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Non-Planar Two-Loop Amplitudes for Five-Parton Scattering

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We review the current status of high-multiplicity double-virtual QCD corrections to processes relevant for LHC phenomenology. In particular, we discuss the recent full-color calculation of the five-parton process, whose two-loop amplitudes are required to obtain next-to-next-to-leading order predictions for three-jet production at the LHC. We address various aspects of the computation, including color decomposition, renormalization, partial amplitudes, color identities and the construction of the finite remainder. We review the method of numerical unitarity, which is used to generate finite-field samples of the amplitude. We then focus on the analytic reconstruction of the coefficient functions from these numerical samples via Ansatz techniques. A novel algorithm, based on the correlation of codimension-one residues, helps manage the complexity of the calculation. Little-group rescalings of the gluon amplitude, inspired by supersymmetry Ward identities, facilitate the computation of the quark amplitudes. We conclude with an outlook towards upcoming computations with an increased number of scales, leading to larger Ansätze and more complicated alphabets.

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1. Introduction

In the absence of glaring signals of new physics (NP) at the Large Hadron Collider (LHC), precision studies of the Standard Model (SM) of particle physics have been playing an increasingly important role. The aim is to better understand aspects of the SM, such as the convergence of the first orders in the perturbative expansion, factorization properties, and values of the free parameters, and at the same time increase the sensitivity to subtle deviations that may hint at NP. Precise theoretical predictions play a crucial role in both regards. In particular, next-to-next-to-leading order (NNLO) predictions in quantum chromodynamics (QCD) are essential for precision studies targeting $\sim$1% uncertainties, and in some cases, N^{3}LO predictions may even be desirable [1, 2].

Recent years have seen significant advancements in the computation of higher-order QCD corrections for multi-scale processes. These consist predominantly of two-loop five-point amplitudes, with massless propagators and up to one external massive leg. These amplitudes were first obtained in the leading color (l.c.) approximation. For pure QCD processes, this generally corresponds to planar diagrams. However, especially in the presence of electroweak particles, non-planar diagrams may contribute at leading color. Nevertheless, it is usually possible to identify gauge-invariant planar subamplitudes. As of the end of 2023, all five-point massless processes relevant for hadron-collider phenomenology have been obtained in full color, while no five-point one-mass process is known beyond the planar approximation. Table 1 summarizes the current status.

Processes	**Analytic Results**	**Public Codes**	**Cross Sections**
$pp \to \gamma\gamma\gamma$	[3, 4]‡ **[5]**	[3]‡, **[5]**	[6, 7]‡
$pp \to \gamma\gamma j$	[8–10]† **[11]**	[8, 10]†	[12, 13]†
$pp \to \gamma jj$	**[14]**		**[14]**
$pp \to jjj$	[15]†**[16–18]**	[15]†**[18]**	[19, 20]†
$pp \to Wb\bar{b}$	[21]* [22–24]†	[24]†	[23, 25]†
$pp \to Hb\bar{b}$	[26]*		
$pp \to Wj\gamma$	[27]‡		
$pp \to Wjj$	[22, 24]†	[24]†	
$pp \to (Z/\gamma^\star)jj$	[22, 24]‡	[24]‡	
$pp \to ttH$			[28]*

Table 1: Summary of known two-loop QCD corrections for five-point scattering processes at hadron colliders. † denotes calculations performed in l.c. approximation, where l.c. coincides with planar, while ‡ denotes planar computations that are not l.c. accurate. ⋆ denotes additional approximations, such as on-shell W, $m_b = 0$ but $y_b \neq 0$, or soft Higgs. Bold denotes non-planar, full-color results.

