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No-π schemes for multi-coupling theories

Ian Jack[a,*]

[a] *Theoretical Physics Division, Department of Mathematical Sciences, University of Liverpool, P.O. Box 147, Liverpool L69 3BX, UK*

E-mail: dij@liverpool.ac.uk

We show that even ζ-functions may be removed from the β-functions of general multi-coupling theories up to high loop order by means of coupling redefinitions. For theories whose β-function is determined by the anomalous dimensions of the fields, such as supersymmetric theories, this corresponds to a renormalisation scheme change to a momentum subtraction scheme.

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*Speaker

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1. Introduction

It was noticed some time ago that ζ_4 cancelled in the QCD Adler function up to $O(\alpha_s^3)$[1] and $O(\alpha_s^4)$ [2, 3]. Further empirical support for even-ζ cancellation was provided for instance in Refs. [4–6]. The phenomenon was explained at 3 loops [7] and 4 loops [8, 9], in terms of the dependence of Feynman integrals on certain combinations of ζ-functions, such that even ζs (which may be expressed as powers of π^2) only appeared in a well-defined way in conjunction with odd ζs. However, it was remarked in Refs. [10], [11] that the ζ_4-dependence in the scalar quark and scalar gluonium correlators respectively did not cancel at $O(\alpha_s^4)$, at least in standard minimal subtraction. On the other hand, in a parallel development, a so-called "C-scheme" was introduced[12], in which it was observed that additional RG functions were free of even ζs up to certain orders[13]. Soon afterwards it was proposed that the "no-π" property holds to all orders in the scheme termed the $\hat{G}$ scheme[14–16].

Meanwhile work on momentum subtraction (MOM) schemes has also demonstrated evidence for the no-π property[17–21]. All the developments so far mentioned have been for single-coupling theories except in the case of the supersymmetric Wess-Zumino model, for which the no-π property was proved up to five loops for a general tensor coupling[22].

In this talk we present a proposal (described in more detail in Ref. [23]) which extends the earlier ideas described here to the multicoupling case and also has the potential to set the various schemes mentioned above in a unified context. We suggest that for any (multi-coupling) theory there are at least two, and possibly several, renormalisation schemes in which even ζs are absent. These can all be specified by redefinition of the couplings. Some of these schemes have simple physical definitions. One of them is a minimal scheme where we absorb even-ζ-dependent finite parts of divergent n-point functions; another is a variant of "MOM", i.e. it is specified by absorbing all finite parts of n-point functions in some well-defined way. It is easiest to demonstrate these suggestions for the supersymmetric Wess-Zumino model where the β function is defined by the two-point function, but we believe that the idea has a wider application. The immediate consequence of this choice of model is that the interactions amongst a multiplet of N superfields Φ_i, $i = 1, \ldots N$, are defined by the superpotential

$$W(\Phi) = g^{ijk}\Phi_i\Phi_j\Phi_k + \text{c.c.}, \tag{1}$$

leading to three-point interactions; so we shall make this simplifying assumption in the rest of the talk.

2. Basic ideas

We start by describing the basic features of the subtraction procedure in order to establish notation. We define counterterms recursively by the standard R-operation. We define the counterterm F by

$$F = \sum F(G), \quad F(G) = -\overline{R}(G). \tag{2}$$

Here the sum is over all the relevant graphs G (in this case the two-point graphs with three-point vertices which contribute to the anomalous dimension), and $\overline{R}$ denotes subtractions of diagrams with

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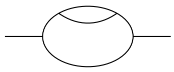
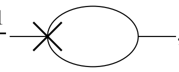
counterterm insertions corresponding to divergent subgraphs. To give a simple two-loop example, for the single two-loop graph G_2 contributing to the anomalous dimension we have

$\overline{R}(G_2) =$  $- \frac{F_{1,1}}{\epsilon}$  , (3)

where $F_{1,1}$ is the one-loop simple pole contribution to F (shortly to be defined in general). We are using here a somewhat schematic notation in which the diagram stands both for the Feynman integral and the associated product of couplings. We write $d = 4 - \epsilon$ so that divergences appear as poles in ϵ, and denote the usual dimensional regularisation mass parameter by μ. We include finite parts in the definition of F in Eq. (2); though standard minimal subtraction involves subtraction only of the pole term. We write the bare coupling g_B (assuming now the use of minimal subtraction) as

$$g_B = \mu^{\frac{1}{2}\epsilon}\left(g + \frac{1}{\epsilon}\left\{F_{1,1} + F_{2,1} + F_{3,1}^{\zeta_3}\zeta_3 + F_{4,1}^{\zeta_4}\zeta_4\right\} + \frac{1}{\epsilon^2}\left\{F_{2,2} + F_{4,2}^{\zeta_3}\zeta_3\right\}\right) + \ldots \tag{4}$$

