

Worldline integration of photon amplitudes

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It has been known for many years that methods inspired by string theory, such as the worldline formalism, allow one to write down integral representations that combine large numbers of Feynman diagrams of different topologies. However, to make this fact useful for state-of-the-art calculations one has to confront non-standard integration problems where neither the known integration techniques for Feynman diagrams nor algebraic manipulation programs are of much help. Here I will give a progress report on this long-term project focussing on photon amplitudes at one and two loops, in vacuum and in external fields.

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1. Worldline representation of the QED S-matrix

After Feynman's seminal work on the quantum-mechanical path integral in 1948, two years later he used a relativistic generalization of the path integral for a perturbative construction of the S-matrices for scalar [1] and spinor [2] QED. This leads to a decomposition of arbitrary QED amplitudes into contributions with a fixed number of scalar/spinor lines and loops, each one represented by a single path integral, interconnected by photon propagators in all possible ways (Fig. 1).

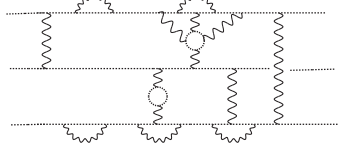


Figure 1: Contributions to the QED S-matrix with a fixed number of matter lines and loops.

However, after deriving the Feynman rules from this construction he seems to have concluded that they provide a much more efficient way of constructing QED perturbation theory than the worldline path-integral representation. This judgement was shared by the community, so that for several decades to come the worldline path-integral representation of QED was rarely if ever considered as a tool for everyday-life calculations of scattering amplitudes or effective actions. It is only since the nineties, and as a spin-off of developments in string theory and supersymmetry, that it became appreciated that this representation, although completely equivalent to the Feynman diagram one, has two principal advantages over it:

1. It avoids the break-up of the scalar/spinor lines or loops into individual propagators.
2. A priori it does not require a fixed ordering of the photon legs along a line or loop.

The first property becomes important when those propagators are already complicated objects by themselves, which typically happens if one wishes to absorb an external field non-perturbatively. The second property is relevant for multi-loop calculations, since in QED the rapid growth in the number of diagrams with increasing number of loops is mainly due to the many different ways of inserting photons into or between fixed loops or lines.

Let us start with the one-loop N -photon amplitudes in scalar QED, which in the modern “string-inspired” approach to Feynman’s worldline formalism are expressed as follows [3–6] (for details, see [7, 8], for generalization to the open line case [9]):

$$\Gamma[\{k_i, \varepsilon_i\}] = (-ie)^N \int \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x V_{\text{scal}}^A[k_1, \varepsilon_1] \dots V_{\text{scal}}^A[k_N, \varepsilon_N] e^{-\int_0^T d\tau \frac{\dot{x}^2}{4}}. \quad (1)$$

Here m is the mass of the loop scalar, and the path integral runs over the space of closed loops in (euclidean) space-time with periodicity T in proper-time. Each photon is represented by a vertex operator

$$V_{\text{scal}}^A[k, \varepsilon] = \int_0^T d\tau \varepsilon \cdot \dot{x}(\tau) e^{ikx(\tau)}. \quad (2)$$

The zero mode $x_0 = \frac{1}{T} \int_0^T d\tau x(\tau)$ factors out of the path integral and produces the momentum conservation factor $(2\pi)^D \delta(\sum k_i)$ which we suppress as usual.

Since the path integral (1) is gaussian, it can be evaluated in closed form, leading to the ‘‘Bern-Kosower master formula’’

$$\begin{aligned} \Gamma[\{k_i, \varepsilon_i\}] &= (-ie)^N \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \\ &\times \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} G_{ij} k_i \cdot k_j + i \dot{G}_{ij} k_i \cdot \varepsilon_j + \frac{1}{2} \ddot{G}_{ij} \varepsilon_i \cdot \varepsilon_j \right] \right\} \Big|_{\text{lin}(\varepsilon_1, \dots, \varepsilon_N)} \end{aligned} \quad (3)$$

written in terms of the ‘‘worldline Green’s function’’ $G(\tau, \tau')$ and its derivatives,

$$\begin{aligned} G(\tau_1, \tau_2) &= |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T}, \\ \dot{G}(\tau_1, \tau_2) &= \text{sgn}(\tau_1 - \tau_2) - 2 \frac{(\tau_1 - \tau_2)}{T}, \\ \ddot{G}(\tau_1, \tau_2) &= 2\delta(\tau_1 - \tau_2) - \frac{2}{T}. \end{aligned}$$

The notation $\Big|_{\text{lin}(\varepsilon_1, \dots, \varepsilon_N)}$ means projection on the terms linear in each polarization vector.