The computation of these amplitudes was made possible thanks to developments on two main fronts. First, regarding the transcendental part of loop amplitudes, this includes the application of differential equations to Feynman integrals [29–31], their ϵ-factorized form [32], and progress on integration-by-parts (IBP) reduction [33] using algebro-geometric techniques [34–36]. To enable phenomenological applications, the development of numerically efficient and stable implementations of master integrals was crucial. This was achieved through Chen's iterated in-

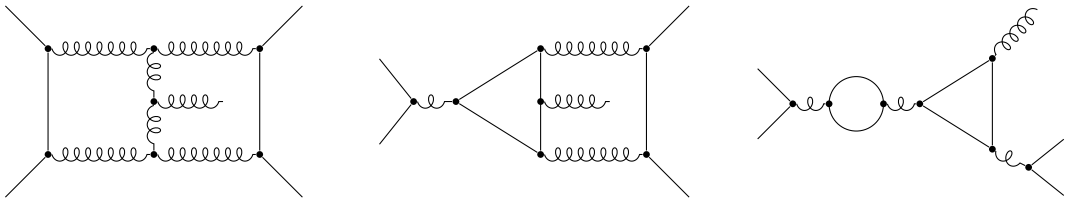

Figure 1: Representative Feynman diagrams for two-loop four-quark one-gluon amplitudes, contributing at N_f^0, N_f^1 and N_f^2 respectively. Solid lines represent massless quarks.

tegrals [37], implemented in so-called pentagon functions [38] now available in the public code `PentagonFunctions++` for both massless [39] and one-mass [40, 41] five-point processes.

Second, on the rational coefficient side, a crucial development was the introduction of exact numerical arithmetic, in the form of finite fields [42, 43], and reconstruction techniques based on interpolation [43, 44] or Ansatz fitting [45, 46]. To achieve compact results, it is important to control the evaluation of the coefficients in degenerate kinematic limits in complex kinematics [45], which generalize the familiar concepts of soft and collinear limits. To reconcile this type of evaluations with exact arithmetic the use of p-adic numbers was proposed [46] and then further investigated in relation to multivariate partial fraction decompositions [47, 48]. Another recent development was the use of $\mathbb{Q}[\vec{x}]$ linear relations [49] to facilitate reconstruction. In this work, we made use of a similar concept, namely $\mathbb{Q}$-linear relations valid in singular limits.

2. Scattering Amplitudes and Finite Remainders

The two-loop amplitudes required for NNLO predictions of $pp \to jjj$ in full color were recently computed in [16–18]. We follow in detail the approach of [17, 18]. Three partonic processes are required,

$$g^{-h_1}_{-p_1} + g^{-h_2}_{-p_2} \to g^{h_3}_{p_3} + g^{h_4}_{p_4} + g^{h_5}_{p_5}\,, \tag{1}$$

$$\bar{u}^{-h_1}_{-p_1} + u^{-h_2}_{-p_2} \to g^{h_3}_{p_3} + g^{h_4}_{p_4} + g^{h_5}_{p_5}\,, \tag{2}$$

$$\bar{u}^{-h_1}_{-p_1} + u^{-h_2}_{-p_2} \to d^{h_3}_{p_3} + \bar{d}^{h_4}_{p_4} + g^{h_5}_{p_5}\,, \tag{3}$$

including all crossings thereof, obtained via analytic continuation, as well as the process with four identical quark flavors, obtained from linear combinations of eq. (3). As indicated by the labels for momenta (p_i) and helicities (h_i), we work in the all-outgoing convention.

Color decomposition The amplitude for each partonic process admits a color decomposition in terms of fundamental $SU(N_c)$ generators. For the four-quark one-gluon process, it reads,

$$\begin{aligned}\mathcal{A}_{\bar{d}}(1_u, 2_{\bar{u}}, 3_d, 4_{\bar{d}}, 5_g) = &\sum_{\sigma\in\mathcal{Z}_2(\{1,2\},\{3,4\})} \sigma\Big(\delta^{\bar{i}_4}_{i_1} (T^{a_5})^{\bar{i}_2}_{i_3}\, A_d(1,2,3,4,5)\Big) + \\ &\sum_{\sigma\in\mathcal{Z}_2(\{1,2\},\{3,4\})} \sigma\Big(\delta^{\bar{i}_2}_{i_1} (T^{a_5})^{\bar{i}_4}_{i_3}\, A_s(1,2,3,4,5)\Big)\,,\end{aligned} \tag{4}$$

where the color-ordered sub-amplitudes A_d and A_s are labelled depending on whether the color-space Kronecker delta is between different or the same quark line, respectively. Permutations are denoted as σ, and $\mathcal{Z}_n$ is the cyclic group of order n.

