D. F. Rentería Estrada, M. Grossi and G. Rodrigo, *Quantum integration of decay rates at second order in perturbation theory*, .