The generalization to spinor QED can be done in various ways. Feynman’s original formalism used a ‘‘spin factor’’ under the path integral involving gamma matrices. Nowadays this spin factor is usually used only in numerical path integration, while for analytic purposes one prefers to rewrite it as a Grassmann path integral [10, 11]. One can then introduce worldline superfields to write down a master formula analogous to (3), but in practice it is usually preferable to use a certain integration-by-parts algorithm for removing the \ddot{G}_{ij} ’s, after which the effect of spin can be implemented by the application of the ‘‘Bern-Kosower replacement rule’’

$$\dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_n i_1} \rightarrow \dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_n i_1} - G_{F i_1 i_2} G_{F i_2 i_3} \cdots G_{F i_n i_1}, \quad (4)$$

involving the ‘‘ τ -cycles’’ $\dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_n i_1}$ and the fermionic worldline Green’s function G_F ,

$$G_F(\tau, \tau') \equiv \text{sgn}(\tau - \tau'). \quad (5)$$

This rule was first derived by Bern and Kosower from string theory for the case of the one-loop N -gluon amplitudes on-shell [4]. Strassler [6] then studied the systematics of the integration-by-parts for the off-shell case, which allowed him to see that it leads to the emergence of photon field strength tensors $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$, and thus to the emergence of gauge invariance at the integrand level. Remaining ambiguities were resolved in [12] in a permutation-invariant manner.

No such replacement rule is available for the open fermion line, for which a worldline representation suitable for state-of-the-art calculations was found only quite recently [13, 14].

Note that the master formula (3) gives the whole amplitudes, without the need of adding ‘‘crossed’’ terms, which is made possible by the signum and absolute-value functions appearing in the worldline Green’s functions. To take full advantage of this property of the worldline formalism

it is therefore necessary to develop methods for the calculation of integrals of this type *without splitting them into ordered sectors*, a non-standard integration problem for which neither existing tables of integrals nor algebraic manipulation programs are of much help. The main purpose of my talk is to give an update on the state-of-the-art of this long-term endeavour (see also [15]).

2. Worldline representation of the four-photon amplitudes

After the integration-by-parts, the four-photon amplitude in spinor QED appears naturally decomposed as follows:

$$\Gamma_{\text{spin}}(k_1, \varepsilon_1, \dots, k_4, \varepsilon_4) = -\frac{e^4}{8\pi^2} \left(\Gamma^{(1)} + \Gamma^{(2)} + \Gamma^{(3)} + \Gamma^{(4)} + \Gamma^{(5)} \right), \quad (6)$$

$$\begin{aligned} \Gamma^{(1)} &= \Gamma_{(1234)}^{(1)} T_{(1234)}^{(1)} + \Gamma_{(1243)}^{(1)} T_{(1243)}^{(1)} + \Gamma_{(1324)}^{(1)} T_{(1324)}^{(1)}, \\ \Gamma^{(2)} &= \Gamma_{(12)(34)}^{(2)} T_{(12)(34)}^{(2)} + \Gamma_{(13)(24)}^{(2)} T_{(13)(24)}^{(2)} + \Gamma_{(14)(23)}^{(2)} T_{(14)(23)}^{(2)}, \\ \Gamma^{(3)} &= \sum_{i=1,2,3} \Gamma_{(123)i}^{(3)} T_{(123)i}^{(3)r_4} + \sum_{i=2,3,4} \Gamma_{(234)i}^{(3)} T_{(234)i}^{(3)r_1} + \sum_{i=3,4,1} \Gamma_{(341)i}^{(3)} T_{(341)i}^{(3)r_2} + \sum_{i=4,1,2} \Gamma_{(412)i}^{(3)} T_{(412)i}^{(3)r_3}, \\ \Gamma^{(4)} &= \sum_{i<j} \Gamma_{(ij)ii}^{(4)} T_{(ij)ii}^{(4)} + \sum_{i<j} \Gamma_{(ij)jj}^{(4)} T_{(ij)jj}^{(4)}, \\ \Gamma^{(5)} &= \sum_{i<j} \Gamma_{(ij)ij}^{(5)} T_{(ij)ij}^{(5)} + \sum_{i<j} \Gamma_{(ij)ji}^{(5)} T_{(ij)ji}^{(5)}. \end{aligned}$$

Here we have introduced the following tensor basis,

$$\begin{aligned} T_{(1234)}^{(1)} &\equiv Z_4(1234), \\ T_{(12)(34)}^{(2)} &\equiv Z_2(12)Z_2(34), \\ T_{(123)i}^{(3)r_4} &\equiv Z_3(123) \frac{r_4 \cdot f_4 \cdot k_i}{r_4 \cdot k_4}, \quad i = 1, 2, 3, \\ T_{(12)ii}^{(4)} &\equiv Z_2(12) \frac{k_i \cdot f_3 \cdot f_4 \cdot k_i}{k_3 \cdot k_4}, \quad i = 1, 2, \\ T_{(12)ij}^{(5)} &\equiv Z_2(12) \frac{k_i \cdot f_3 \cdot f_4 \cdot k_j}{k_3 \cdot k_4}, \quad (i, j) = (1, 2), (2, 1) \end{aligned}$$

as well as

$$\begin{aligned} Z_2(ij) &\equiv \frac{1}{2} \text{tr}(f_i f_j) = \varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_j, \\ Z_n(i_1 i_2 \dots i_n) &\equiv \text{tr} \left(\prod_{j=1}^n f_{i_j} \right), \quad (n \geq 3). \quad (\text{“Lorentz cycle”}) \end{aligned}$$