Renormalization Each color-ordered amplitude further admits an asymptotic expansion in powers of the bare QCD coupling α_s^0. Representative diagrams with a maximum number of propagators for the four-quark one-gluon process at order $(\alpha_s^0)^2$ are displayed in fig. 1. We perform renormalization in the $\overline{\text{MS}}$ scheme, via the substitution,

$$\alpha_s^0 \mu_0^{2\epsilon} S_\epsilon = \alpha_s \mu^{2\epsilon} \left(1 - \frac{\beta_0}{2\epsilon}\frac{\alpha_s}{2\pi} + \left(\frac{\beta_0^2}{4\epsilon^2} - \frac{\beta_1}{8\epsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + O\left(\alpha_s^3\right)\right),$$

Partial Amplitudes At each order in the α_s^0 (or α_s) expansion, amplitudes can be further expanded in powers of the number of colors, $N_c^{n_c}$, and of the number of light quark flavors, $N_f^{n_f}$. We label these gauge-invariant building blocks with the notation $A^{(L),(n_c,n_f)}$. For the two four-quark one-gluon amplitudes of eq. 4 at two-loop this reads,

$$A_d^{(2)} = N_c^2 A_d^{(2),(2,0)} + A_d^{(2),(0,0)} + \frac{1}{N_c^2} A_d^{(2),(-2,0)} + N_f N_c A_d^{(2),(1,1)} + \frac{N_f}{N_c} A_d^{(2),(-1,1)} + N_f^2 A_d^{(2),(0,2)}, \tag{5}$$

$$A_s^{(2)} = N_c A_s^{(2),(1,0)} + \frac{1}{N_c} A_s^{(2),(-1,0)} + \frac{1}{N_c^3} A_s^{(2),(-3,0)} + N_f A_s^{(2),(0,1)} + \frac{N_f}{N_c^2} A_s^{(2),(-2,1)} + \frac{N_f^2}{N_c} A_s^{(2),(-1,2)}. \tag{6}$$

The subleading-color partial amplitudes displayed in red were obtained recently [16, 18].

Color identities It is well known that the color decomposition of eq. 4 is not optimal in terms of having independent partial amplitudes. This is not an issue for our computation strategy, since its cost is largely unaffected by redundancies among partials. Nevertheless, we were interested in uncovering potentially new relations, such as

$$\begin{aligned} &\big[32\, A_d^{(2),(2,0)}(1,2,3,4,5) + 8\, A_d^{(2),(0,0)}(1,2,3,4,5) + 2A_d^{(2),(-2,0)}(1,2,3,4,5) \\ &+16\, A_s^{(2),(1,0)}(1,2,3,4,5) + 4A_s^{(2),(-1,0)}(1,2,3,4,5) + A_s^{(2),(-3,0)}(1,2,3,4,5)\big] - \big[\dots\big]_{3\leftrightarrow 4} = 0\,. \end{aligned} \tag{7}$$

We obtain such relations via numerical finite-field evaluations and linear algebra (null-space computation and intersection). Note that while an identity exists among partials, it does not allow to express one amplitude as a linear combination of others, unlike in the pure Yang-Mills channel. This is due to the fact that each partial amplitude appears with multiple permutations of the arguments.

Finite remainder We subtract the infrared singularities via a color-space operator $\mathbf{Z}(\epsilon,\mu)$ [50] (see also [51, 52]),

$$\mathcal{R}(\lambda,\tilde\lambda,\mu) = \mathbf{Z}(\epsilon,\lambda\tilde\lambda,\mu)\mathcal{A}(\lambda,\tilde\lambda,\mu) \ + \ O(\epsilon)\,. \tag{8}$$

$\mathcal{R}$ denotes the so-called finite remainder, i.e. it is the finite part of the amplitude carrying the new physical information at a given loop order. It can be written as a weighted sum of pentagon functions h_i with rational function coefficients r_i,

$$\mathcal{R}(\lambda,\tilde\lambda,\mu) = \sum_i r_i(\lambda,\tilde\lambda)\, h_i(\lambda\tilde\lambda,\mu)\,. \tag{9}$$

Note that knowledge of the one-loop amplitude $\mathcal{A}^{(1)}$ at higher orders in the dimensional regulator ϵ is only required to build the two-loop finite remainder $\mathcal{R}^{(2)}$. For the computation of NNLO observables, such as differential cross-sections, the finite remainders $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ suffice [53] (besides the tree $\mathcal{A}^{(0)}$).

