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Elliptic Feynman integrals needed for NNLO QCD corrections to dijet production

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In this talk, we discuss the progress made for a set of two-loop non-planar Feynman integrals containing massive internal propagators. These integrals are needed to compute NNLO QCD corrections to dijet and diphoton production. The corresponding Feynman integral topology contains a top-quark loop and is known to contain an elliptic curve.

Loops and Legs in Quantum Field Theory (LL2024)
14-19, April, 2024
Wittenberg, Germany

*Speaker

PoS(LL2024)007

1. Introduction

The Large Hadron Collider (LHC) operates at unprecedentedly high energies, giving rise to exceptionally precise experimental data. There is a growing demand for equally precise theoretical calculations to match the experimental precision achievable by the present and future colliders. To achieve this goal, it is imperative to compute precise higher-order perturbative corrections including contributions arising from massive quark loops. However inclusion of these massive quark loops presents us with a new challenge: appearance of complicated analytic structure.

Feynman integrals appearing at various perturbative orders are known to exhibit rich mathematical structure. It is an interesting question to ask, *what is the class of functions that appears at various loop orders in these perturbative computations?* With only massless internal particles, these perturbative corrections mostly give rise to the class of functions known as the multiple polylogarithms. However, already at two-loop, the inclusion of massive internal propagators brings us to a family of Feynman integrals notorious for containing elliptic functions: the famous sunrise integral. Beyond two-loops, we quickly reach the class of functions that live on $K3$ surfaces [1] or even higher-dimensional Calabi-Yau manifolds [2]. To compute analytic corrections with functions living on such manifolds, it is important to understand their mathematical properties. Naturally, in the context of analytic perturbative computations, mathematics and physics go hand in hand.

Scattering amplitudes lie at the core of these perturbative calculations, serving as a building blocks connecting theoretical results with experiments. These scattering amplitudes can be expressed in terms of Feynman integrals. Analytic computation of these Feynman integrals is hence required to compute the higher-order corrections for the scattering amplitudes. A good control on the analytic function space of the Feynman integrals gives rise to very precise theoretical data to compare against the experimental data, see for instance [3, 4].

2. Preliminaries

One of the most powerful modern techniques to analytically compute multi-loop Feynman integrals is the method of differential equations (DE). Feynman integrals in dimensional regularisation satisfy the integration-by-parts identities (IBP) [5, 6], which can be used to obtain a minimal set of master integrals (MI). To solve these MI analytically we can use their DE. Using the same IBP, one can obtain a linear system of DE with respect to all the invariants and the masses. Finding the analytic results of the MI then corresponds to integrating this system of DE. Instead of using the basis that we arrive at when using IBP reductions (the so-called pre-canonical basis), we can transform our pre-canonical basis using a transformation matrix T to a better basis $\vec{J}$

$$\vec{J} = T\vec{I}, \tag{1}$$

such that the corresponding DE attain a canonical form given by

$$d\vec{J} = \epsilon A\vec{J}. \tag{2}$$

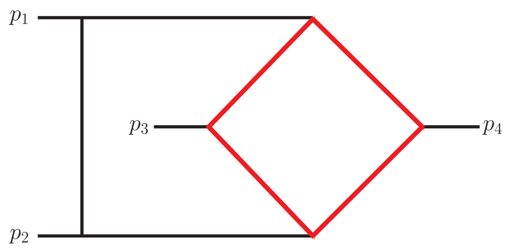

Figure 1: The non-planar Feynman integral topology with massive internal loops needed for NNLO QCD corrections for dijet production. Red and black lines represent massive and massless quarks respectively. This figure is from [10] licensed under CC-BY 4.0.

Here the differential equation matrix A is free of the dimensional regularisation parameter ϵ. With such a choice of basis, solving the set of DE to obtain the Laurent's expansion in ϵ becomes trivial for $\vec{J}$. Particularly, each $J_k^{(j)}$ in

$$J_k = \sum_{j=0} \epsilon^j J_k^{(j)} \tag{3}$$

can be conveniently expressed in terms of Chen's iterated integrals [16]. The important question here is: which kind of functions appear in the J_k^j for different Feynman integrals with varying number of loops and legs? To answer this question a powerful tool is the maximal cut. Cutting a propagator means putting that propagator on-shell. This is achieved by adding a delta function, within the Feynman integral, corresponding to the vanishing of that propagator. Maximal cut mathematically means computing an n-fold residue in the complex plane around every pole for the n number of propagator for a Feynman integral. Maximal cuts are important as they are the solution of the homogeneous DE and hence informs us about the kind of functions that can appear in the analytic solution of the DE.