The coefficient functions are given by the following integrals,

$$\Gamma_{\dots}^{(k)} = \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-m^2 T} \int_0^1 \prod_{i=1}^4 du_i \Gamma_{\dots}^{(k)}(\dot{G}_{ij}) e^{\frac{1}{2} T \sum_{i,j=1}^4 G_{ij} k_i \cdot k_j}$$

where we have rescaled $\tau_i = Tu_i$, $i = 1, \dots, 4$ and

$$\begin{aligned}
\Gamma_{(1234)}^{(1)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41}, \\
\Gamma_{(12)(34)}^{(2)} &= (\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21})(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}), \\
\Gamma_{(123)1}^{(3)} &= (\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31})\dot{G}_{41}, \\
\Gamma_{(12)11}^{(4)} &= (\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21})\dot{G}_{13}\dot{G}_{41}, \\
\Gamma_{(12)12}^{(5)} &= (\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21})\dot{G}_{13}\dot{G}_{42}.
\end{aligned} \tag{7}$$

In previous work, the coefficient functions have been evaluated for the off-shell case, but with two legs taken in the low-energy limit [16, 17]. In a forthcoming article [18], we instead compute them for the on-shell case at full momentum, for both scalar and spinor QED. Although this could be done using existing methods for the calculation of one-loop on-shell integrals, with a view to eventual multi-loop generalization we prefer to do it “the hard way”, that is, by building tables of worldline integrals that allow one to integrate out one photon leg *without fixing any ordering between the photons*. For starters, let us consider the basic scalar box integral. Keeping all three Mandelstam variables to maintain manifest permutation invariance, the universal exponent of the master formula (3) becomes

$$\Lambda \equiv \frac{1}{2} \sum_{i,j=1}^4 G_{ij} k_i \cdot k_j = -\frac{T}{2} [(G_{12} + G_{34})s + (G_{13} + G_{24})t + (G_{14} + G_{23})u] \tag{8}$$

Integrating out leg number 4 without fixing an ordering for the remaining three legs can be done with the formula

$$\begin{aligned}
\int_0^1 du_4 e^\Lambda &= \frac{1}{T} \left[\frac{2}{u + \dot{G}_{12}t + \dot{G}_{13}s} + \frac{2}{u - \dot{G}_{12}t - \dot{G}_{13}s} \right] e^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT} \\
&+ \frac{1}{T} \left[\frac{2}{t + \dot{G}_{23}s + \dot{G}_{21}u} + \frac{2}{t - \dot{G}_{23}s - \dot{G}_{21}u} \right] e^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT} \\
&+ \frac{1}{T} \left[\frac{2}{s + \dot{G}_{31}u + \dot{G}_{32}t} + \frac{2}{s - \dot{G}_{31}u - \dot{G}_{32}t} \right] e^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT}.
\end{aligned} \tag{9}$$

In the four-photon case, we need a generalization of this formula with additional factors of \dot{G} in the numerator. For example, an additional factor of \dot{G}_{41} leads to

$$\begin{aligned}
\int_0^1 du_4 \dot{G}_{41} e^\Lambda &= \left[-\frac{8}{T^2(u + \dot{G}_{12}t + \dot{G}_{13}s)^2} + \frac{8}{T^2(u - \dot{G}_{12}t - \dot{G}_{13}s)^2} \right. \\
&+ \left. \frac{2}{T(u + \dot{G}_{12}t + \dot{G}_{13}s)} - \frac{2}{T(u - \dot{G}_{12}t - \dot{G}_{13}s)} \right] e^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT} \\
&+ 2 \text{Perm.}
\end{aligned} \tag{10}$$

In [18] along these lines we obtain the known on-shell four-photon amplitudes for spinor QED, as well as the ones for scalar QED that apparently have never been computed in full generality. As an

example, let us show the result for the integral of the coefficient function $\Gamma_{(12)12}^{(5)}$ in scalar QED:

$$\begin{aligned} \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} e^{-m^2 T} \int_{(4)} e^{T\Lambda} \dot{G}_{12} \dot{G}_{21} \dot{G}_{13} \dot{G}_{42} &= r_{(12)12}^{(1)} + r_{(12)12}^{(2)} \ln \left(\frac{\beta_{\hat{s}} - 1}{\beta_{\hat{s}} + 1} \right) + r_{(12)12}^{(3)} \ln \left(\frac{\beta_{\hat{t}} - 1}{\beta_{\hat{t}} + 1} \right) \\ &+ r_{(12)12}^{(4)} \ln \left(\frac{\beta_{\hat{u}} - 1}{\beta_{\hat{u}} + 1} \right) + r_{(12)12}^{(5)} \left[\ln \left(\frac{\beta_{\hat{s}} - 1}{\beta_{\hat{s}} + 1} \right) \right]^2 + r_{(12)12}^{(6)} \left[\ln \left(\frac{\beta_{\hat{t}} - 1}{\beta_{\hat{t}} + 1} \right) \right]^2 + r_{(12)12}^{(7)} \left[\ln \left(\frac{\beta_{\hat{u}} - 1}{\beta_{\hat{u}} + 1} \right) \right]^2 \\ &+ r_{(12)12}^{(8)} \bar{B}(s, t, u) + r_{(12)12}^{(9)} \bar{B}(s, u, t) + r_{(12)12}^{(10)} \bar{B}(t, u, s). \end{aligned}$$