3. Numerical Computation over Finite Fields

To circumvent the swell of complexity in intermediate stages of the computation, we first compute partial amplitudes and associated finite remainders numerically over finite fields. This is achieved via the method of numerical unitarity [35, 54–56], as implemented in the public code Caravel [57]. The amplitude integrand $A(\lambda, \tilde{\lambda}, \ell)$ is expressed as a linear combination of topologies Γ and, for each topology, a set of master integrands M_Γ and surface terms S_Γ, with coefficients $c_{\Gamma,i}(\lambda, \tilde{\lambda}, \epsilon)$, dependent on external kinematics and epsilon, namely,

$$A(\lambda, \tilde{\lambda}, \ell) \quad = \sum_{\Gamma,\, i\in M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(\lambda, \tilde{\lambda}, \epsilon)\, \frac{m_{\Gamma,i}(\lambda\tilde{\lambda}, \ell)}{\prod_j \rho_{\Gamma,j}(\lambda\tilde{\lambda}, \ell)}\,. \tag{10}$$

After integration, surface terms drop out (they integrate to zero), master integrands yield master integrals $I_{\Gamma,i}$, and, for a suitable choice of these, the location of the poles in the dimensional regulator ϵ take values independent of the external kinematics ($a_{ij} \in \mathbb{Q}$),

$$A(\lambda, \tilde{\lambda}) = \int d^D\ell\; A(\lambda, \tilde{\lambda}, \ell) = \sum_{\substack{\Gamma,\\ i\in M_\Gamma}} \frac{\sum_{k=0}^{k_{\max}} c^{(k)}_{\Gamma,i}(\lambda, \tilde{\lambda})\, \epsilon^k}{\prod_j (\epsilon - a_{ij})}\, I_{\Gamma,i}(\lambda\tilde{\lambda}, \epsilon) \tag{11}$$

In a procedure equivalent to IBP reduction, the coefficients of the master integrands (and surface terms) are obtained by solving linear systems of equations to match the right-hand side parametrization of the integrand to products of trees on cuts,

$$\sum_{\text{states}} \prod_{\text{trees}} A^{\text{tree}}(\lambda, \tilde{\lambda}, \ell)\Big|_{\Gamma_{\text{cut}}} = \sum_{\substack{\Gamma' \geq \Gamma_{\text{cut}},\\ i\in M'_\Gamma \cup S'_\Gamma}} c_{\Gamma',i}(\lambda, \tilde{\lambda})\, \frac{m_{\Gamma',i}(\lambda\tilde{\lambda}, \ell)}{\prod\limits_{j\in P_{\Gamma'}/P_{\Gamma_{\text{cut}}}} \rho_j(\lambda\tilde{\lambda}, \ell)}\Bigg|_{\Gamma_{\text{cut}}}\,. \tag{12}$$

This requires prior knowledge of the integrand decomposition $M_\Gamma \cup S_\Gamma$. For the computation of $pp \to jjj$ at two-loop this decomposition was obtained by extending to higher power counting the decomposition previously obtained for $pp \to \gamma\gamma\gamma$ using the embedding space formalism [5].

4. Analytic Reconstruction

To enable phenomenological studies, we must have access to reasonably fast and stable evaluations of the amplitude (or finite remainder) with floating-point numbers. This is achieved by reconstructing compact analytic form from numerical evaluations in $\mathbb{F}_p$ [42, 43].