3. A two-loop non-planar topology with elliptic sectors

We now discuss a set of two-loop non-planar Feynman integrals that appear in perturbative computations for precision physics. Particularly, the diagram shown in figure 1 is needed for computing the next-to-next-to leading order (NNLO) corrections in QCD to dijet as well as diphoton productions. Both the top sector and a subsector, shown in figure 2, are known to evaluate to functions depending on an elliptic curve. This diagram has been investigated multiple times in the literature [7–9], however, due to the algebraic complexity of the function spaces, a fully compact analytic representation is missing.

The Feynman integral family is defined as follows:

$$I_{a_1,\cdots,a_9} = \left(\frac{e^{\epsilon\gamma_E}}{i\pi^{\frac{d}{2}}}\right)^2 \int \prod_{i=1}^{2} d^d k_i \frac{D_8^{a_8} D_9^{a_9}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7}}, \quad a_j \in \mathbb{Z}. \tag{4}$$

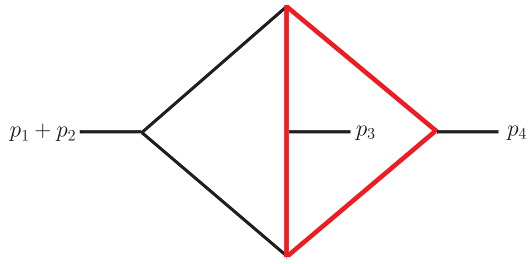

Figure 2: One of the subsectors that appear in the set of master integrals for figure 1. This figure is from [10] licensed under CC-BY 4.0.

with the list of propagators D given as

$$\begin{aligned}&\{k_1^2, (k_1-p_1)^2, (k_1-p_1-p_2)^2, k_2^2-m_t^2, (k_2-p_1-p_2-p_3)^2-m_t^2,\\&(k_1-k_2)^2-m_t^2, (k_1-k_2+p_3)^2-m_t^2, (k_1-p_1-p_2-p_3)^2, (k_2+p_1)^2-m_t^2\}\,.\end{aligned} \tag{5}$$

We start with setting up the system of IBP, which gives us 36 master integrals. We then set up a system of DE with respect to both the Mandelstam variables s and t (we set $m^2 = 1$) given by

$$s = (p_1+p_2)^2 \qquad t = (p_1+p_3)^2. \tag{6}$$

Following the bottom-up approach, we begin solving this triangular system of DE by first bringing the DE corresponding to the subsectors to an ϵ-factorised form. All the subsector integrals of this family of Feynman integrals are well-known in the literature [7, 8, 11–15]. While the integrals J_1 to J_{30} are polylogarithmic in nature, the J_{31}-J_{32} integrals are known to contain an elliptic curve. The top-sector integrals J_{33}-J_{36} were also known to evaluate to elliptic functions even before [10]; however, due to multi-scale dependence, obtaining an ϵ-factorised DE could not be achieved. We now briefly describe how an ϵ-factorised DE of the top sector on the maximal cut was obtained in [10].

To achieve this, we start by studying the maximal cuts of the two elliptic sectors present in the set of master integrals within the loop-by-loop Baikov parametrisation [19, 20]. In d=4-2ϵ dimensions, the maximal cut of the sector $J_{31} - J_{32}$ is given by

$$\frac{8}{\pi^3}\int_C \frac{dP}{s\,\sqrt{P}\,\sqrt{s+P}\,\sqrt{-4m_t^2 s + sP + P^2}} + O(\epsilon). \tag{7}$$

On the other hand, the maximal cut of the top sector is given as

$$\frac{16}{\pi^4}\int_C \frac{dP}{s\,(s+t+P)\,\sqrt{P}\,\sqrt{s+P}\,\sqrt{-4m_t^2 s + sP + P^2}} + O(\epsilon). \tag{8}$$

Here, C is the contour over the Baikov variable P, which is left after integrating out all the delta-distributions corresponding to putting all the propagators on-shell. From these equations, it becomes clear that both the maximal cuts contain the same quartic square root, with the t-dependence factoring out in equation (8). A square root of a quartic polynomial with distinct zeroes, defines an elliptic curve. Hence both these sectors correspond to the same elliptic curve. This information plays an important role in ϵ-factorising the DE for the top-sector, as will be clear shortly.