Here we have introduced the variables [19]

$$\beta_{\hat{s}} \equiv \sqrt{1 - \frac{1}{\hat{s}}}, \quad \beta_{\hat{t}\hat{u}} \equiv \sqrt{1 + \frac{\hat{s}}{\hat{t}\hat{u}}} = \sqrt{1 - \frac{1}{\hat{t}} - \frac{1}{\hat{u}}}. \quad (11)$$

with $\hat{s} \equiv \frac{s}{4m^2}$, $\hat{t} \equiv \frac{t}{4m^2}$, $\hat{u} \equiv \frac{u}{4m^2}$. The functions $r_{(12)12}^{(1)}, \dots, r_{(12)12}^{(10)}$ are algebraic, and all dilogarithms are contained in a single integral $\bar{B}(s, t, u)$,

$$\bar{B}(s, t, u) = \frac{1}{s-t} \int_0^1 dx \left[\frac{s(2t+u)(1-2x)}{m^2 - s(1-x)x} - \frac{t(2s+u)(1-2x)}{m^2 - t(1-x)x} \right] \ln \left(\frac{x - \frac{1+\beta_{\hat{s}\hat{t}}}{2}}{x - \frac{1-\beta_{\hat{s}\hat{t}}}{2}} \right). \quad (12)$$

3. Incorporating a constant external field

The generalization of all the previous formulas from vacuum QED to the inclusion of a constant external field requires only the following changes:

1. Replacing the worldline Green's functions G_B, G_F by field-dependent ones $\mathcal{G}_B, \mathcal{G}_F$,

$$\begin{aligned} G_B(\tau_1, \tau_2) &\rightarrow \mathcal{G}_B(\tau_1, \tau_2) = \frac{T}{2\mathcal{Z}^2} \left(\frac{\mathcal{Z}}{\sin \mathcal{Z}} e^{-i\mathcal{Z}\dot{G}_{B12}} + i\mathcal{Z}\dot{G}_{B12} - 1 \right), \\ G_F(\tau_1, \tau_2) &\rightarrow \mathcal{G}_F(\tau_1, \tau_2) = G_{F12} \frac{e^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos \mathcal{Z}}, \end{aligned}$$

where $\mathcal{Z}_{\mu\nu} \equiv eF_{\mu\nu}T$.

2. Adding global determinant factors

$$\det^{\frac{1}{2}} \left[\frac{\mathcal{Z}}{\sin \mathcal{Z}} \right] \quad (\text{Scalar QED}), \quad \det^{\frac{1}{2}} \left[\frac{\mathcal{Z}}{\tan \mathcal{Z}} \right] \quad (\text{Spinor QED}).$$

In particular, the master formula (3) generalizes to the N -photon amplitudes in a constant field as [20, 21]

$$\begin{aligned} \Gamma_{\text{scal}}(k_1, \varepsilon_1; \dots; k_N, \varepsilon_N | F) &= (-ie)^N \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \det^{\frac{1}{2}} \left[\frac{\mathcal{Z}}{\sin \mathcal{Z}} \right] \prod_{i=1}^N \int_0^T d\tau_i \\ &\times \exp \left\{ \sum_{i,j=1}^N \left[\frac{1}{2} k_i \cdot \mathcal{G}_{Bij} \cdot k_j - i\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j + \frac{1}{2} \varepsilon_i \cdot \ddot{\mathcal{G}}_{Bij} \cdot \varepsilon_j \right] \right\} \Big|_{\text{lin}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)}. \end{aligned} \quad (13)$$

The loop replacement rule (4) can also be generalized in a straightforward way.

This formalism for calculations in constant fields has already found many applications in strong-field QED. This includes the photon propagator in a constant field [22, 23], magnetic photon splitting [24], one and two-loop Euler-Heisenberg Lagrangians [21, 25, 26], magnetic photon-graviton conversion [27, 28], and the low-energy limit of the N -photon amplitudes in a constant field [29].