Here, $F_{L,m}^{\zeta_n}$ denotes the L-loop, order ϵ^{-m} ζ_n-dependent contribution to F, while $F_{L,m}$ similarly denotes the L-loop, order ϵ^{-m} non-ζ-dependent contribution to F. We have

$$\mu\frac{d}{d\mu}g_B = 0, \tag{5}$$

and the β-function is defined by

$$\hat{\beta}(g) = \mu\frac{d}{d\mu}g = -\epsilon g + \beta(g) \tag{6}$$

which entails

$$\beta_L^{\zeta_n} = LF_{L,1}^{\zeta_n}, \quad \beta_L = LF_{L,1}, \tag{7}$$

where $\beta_L^{\zeta_n}$, β_L are the ζ_n-dependent and purely rational L-loop contributions, respectively, to $\beta(g)$. We also have from Eq. (5)

$$F_{4,2}^{\zeta_3} = \frac{1}{4}\left(\beta_3^{\zeta_3}\cdot F_{1,1} + \beta_1\cdot F_{3,1}^{\zeta_3}\right) = \frac{1}{4}\left(3F_{3,1}^{\zeta_3}\cdot F_{1,1} + F_{1,1}\cdot F_{3,1}^{\zeta_3}\right), \tag{8}$$

where

$$f\cdot \equiv f\frac{\partial}{\partial g} \quad \text{or} \quad f\cdot \equiv f^{ijk}\frac{\partial}{\partial g^{ijk}} \tag{9}$$

depending on whether we have a single coupling g or a tensor coupling g^{ijk}. In minimal subtraction the first appearance of ζ_4 is at four loops, so we now want to find a renormalisation scheme where $\beta_4^{\zeta_4} = 0$. A change of scheme corresponds to a coupling redefinition $g \to g'(g)$ which leads to a variation $\beta(g) \to \beta'(g')$ given by

$$\beta'(g') = \mu\frac{d}{d\mu}g' = (\beta(g))^{klm}\frac{\partial}{\partial g^{klm}}g'(g) = \beta(g)\cdot g'(g). \tag{10}$$

An infinitesimal variation $g' = g + \delta g$ then gives $\beta' = \beta + \delta\beta$ with

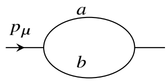
$$\delta\beta = [\beta, \delta g] + \ldots \tag{11}$$

with the commutator defined by

$$[X, Y] = X\cdot Y - Y\cdot X. \tag{12}$$

3. Properties of Feynman diagrams

In this Section we give a brief introduction to the properties of Feynman diagrams with regard to even ζ-functions which lead to the existence of schemes in which these even ζs are absent. It turns out that the even-ζ terms in (at least a large class of) Feynman diagrams are not independent; in fact they only occur in certain combinations with the odd ζ-functions. For an illustration, consider the simple generic loop



(13)

(where a, b are the powers of the propagators and p_μ is the incoming momentum) for which we have the well-known result

$$\frac{1}{(p^2)^{a+b-\frac{d}{2}}}L(a,b), \quad \text{where} \quad L(a,b) = (4\pi)^{\frac{\epsilon}{2}}\frac{\Gamma(\frac{d}{2}-a)\Gamma(\frac{d}{2}-b)\Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}. \tag{14}$$

A large class of Feynman integrals may be written as a product of $L(a,b)$ for various a, b. For logarithmically divergent integrals the arguments a, b are of the form $a = 1+\alpha\epsilon$, $b = 1+\beta\epsilon$. There is an expansion

$$\begin{aligned} L(1+\alpha\epsilon, 1+\beta\epsilon) =& \frac{2}{\epsilon(2\alpha+2\beta+1)}\exp\left\{\frac{\epsilon}{2}\left[\ln 4\pi - \gamma - \frac{\epsilon}{4}\zeta_2\right]\right\} \\ &\times \exp\left\{\sum_{j=1}^{\infty}(\alpha+\beta+1)^j\frac{\epsilon^j}{j} + \sum_{j=3}^{\infty}h_j(\alpha,\beta)\zeta_j\frac{\epsilon^j}{j}\right\} \end{aligned} \tag{15}$$

where

$$h_j(\alpha,\beta) = (\alpha+\tfrac{1}{2})^j + (\beta+\tfrac{1}{2})^j + (-\alpha-\beta-\tfrac{1}{2})^j - (-\alpha)^j - (-\beta)^j - (\alpha+\beta+1)^j. \tag{16}$$