The coefficient $r_i(\lambda, \tilde{\lambda})$ in the finite remainder are rational functions of the external kinematics. That is, they belong to the field of fractions of the following polynomial quotient ring [46],

$$R_n = \mathbb{F}\big[|1\rangle, [1|, \ldots, |n\rangle, [n|\big] \Big/ \Big\langle \sum_i |i\rangle[i| \Big\rangle\,, \tag{13}$$

where the big angle brackets denote an ideal, specifically the ideal generated by the four momentum-conservation equations. The small angle and square brackets denote spinors in the spinor-helicity

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notation, $|i\rangle = \lambda_{i,\alpha}$ and $[i| = \tilde\lambda_{i,\dot\alpha}$. We refer to R_n as the Lorentz covariant ring. Since the $r_i(\lambda,\tilde\lambda)$ are Lorentz invariant, they belong to a sub-ring of R_n, namely

$$\mathcal{R}_n = \mathbb{F}\big[\langle 1|2\rangle, \dots, [n-1|n]\big] \big/ (\mathcal{J}_n + \mathcal{K}_n + \bar{\mathcal{K}}_n)\,, \tag{14}$$

where $\mathcal{J}_n$ denotes the Lorentz-invariant momentum-conservation ideal, and $\mathcal{K}_n$ and $\bar{\mathcal{K}}_n$ denote the ideals generated by holomorphic and anti-holomorphic Schouten identities respectively.

Least common denominator In common-denominator form, the rational functions read,

$$r_i(\lambda,\tilde\lambda) = \frac{\mathcal{N}_i(\lambda,\tilde\lambda)}{\prod_j \mathcal{D}_j^{\alpha_{ij}}(\lambda,\tilde\lambda)} \tag{15}$$

where $\mathcal{N}_i(\lambda,\tilde\lambda)$ is a polynomial and the denominator factors $\mathcal{D}_j(\lambda,\tilde\lambda)$ are irreducible. If the powers $\alpha_{ij} \in \mathbb{Z}$ are as low as possible, then the denominator is the least common denominator (LCD).

The poles $\mathcal{D}_j$ can be thought of as codimension-one varieties $V(\langle \mathcal{D}_j \rangle)$ in R_n (or $\mathcal{R}_n$). At five points and up to two loops there are 35 possible poles, unchanged from one loop,

$$\{\mathcal{D}_{\{1,\dots,35\}}\} = \bigcup_{\sigma\,\in\,\mathrm{Aut}(R_5)} \sigma \circ \big\{\langle 12\rangle, \langle 1|2+3|1]\big\} \tag{16}$$

where the automorphisms of R_5 are composed of permutations of the legs $\{1,\dots,5\}$ and a possible swap of the left- and right-handed representations of the Lorentz group, $\langle\rangle \leftrightarrow []$. The latter is related to parity. Note that $\mathrm{tr}_5 = \mathrm{tr}(\gamma_5 p_1 p_2 p_3 p_4)$ is not a singularity of the pentagon-function coefficients in the finite remainder. It is manifestly absent from the denominators of our results.

Basis change The form of eq. 18 is neither compact nor efficient for reconstruction. In this form, we would have required around a quarter million numerical probes. This is computationally prohibitive. In the end, we managed to reconstruct the amplitude from around $35k$ numerical evaluations. To achieve this, we performed a basis change in the vector space of pentagon coefficients, $\mathrm{span}(r_i)$,

$$\tilde r_i = O_{ij}\, r_{j\in\mathcal{B}}\,,\ O_{ij} \in \mathbb{Q}\,,\ \text{and } \mathcal{B} \text{ a basis of } \mathrm{span}(r_i)\,. \tag{17}$$

The aim is to minimise the denominator powers α_{ij} of the basis functions. To achieve this, we require information regarding the correlation of residues among the r_i's at the different poles $\mathcal{D}_k$. To obtain residues in $\mathbb{F}_p$ we perform an univariate reconstruction and subsequent formal Laurent expansion,

$$r_{i\in\mathcal{B}} = \sum_{m=1}^{q_k=\max_i(q_{ik})} \frac{e^k_{im}}{(t-t_{\mathcal{D}_k})^m} + \mathcal{O}((t-t_{\mathcal{D}_k})^0) \tag{18}$$