4. Low-energy limit of N -photon amplitudes in vacuum and in a constant field

Let us discuss here the last-mentioned calculation, since it is recent and relevant for our main topic, worldline integration. We start with the N -photon amplitudes in vacuum as a warm-up. After expanding out the exponential in the master formula (3), and the removal of the \ddot{G}_{ij} 's by integration-by-parts, the low-energy limit can simply be taken by replacing the remaining exponential factor $e^{\frac{1}{2} \sum_{i,j=1}^N G_{ij} k_i \cdot k_j}$ by unity. It turns out that then all terms in the integrand that are not just products of cycles turn into total derivatives, and integrate to zero. The cycle-integrals can be done in closed form, leading to Bernoulli numbers B_n :

$$\int_0^1 du_1 du_2 \dots du_n \left(\dot{G}_{12} \dot{G}_{23} \dots \dot{G}_{n1} - G_{F12} G_{F23} \dots G_{Fn1} \right) = \begin{cases} 2^n (2^n - 2) \frac{B_n}{n!} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

This leads to the closed-form expression [26]

$$\Gamma_{\text{spin}}^{(\text{LE})}(k_1, \varepsilon_1; \dots; k_N, \varepsilon_N) = (-2) \frac{e^N \Gamma(N - \frac{D}{2})}{(4\pi)^2 m^{2N-D}} \exp \left\{ \sum_{m=1}^{\infty} (1 - 2^{2m-1}) \frac{b_{2m}}{2m} \text{tr}(f_{\text{tot}}^{2m}) \right\} \Big|_{\text{lin}(f_1 \dots f_N)} \quad (14)$$

where $f_{\text{tot}} \equiv \sum_{i=1}^N f_i$, $b_n = -2^n \frac{B_n}{n!}$. For the projection on individual helicity components, see [30].

In the constant-field background, it is still true that the N -photon amplitudes can, in the weak-field limit, be reduced to terms that factorize into cycles. However, since the generalized worldline Green's functions are non-trivial Lorentz matrices, these cycles do not any more factorize into “ τ -cycles” and “Lorentz-cycles”, instead they combine as

$$\dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \dots \dot{G}_{i_n i_1} Z_n(i_1 i_2 \dots i_n) \rightarrow \text{tr} \left(f_{i_1} \cdot \dot{\mathcal{G}}_{B i_1 i_2} \cdot f_{i_2} \cdot \dot{\mathcal{G}}_{B i_2 i_3} \dots f_{i_n} \cdot \dot{\mathcal{G}}_{B i_n i_1} \right). \quad (15)$$

Thus the basic mathematical problem becomes the computation of the “open-index cycle integral”

$$\int_0^1 du_1 \dots \int_0^1 du_n \dot{\mathcal{G}}_{B12} \otimes \dot{\mathcal{G}}_{B23} \otimes \dots \otimes \dot{\mathcal{G}}_{Bn1} \quad (16)$$

which at first sight seem to generate a large number of component integrals. However, it turns out that there is a nice way of calculating them all in one go. Let us show this for the purely magnetic case. The magnetic worldline Green's function $\dot{\mathcal{G}}_B$ has the matrix decomposition [23]

$$\dot{\mathcal{G}}_B(\tau_1, \tau_2) = \dot{G}_{12} g_- + S_{B12}(z) g_+ - A_{B12}(z) i r_+ \quad (17)$$

where $z = eBT$, $\dot{G}_{12} = \text{sgn}(\tau_1 - \tau_2)$,

$$\begin{aligned}
g_+ &\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & g_- &\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
r_+ &\equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & r_- &\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\end{aligned} \tag{18}$$

and

$$S_{B12}(z) = \frac{\sinh(z \dot{G}_{12})}{\sinh z}, \quad A_{B12}(z) = \frac{\cosh(z \dot{G}_{12})}{\sinh z} - \frac{1}{z}.$$

Introducing the function

$$H_{ij}(z) \equiv \frac{e^{z \dot{G}_{ij}}}{\sinh z} - \frac{1}{z} \tag{19}$$

the three component functions of the Green's function can be written as

$$\dot{G}_{ij} = H_{ij}(0), \quad S_{Bij}(z) = \frac{1}{2} [H_{ij}(z) + H_{ij}(-z)], \quad A_{Bij}(z) = \frac{1}{2} [H_{ij}(z) - H_{ij}(-z)]. \tag{20}$$

In this way the multi-component integral (16) can be reduced to a single iterated integral, which moreover turns out to have the following remarkable self-reproducing property:

$$\begin{aligned}
H_{ik}^{(2)}(z, z') &\equiv \int_0^T d\tau_j H_{ij}(z) H_{jk}(z') = \frac{H_{ik}(z)}{z' - z} + \frac{H_{ik}(z')}{z - z'}, \\
H_{il}^{(3)}(z, z', z'') &\equiv \int_0^T d\tau_j \int_0^T d\tau_k H_{ij}(z) H_{jk}(z') H_{kl}(z'') \\
&= \frac{H_{il}(z)}{(z' - z)(z'' - z)} + \frac{H_{il}(z')}{(z - z')(z'' - z')} + \frac{H_{il}(z'')}{(z - z'')(z' - z'')}, \\
&\vdots \\
H_{i_1 i_{n+1}}^{(n)}(z_1, \dots, z_n) &= \sum_{k=1}^n \frac{H_{i_1 i_{n+1}}(z_k)}{\prod_{l \neq k} (z_l - z_k)}.
\end{aligned} \tag{21}$$