We begin our construction of an ϵ-basis for the top-sector by first decoupling the DE at ϵ^0 using the methodology described in [17]. This brings the DE at order ϵ^0 to a form

$$\begin{pmatrix} F_1(s) & F_2(s) & 0 & 0 \\ F_3(s) & F_4(s) & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

where $F_i(s)$ are some functions with no dependence on t while $*$ represent non-zero entries (which could depend on both s and t). In particular, the first 2×2 block for the partial DE with respect to t are already ϵ-factorised. Using the transformed DE we can compute the Picard-Fuchs operator for the integral

$$I_1 = \frac{2\epsilon^4}{(16+s)(s+2t)}\left(sI_{1,1,1,1,1,1,1,-1,-1} - sI_{1,1,1,1,1,1,1,0,-2} + (4s+s^2+8t+st)I_{1,1,1,1,1,1,1,0,-1}\right). \tag{10}$$

This integral is simply the first integral from the transformed basis, i.e. the basis with the DE of the form 9. The corresponding second-order Picard-Fuchs operator L_0, where

$$L_0 = \partial_s^2 - \left(-\frac{4}{s} - \frac{2}{16+s}\right)\partial_s + \frac{6(6+s)}{s^2(16+s)}, \tag{11}$$

is associated to an elliptic curve and has two independent solutions. They can be chosen as

$$\psi_0(s) = \frac{32\, E(-\frac{s}{16})}{\pi s^{3/2}(s+16)}, \tag{12}$$

$$\psi_1(s) = \frac{32\,\left(E(\frac{s}{16}+1) - K(\frac{s}{16}+1)\right)}{s^{3/2}(s+16)}. \tag{13}$$

We further factorise ϵ from the DE by extending the methods of [18] to multiple scales. In particular, we look for new MIs

$$M_1 = \frac{1}{h(s,t)}I_1, \tag{14}$$

$$M_2 = \frac{g_1(s,t)}{\epsilon}\frac{d}{ds}M_1 + \frac{g_2(s,t)}{\epsilon}\frac{d}{dt}M_1 - f_1(s,t)M_1, \tag{15}$$

constructed as the most general ansatz with four rational functions $h(s,t), g_1(s,t), g_2(s,t)$ and $f_1(s,t)$, with the motivation of Hodge filtration of the Hodge structure. We find all the unknown functions in equation (14) by requiring that the transformed DE have terms proportional to ϵ and the constant terms are as simple as possible. The following is one such choice:

$$h(s,t) = \frac{1}{s^2+16s}, \quad g_1(s,t) = 0, \tag{16}$$

$$g_2(s,t) = \frac{16\, s\, (s+2t)^2}{16\, t + s\, (8+t)}, \quad f_1(s,t) = \frac{16\, s^3\, (s+2t)}{t\, (s+t)\, (16\, t+s\, (8+t))}. \tag{17}$$

This gives us the transformation matrix

$$U_1 = \begin{pmatrix} s\ (16+s) & 0 \\ -64\ s^2\ (12+s) & -16\ s^2\ (16+s) \end{pmatrix}. \tag{18}$$

We then perform one more transformation with the matrix U_2 where the entries are constrained to eliminate the constant terms in the DE. This gives us an ϵ-form for the 2×2 block.

$$U_2 = \frac{s^3(s+16)^2}{16} \begin{pmatrix} s\psi_1' - 2\psi_1 & -\psi_1 \\ -2(s\psi_0' - 2\psi_0) & 2\psi_0 \end{pmatrix}. \tag{19}$$

Using these two transformations we have already attained a special linear form [21] for the DE for the top-sector. From here, to obtain a fully ϵ-factorised DE for the 4×4 block, we can construct yet another (trivial) transformation by putting constraints to eliminate the constant terms in the DE matrices, as explained in [10]. Nevertheless, already as this stage one can integrate out the DE to express the analytic results of the master integrals order by order in ϵ, albeit with functions having different weights appearing at a particular ϵ order.

We have progressed significantly in understanding the analytic nature of this family of Feynman integrals, by finding conclusive evidences of its dependence on only one elliptic curve, contrary to previous findings. Nevertheless, there is a lot of work remaining to be done for in choosing a function space for the analytic results that can be used to obtain compact representations of higher-order perturbative corrections. This will be also crucial for obtaining a fast and stable numerical implementation valid in the full phase-space region.

4. Conclusions

The study of mathematical properties of Feynman integrals plays an important role in perturbative computations that are necessary to confront the current precision frontier. In this talk, we have highlighted the crucial steps that have been taken in understanding the algebraic spaces corresponding to the non-planar topology contributing to NNLO QCD corrections to dijet and diphoton production, with massive internal propagators. Particularly, the algebraic properties play a crucial role in finding an ϵ-form for the DE of the master integrals. The next obvious steps are obtaining a stable numerical evaluation of these Feynman integrals, that is valid in the full phase-space region.

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PoS(LL2024)007