It is easy to check that

$$h_4 = \frac{3}{4}h_3 + \frac{1}{64}, \tag{17}$$

irrespective of α, β; and there are similar expressions for h_j for higher even values of j. We would like to use this to derive a relation between $F^{\zeta_4}_{L,m}$ and $F^{\zeta_3}_{L,m+1}$ which we will be able to exploit in Eqs. (7), (8); but these quantities may include ζ-dependent counterterms which do not obey this relation between h_4 and h_3. Accordingly, we define $G^{\zeta_n}_{L,m}$ to be the value of $F^{\zeta_n}_{L,m}$ after omitting ζ_n-dependent counterterms. Then we do have

$$G^{\zeta_4}_{L,m} = \frac{3}{4}G^{\zeta_3}_{L,m+1} \quad (m \geq 0). \tag{18}$$

We further obtain

$$\begin{aligned} G^{\zeta_6}_{L,m} =& \frac{5}{4}\left(G^{\zeta_5}_{L,m+1} - \frac{1}{3}G^{\zeta_4}_{L,m+2}\right), \\ G^{\zeta_8}_{L,m} =& \frac{7}{4}\left(G^{\zeta_7}_{L,m+1} - \frac{1}{2}G^{\zeta_6}_{L,m+2} + \frac{1}{24}G^{\zeta_4}_{L,m+4}\right). \end{aligned} \tag{19}$$

Relations like this can be extended to higher even ζ and higher loops (in fact for almost[1] all known "p-integrals" - two-point integrals with a single momentum dependence). Consequently, the $G^{\zeta_k}_{L,m}$ for even k may be recursively written in terms of the $G^{\zeta_k}_{L,m}$ for lower odd values of k.

[1] See the Conclusions for a clarification of this *caveat*.

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4. Scheme redefinitions

In this Section we shall show how relations such as those in Eqs. (18) and (19) may be exploited to define scheme redefinitions making a transformation from $\overline{\text{MS}}$ to a scheme in which even ζs are absent.

We have from Eq. (18) combined with the definition of the $G^{\zeta_n}_{l,m}$,

$$G^{\zeta_3}_{3,1} = F^{\zeta_3}_{3,1}, \quad G^{\zeta_4}_{3,0} = F^{\zeta_4}_{3,0}, \quad G^{\zeta_4}_{4,1} = F^{\zeta_4}_{4,1},$$
$$G^{\zeta_3}_{4,2} = F^{\zeta_3}_{4,2} - F^{\zeta_3}_{3,1} \cdot F_{1,1}. \tag{20}$$

Combining this with Eq. (8), we obtain

$$G^{\zeta_3}_{4,2} = \frac{1}{4}[F_{1,1}, F^{\zeta_3}_{3,1}] = \frac{1}{4}[\beta_1, G^{\zeta_3}_{3,1}]. \tag{21}$$

Using

$$G^{\zeta_4}_{4,1} = \frac{3}{4} G^{\zeta_3}_{4,2},$$
$$G^{\zeta_4}_{3,0} = \frac{3}{4} G^{\zeta_3}_{3,1}. \tag{22}$$

we then find that

$$\beta^{\zeta_4}_4 = 4F^{\zeta_4}_{4,1} = 4G^{\zeta_4}_{4,1} = [\beta_1, G^{\zeta_4}_{3,0}] = [\beta_1, F^{\zeta_4}_{3,0}]. \tag{23}$$

In the light of Eq. (11), this may be removed by a coupling redefinition with

$$\delta g = -F^{\zeta_4}_{3,0}\zeta_4 \tag{24}$$

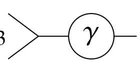
i.e. removing the ζ_4-dependent finite part of F. This idea may clearly be extended to higher orders; using the further relations in Eq. (19), our proposal is that all even ζ may be removed by

$$\delta g = -\left(\left[F^{\zeta_4}_{3,0} + F^{\zeta_4}_{4,0}\right]\zeta_4 + \left[F^{\zeta_6}_{4,0} + F^{\zeta_6}_{5,0}\right]\zeta_6 + \left[F^{\zeta_8}_{5,0} + F^{\zeta_8}_{6,0}\right]\zeta_8\right) + \ldots \tag{25}$$

This is a scheme where we subtract all even-ζ-dependent finite parts in F- we call it MOM′. We have shown cancellation of even ζs in the MOM′ scheme up to a loop order beyond their first appearance for ζ_4 (i.e. 5 loops) and ζ_6 (i.e. 6 loops). We have also examined the MOM scheme in which we subtract *all* finite parts. As we have mentioned, there is a considerable literature devoted to this scheme, though largely in the single-coupling case. We have shown (at least up to five loops for ζ_4) that even ζs also cancel in MOM for a general multi-coupling theory; it seems likely that MOM shares the no-π property with MOM′ at higher orders as well.