Note that the right-hand sides have the full permutation symmetry. Defining

$$z_0 \equiv 0, \quad z_+ \equiv z, \quad z_- \equiv -z, \tag{22}$$

$$\mathfrak{g}_0 \equiv g_-, \quad \mathfrak{g}_+ \equiv \frac{1}{2}(g_+ - ir_+), \quad \mathfrak{g}_- \equiv \frac{1}{2}(g_+ + ir_+) \tag{23}$$

we can then write

$$\int_0^1 du_2 \cdots \int_0^1 du_n \dot{G}_{B12} \otimes \dot{G}_{B23} \otimes \cdots \otimes \dot{G}_{Bn(n+1)} = \sum_{\alpha_1, \dots, \alpha_n} H_{1(n+1)}^{(n)}(z_{\alpha_1}, \dots, z_{\alpha_n}) \mathfrak{g}_{\alpha_1} \otimes \cdots \otimes \mathfrak{g}_{\alpha_n}$$

where each index α_i runs over 0, +, -. This can be extended to the spinor QED case [29], and reduces the calculation of the low-energy limit of the magnetic N -photon amplitudes to simple algebra and a single global proper-time integral with trigonometric integrand.

5. N -photon amplitudes in a plane-wave background

The plane-wave background can be defined by a vector potential $A(x)$ of the form

$$eA_\mu(x) = a_\mu(n \cdot x) \quad (24)$$

where n^μ is a null vector, $n^2 = 0$, and as is usual we will further impose the *light-front gauge condition* $n \cdot a = 0$. Despite of many similarities with the constant-field case, it is far from straightforward to reduce the path integrals for the N -photon amplitudes to gaussian ones. Building on work of [31] for the two-point case, in [32] the following master formula for the scalar QED N -photon amplitude in a plane-wave background was obtained:

$$\begin{aligned} \Gamma_{\text{scal}}(\{k_i, \varepsilon_i\}; a) &= (-ie)^N (2\pi)^3 \delta\left(\sum_{i=1}^N k_i^1\right) \delta\left(\sum_{i=1}^N k_i^2\right) \delta\left(\sum_{i=1}^N k_i^+\right) \int_{-\infty}^{\infty} dx_0^+ e^{-ix_0^+ \sum_{i=1}^N k_i^-} \\ &\times \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} \prod_{i=1}^N \int_0^T d\tau_i e^{\sum_{i,j=1}^N \left[\frac{1}{2} G_{ij} k_i \cdot k_j - i \dot{G}_{ij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{ij} \varepsilon_i \cdot \varepsilon_j \right]} \\ &\times e^{-\left(m^2 + \langle (a^2) \rangle - \langle (a) \rangle^2\right) T + 2 \sum_{i=1}^N k_i \cdot (I(\tau_i) - \langle I \rangle) - 2i \sum_{i=1}^N (a(\tau_i) - \langle a \rangle) \cdot \varepsilon_i} \Big|_{\varepsilon_1 \dots \varepsilon_N} \end{aligned} \quad (25)$$

where we have introduced light-cone coordinates and the worldloop average

$$\langle \langle f \rangle \rangle \equiv \frac{1}{T} \int_0^T d\tau f(\tau). \quad (26)$$

Spin was incorporated in [32] using the following generalization of the vacuum correlator (5),

$$\langle \psi^\mu(\tau) \psi^\nu(\tau') \rangle = \frac{1}{2} \mathfrak{G}_F^{\mu\nu}(\tau, \tau'),$$

where

$$\mathfrak{G}_F^{\mu\nu}(\tau, \tau') \equiv \left\{ \delta^{\mu\nu} + 2in^\mu \mathcal{J}^\nu(\tau, \tau') + 2i\mathcal{J}^\mu(\tau', \tau)n^\nu + 2 \left[\mathcal{J}^2(\tau, \tau') - \frac{T^2}{4} \langle (a') \rangle^2 \right] n^\mu n^\nu \right\} G_F(\tau, \tau') \quad (27)$$

and we have further defined

$$J_\mu(\tau) \equiv \int_0^\tau d\tau' \left(a'_\mu(\tau') - \langle \langle a'_\mu \rangle \rangle \right), \quad \mathcal{J}_\mu(\tau, \tau') \equiv J_\mu(\tau) - J_\mu(\tau') - \frac{T}{2} \dot{G}(\tau, \tau') \langle \langle a'_\mu \rangle \rangle.$$

See [33] for generalization to the open-line case.