If the denominator factors are not linear in t, this expansion can be formulated as a $p(t)$-adic series [58]. Through Gaussian elimination on the residues e^k_{im} on a set of different slices we construct the following decomposition of the vector space,

$$\mathrm{span}(r_{i\in\mathcal{B}}) = \underbrace{\mathrm{column}(\mathrm{Res}(r_{i\in\mathcal{B}}, \mathcal{D}_k^m))}_{\text{functions with the singularity } \mathcal{D}_k^m} \oplus \underbrace{\mathrm{null}(\mathrm{Res}(r_{i\in\mathcal{B}}, \mathcal{D}_k^m))}_{\text{functions without the singularity } \mathcal{D}_k^m} \tag{19}$$

We observe that the rational numbers appearing in the null spaces are very simple and can be reconstructed with just a couple of primes. Furthermore, the lifting $\mathbb{F}_p \to \mathbb{Q}$ of matrices in reduced row echelon form can be efficiently performed by iterating on subsequent values of p only those rows containing entries deemed incorrect. A simple way to deem the correctness of a reconstructed rational number is to re-scale it by a constant and compare reconstructed results.

The matrix O_{ij} is constructed by searching the space of intersections of null spaces,

$$O_{ij} = \bigcap_{k,m} \text{null}(\text{Res}(r_{i\in\mathcal{B}}, \mathcal{D}_k^m)) \,. \tag{20}$$

We performed the basis change one sub-amplitude at a time for the gluon channel, before merging them into all-plus, single-minus and MHV vector spaces. The basis functions in LCD form after the basis change are more than an order of magnitude simpler than the original ones.

Ansatz reconstruction Having determined a good set of functions to reconstruct, namely the $\tilde{r}_i$ of eq. 17, we fit their spinor-helicity Ansatz [45, 46] through $\mathbb{F}_p$ evaluations. The simplification stemming from the basis change is compounded with that from a partial-fraction decomposition (PFD). In our PFD we impose (and thus verify) that no denominator contains more than a single letter of the equivalence class with representative $\langle 1|2+3|1]$.

Little-group rescalings Having obtained a compact basis for the gluonic amplitude and inspired by supersymmetry Ward identities, we note that candidate basis functions for the quark amplitudes can be easily obtained by rescaling the gluon basis functions by factors with suitable Little-group weights [18]. Their validity is checked through a far simpler computation than the full one.

5. Conclusions and Outlook

With the recent computation of double-virtual corrections to $pp \to jjj$ [16–18] all two-loop five-point amplitudes with any combination of partons and photons are known in full color in QCD (excluding effects from quark masses, particularly contributions from top quarks). This was possible thanks to three decades of progress in perturbation theory since as the previous perturbative order for $pp \to jjj$ was computed in 1993 [59–61].

Our results, summarized in table 2, demonstrate the potential simplicity of scattering amplitudes even at second order in perturbation theory thanks to the spinor-helicity formalism, partial fraction decompositions, the manifestation of symmetries, and a carefully chosen basis change in the vector space of rational coefficients.

Future computations for multi-loop amplitudes with external or internal masses, or higher multiplicities, will have to contend with a steep increase in complexity not only due to the increased number of scales but also due to more complicated alphabets and additional redundancies in kinematic representations. In an upcoming publication [24] we demonstrate the efficacy of the techniques described here, along with others, in achieving a drastic simplification for the two-loop leading-color amplitudes for $pp \to Wjj$.

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Particle Helicities	Vector-space dimension	Generating set size	File size
$g^+g^+g^+g^+g^+$	24	3	2 KB
$g^+g^+g^+g^+g^-$	440	33	24 KB
$g^+g^+g^+g^-g^-$	937	115	68 KB
$u^+\bar{u}^-g^+g^+g^+$	424	91	45 KB
$u^+\bar{u}^-g^+g^+g^-$	844	449	200 KB
$u^+\bar{u}^-d^+\bar{d}^-g^-$	435	124	56 KB

Table 2: For each helicity configuration, this table shows the dimension of the vector space of rational functions, the number of functions in the generating set that spans the space upon closure under the symmetries of the vector of Little-group scalings of the given process, and the file size where the generating set is stored. This does not include the matrices of rational numbers needed to express the pentagon-function coefficients in terms of the basis of the vector space that they span.

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