5. Example: the Wess-Zumino model

The supersymmetric Wess-Zumino model is naturally written in terms of superfields Φ with a cubic superpotential given by Eq. (1). As we have suggested already, the Wess-Zumino model is a useful test-bed since the β-function is determined by the anomalous dimensions. Schematically,

$\beta = S_3$  (26)

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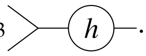
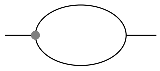
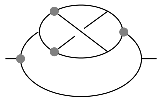
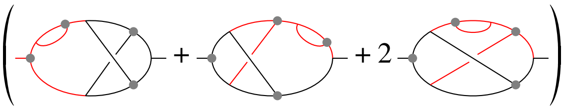
where $\mathcal{S}_3$ denotes the sum over the three terms where γ is attached to each external line. Likewise we consider variations δg given by

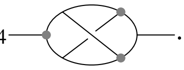
$$\delta g = \mathcal{S}_3$$  (27)

The anomalous dimension is given up to four loops by

$$\gamma = \frac{1}{2}$$  $$+ \ldots - \frac{3}{4}(2\zeta_3 + \zeta_4)$$ 

$$-\frac{3}{4}(2\zeta_3 - \zeta_4)$$  $$+ \ldots .$$ (28)

Here, in contrast to Eq. (3), each diagram denotes purely a combination of contracted tensor couplings and the results of Feynman integrals are subsumed into the coefficients multiplying the diagrams. The dots at vertices denote the complex conjugated terms in Eq. (1); since in this theory the propagators link a Φ to a $\bar{\Phi}$, the contractions are always between conjugated and unconjugated tensors. We only show the terms relevant for our discussion, omitting all the two-and three-loop terms and the non-ζ_4-dependent four-loop terms. In line with Eq. (24) and in the light of Eq. (27), we have

$$h^{(3)} = -\frac{1}{4}\gamma_3^{\zeta_3}\zeta_4 = -\frac{3}{8}\zeta_4$$  (29)

Eq. (11) then leads to (once again using Eq. (27))

$$\begin{aligned}\delta\beta =&[\delta g + \delta g^*, \beta] \Rightarrow \delta\gamma = (\delta g + \delta g^*) \cdot \gamma - (\beta + \beta^*) \cdot h\\ =&(\delta g)^{(3)klm} \frac{\partial}{\partial g^{klm}}\gamma^{(1)} - \beta^{(1)klm}\frac{\partial}{\partial g^{klm}} h^{(3)} + * \text{ terms.}\end{aligned} \quad (30)$$

It is then easy to check using Eqs. (26), (28) and (29) that the ζ_4 terms cancel. We have indicated in red the terms in Eq. (28) which form up into an insertion of $\beta^{(1)}$ on $h^{(3)}$ and are therefore cancelled by the second term in Eq. (30).

6. Conclusions

Our goal in this talk has been to seek renormalisation schemes in which even ζ-functions are absent from renormalisation group functions for a general theory. To this end we have considered two renormalisation schemes, MOM and MOM′. The MOM scheme is defined by subtracting $O(\epsilon^0)$ parts of n-point functions in addition to poles in ϵ, while in the MOM′ scheme we only subtract even-ζ finite parts. We have considered a general multi-coupling theory, focussing attention on one where the β-function is defined by the anomalous dimensions; the supersymmetric Wess Zumino model is a concrete example. In this context we have shown (for more details see Ref. [23]) that the no-π theorem holds for ζ_4 up 5 loops in both the MOM′ and MOM schemes and for ζ_6 up to 6 loops in the MOM′ scheme. We believe that these results may be extended to higher loops and higher even-ζs for both MOM and MOM′, but evidently this will require more work. The theoretical underpinning of our work is based on p-integrals, i.e. integrals with a single momentum dependence; these naturally arise in two-point functions, hence our primary interest in theories whose renormalisation is determined by anomalous dimensions. However we believe that an extension to 3-point and 4-point graphs is feasible, based on nullification of one or two (respectively) of the external momenta. There are certainly well-motivated nullification procedures for 3-point vertices in QCD, as described in Ref. [21]. Furthermore this nullification procedure has also been carried out for ϕ^3 theory in six dimensions[24]. The implementation of a MOM-type scheme in both these cases leads to the expected absence of even ζs up to the loop order considered. It seems likely that we shall be able to apply the same procedure to ϕ^4 theory in four dimensions, despite additional complications arising in this case from potential infra-red issues when nullifying momenta in 4-point graphs. Finally there is evidence that the expected behaviour of p-integrals on which we have been relying breaks down at high loop orders[25][2]-in fact at eight loops, for ζ_{12}.

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