6. The combined constant and plane-wave field

The arguably most complex known background field for which the Klein-Gordon and Dirac equations can be solved in closed form is the combination of a constant and a plane-wave field where the directions of the magnetic and of the electric field coincide with each other and the direction

of the wave propagation [34, 35]. Thus this background should also permit a Bern-Kosower type master formula, and indeed it was derived in the recent [36]:

$$\begin{aligned} \Gamma_{\text{scal}}(\{k_i, \varepsilon_i\}; a, F) &= (-ie)^N (2\pi)^3 \delta\left(\sum_{i=1}^N k_i^1\right) \delta\left(\sum_{i=1}^N k_i^2\right) \delta\left(\sum_{i=1}^N k_i^3\right) \int_{-\infty}^{\infty} dx_0^+ e^{-ix_0^+ \sum_{i=1}^N k_i^-} \\ &\times \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} \det^{\frac{1}{2}} \left[\frac{\mathcal{Z}}{\sin \mathcal{Z}} \right] \prod_{i=1}^N \int_0^T d\tau_i e^{\sum_{i,j=1}^N \left[\frac{1}{2} k_i \cdot \mathcal{G}_{Bij} \cdot k_j - i \varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j + \frac{1}{2} \varepsilon_i \cdot \ddot{\mathcal{G}}_{Bij} \cdot \varepsilon_j \right]} \\ &\times e^{-\left[m^2 + \frac{1}{2} \int_0^T d\tau \int_0^T d\tau' \bar{a}(\tau) \cdot \ddot{\mathcal{G}}_B(\tau, \tau') \cdot \bar{a}(\tau') \right] T - \sum_{i=1}^N \int_0^T d\tau \left[\bar{a}(\tau) \cdot \dot{\mathcal{G}}_B(\tau, \tau_i) \cdot k_i + i \bar{a}(\tau) \cdot \ddot{\mathcal{G}}_B(\tau, \tau_i) \cdot \varepsilon_i \right]} \Big|_{\varepsilon_1 \dots \varepsilon_N} \end{aligned} \quad (28)$$

where now

$$\tilde{a}_\mu(\tau) \equiv a_\mu \left(x_0^+ + n \cdot \sum_{i=1}^N \left[-i \mathcal{G}_B(\tau, \tau_i) \cdot k_i + \dot{\mathcal{G}}_B(\tau, \tau_i) \cdot \varepsilon_i \right] \right) \quad (29)$$

and the fermionic worldline Green's function (27) has been further generalized to

$$\begin{aligned} \tilde{\mathcal{G}}_F(\tau, \tau') &= \mathcal{G}_F(\tau, \tau') + 2i \left[n \otimes \tilde{J}(\tau) \cdot \mathcal{G}_F(\tau, \tau') - \mathcal{G}_F(\tau, \tau') \cdot n \otimes \tilde{J}(\tau') \right] \\ &+ 2i \left[\mathcal{G}_F(\tau, \tau') \cdot \tilde{J}(\tau') \otimes n - \tilde{J}(\tau) \otimes n \cdot \mathcal{G}_F(\tau, \tau') \right] \\ &+ 2\tilde{J}^2(\tau) n \otimes n \cdot \mathcal{G}_F(\tau, \tau') + 2\mathcal{G}_F(\tau, \tau') \cdot n \otimes n \tilde{J}^2(\tau') \\ &- 4\tilde{J}(\tau) \cdot \mathcal{G}_F(\tau, \tau') \cdot \tilde{J}(\tau') n \otimes n \\ &- \frac{iT}{z_{\parallel} + \lambda z_{\perp}} \left[\mathcal{G}_F(\tau, \tau') \cdot \left(n \otimes \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fm\lambda} - \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fm\lambda} \otimes n \right) \right. \\ &\quad \left. - \left(n \otimes \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fm\lambda} - \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fm\lambda} \otimes n \right) \cdot \mathcal{G}_F(\tau, \tau') \right] \\ &+ 2 \frac{T}{z_{\parallel} + \lambda z_{\perp}} \left[\langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fm\lambda} \cdot \tilde{J}(\tau') \mathcal{G}_F(\tau, \tau') \cdot n \otimes n + \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fm\lambda} \cdot \tilde{J}(\tau) n \otimes n \cdot \mathcal{G}_F(\tau, \tau') \right. \\ &\quad \left. - (\tilde{J}(\tau) \cdot \mathcal{G}_F(\tau, \tau') \cdot m_{\lambda} \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_F + \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fm\lambda} \cdot \mathcal{G}_F(\tau, \tau') \cdot \tilde{J}(\tau')) n \otimes n \right] \\ &+ \frac{T^2}{z_{\parallel}^2 - z_{\perp}^2} \left[\frac{\langle \langle \tilde{a}'_{\lambda} \rangle \rangle_F^2}{2} \mathcal{G}_F(\tau, \tau') \cdot n \otimes n + \frac{\langle \langle \tilde{a}'_{\lambda} \rangle \rangle_F^2}{2} n \otimes n \cdot \mathcal{G}_F(\tau, \tau') \right. \\ &\quad \left. - \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_F \cdot \mathcal{G}_F(\tau, \tau') \cdot \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_{Fn} \otimes n \right]. \end{aligned} \quad (30)$$

Here $m_{\pm} \equiv \frac{1}{\sqrt{2}}(1, \pm i, 0, 0)$ and

$$\tilde{J}^{\mu}(\tau) \equiv \sum_{\lambda=\pm} m_{\lambda}^{\mu} e^{-2\frac{\tau}{T}(z_{\parallel} + \lambda z_{\perp})} \int_0^{\tau} d\bar{\tau} \left(\tilde{a}'_{\lambda}(\bar{\tau}) - \langle \langle \tilde{a}'_{\lambda} \rangle \rangle_F \right) e^{2\frac{\tau}{T}(z_{\parallel} + \lambda z_{\perp})}$$

where $z_{\perp} = eBT$, $z_{\parallel} = ieET$, and

$$\langle \langle \tilde{a}'_{\lambda} \rangle \rangle_F \equiv \frac{2(z_{\parallel} + \lambda z_{\perp})}{1 - e^{-2(z_{\parallel} + \lambda z_{\perp})}} \frac{1}{T} \int_0^T d\tau \tilde{a}'_{\lambda}(\tau) e^{-2(z_{\parallel} + \lambda z_{\perp}) \frac{T-\tau}{T}}.$$

7. Multi-loop photon amplitudes

Finally, let us come to the second interesting feature of the worldline formalism mentioned in the introduction: dealing with the amplitude as a whole becomes important when one uses the one-loop amplitudes to construct higher-loop amplitudes by sewing. E.g., from the four-photon amplitude we can construct the two-loop photon propagator,



Figure 2: Summing diagrams for the two-loop photon propagator.

with all three diagrams combined into a single integral. Similarly, from the one-loop six-photon amplitude we get the three-loop quenched propagator

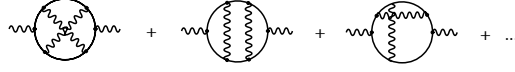


Figure 3: Summing diagrams for the three-loop quenched photon propagator.

etc. And precisely this type of sums of diagrams is known to suffer from particularly extensive cancellations (see, e.g., [15, 37]). More efficient than sewing is the use of multi-loop worldline Green's functions that hold the information on photon insertions [38, 39]. For a single insertion,

$$G^{(1)}(\tau_1, \tau_2) = G(\tau_1, \tau_2) + \frac{1}{2} \frac{[G(\tau_1, \tau_a) - G(\tau_1, \tau_b)][G(\tau_a, \tau_2) - G(\tau_b, \tau_2)]}{\bar{T} + G(\tau_a, \tau_b)} \quad (31)$$

where \bar{T} is the proper-time length of the inserted propagator, and τ_a, τ_b the points on the loop between which the propagator is inserted. It leads to integral representations for the l -loop photon propagator naturally written in the variables $G_{a_1 b_1}, G_{a_2 b_2}, \dots, G_{a_l b_l}, C_{a_1 b_1 a_2 b_2}, \dots, C_{a_{l-1} b_{l-1} a_l b_l}$, where the $G_{a_i b_i}$ depend only on a single propagator, and the $C_{a_i b_i a_j b_j}$ on pairs of propagators.

In continuation of previous work [39] (see also [7]) on the two-loop QED beta functions, we (V.M.B.G. and C.S.) are presently calculating the full two-loop photon propagators for both scalar and spinor QED along the lines outlined above. Let us show here the integral representation obtained using the Green's function (31) for this propagator in scalar QED,

$$\begin{aligned} \Pi_{\text{scal}}^{(2)}(k^2) &= -\frac{e^6}{2(4\pi)^D} \int_0^\infty \frac{dT}{T^{D+1}} e^{-m^2 T} \int_0^\infty d\bar{T} \int_0^{\bar{T}} d\tau_a \int_0^{\bar{T}} d\tau_b \gamma_{ab}^{D/2} \\ &\quad \times \int_0^{\bar{T}} d\tau_1 \int_0^{\bar{T}} d\tau_2 e^{-(G_{12} - \frac{\gamma_{ab}}{4} C^2) k^2} I, \end{aligned} \quad (32)$$

$$\begin{aligned} I &= D(-\ddot{G}_{ab} + \frac{\gamma_{ab}}{2} \dot{G}_{ab}^2)(-\ddot{G}_{12} - \frac{\gamma_{ab}}{2} \partial_1 C \partial_2 C) \frac{1}{k^2} \\ &\quad + \left[(-\ddot{G}_{a1} - \frac{\gamma_{ab}}{2} \dot{G}_{ab} \partial_1 C)(-\ddot{G}_{b2} + \frac{\gamma_{ab}}{2} \dot{G}_{ab} \partial_2 C) + (1 \leftrightarrow 2) \right] \frac{1}{k^2} \\ &\quad - [\partial_a C - \frac{\gamma_{ab}}{2} \dot{G}_{ab} C][\partial_b C + \frac{\gamma_{ab}}{2} \dot{G}_{ab} C] \left(\ddot{G}_{12} + \frac{\gamma_{ab}}{2} \partial_1 C \partial_2 C \right) \end{aligned} \quad (33)$$

$$(C = G_{1a} - G_{1b} - G_{2a} + G_{2b}, \gamma_{ab} = (\bar{T} + G_{ab})^{-1}).$$

8. Some remarks on the non-abelian case

While here we have focussed on the abelian case, let us mention that the integrated-by-parts integrand following from the master formula (3) has recently been found to have other interesting properties that are not visible in the abelian case. In particular, it provides a simple route to the derivation of gluon [40] and graviton [41] Berends-Giele currents.

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