

PoS

Discrete symmetry and 't Hooft anomalies for 3450 model

Tetsuya Onogi,* Hiroki Wada and Tatsuya Yamaoka

Department of Physics, Osaka University, 1-1 Machkaneyama-cho, Toyonaka, 560-0043, Japan E-mail: onogi@het.phys.sci.osaka-u.ac.jp, t_yamaoka@het.phys.sci.osaka-u.ac.jp, hwada@het.phys.sci.osaka-u.ac.jp

We report our study of the discrete symmetry for lattice 3450 model proposed by Wang and Wen [1]. Lattice 3450 model is expected to describe the anomaly free chiral U(1) gauge theory in 1+1 dimension using 2+1 dimensional domain-wall fermion with gapping interactions for the mirror sector. We find that the lattice model has exact discrete symmetry in addition to U(1) x U(1) symmetry. Assuming the Zumino-Stora procedure works also for discrete symmetry, we compute the full 't Hooft anomaly for the target continuum U(1) chiral gauge theory with the same discrete symmetry. We show that the mixed and self anomalies involving the discrete symmetry are absent, which is consistent with the expectation that the lattice 3450 produces chiral U(1) gauge theory in the continuum limit.

The 41st International Symposium on Lattice Field Theory (LATTICE2024) 28 July - 3 August 2024 Liverpool, UK

*Speaker

[©] Copyright owned by the author(s) under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License (CC BY-NC-ND 4.0).

1. Introduction

Many people would agree that chiral gauge theory on the lattice is needed. This is because standard model is chiral, and also, interesting dynamics in chiral gauge theories are not fully understood yet.

What about the status of lattice formulation? There are two approaches. One approach is with overlap fermion which has given partial success. Abelian gauge theory in constructed by Lüscher [2]. For non-abelian case, it is not completed due to the lattice cohomology problem [3]. The other approach is Domain-wall fermion [4] with symmetric gapping interactions [5], [6]. With gapping interaction, one hopes to gap out mirror edge mode to construct chiral gauge theory on the lattice. The lattice 3-4-5-0 model is one of the examples. And this is the topic I would like to focus in our study.

Here we would like to make two remarks. First remark is that for a given interaction, whether mirror edge mode can really be gapped out or not is still unknown and numerical test is needed. Second is that However there is a hint that the existence of edge mode in domain-wall fermion is guaranteed by anomaly inflow. Therefore, symmetry and anomaly give strong restrictions on the symmetric gapping scenario with DW fermion.

The question we would like to ask is whether symmetric gapping scenario is consistent with anomaly inflow. In 3450 model, this is already considered for continuous U(1) symmetries. There is a study on the anomaly using cobordism [7] where it was shown that the (1+1)D nonperturbative global anomalies are classified by cobordism group which turn out to vanish. In Ref.[8], the discrete symmetry of 1+1d 3450 model is studied and explains how to avoid the CT or P problem (like strong CP) can be avoided. It is not yet known whether the anomalies for possible discrete symmetries such kind also vanish or not. However, there may be a possibility that the model may contain other discrete symmetries which are not included in the continuous U(1) symmetry. If they exist it is worth knwoing whether the anomalies for possible discrete symmetries such kind also vanish or not. Therefore, our goal is to study the 3450 model with single flavor as well as two flavors to see 1) whether there are discrete symmetries which are not included in continuous U(1) symmetries. These would give a new consistency check of the mirror edge mode gapping scenario for lattice 3450 model.

2. Symmetric gapping and domain-wall fermion

We will now give a brief review of symmetric gapping with DW fermion. How can we define chiral gauge theory on the lattice? The idea is to gap out mirror edge mode from DW fermion by symmetric interactions V_{int} at the mirror edge. There are several promising proposals such as SO(10), SM in 4 dim or 3450 model in 2-dim with symmetric gapping [1, 9–17]

And numerical test of mirror edge mode decoupling has been carried out.

We would like to consider what conditions are needed for gapping interactions. Free DW fermion has exact vector symmetries G. Interaction Vint breaks G to a subgroup H. There are two necessary conditions for H and V_{int} .

- Bulk anomaly for symmetry H must be absent. Otherwise, anomaly inflow requires massless edge-mode. This gives an obstruction against gapping of mirror fermion edge mode.
- 2. Instanton saturation must be fulfilled by V_{int} . In U(1) instanton sector, there are chiral zero mode at the mirror edge. In order to have nonzero path-integral Vint must saturate mirror zero modes.

3. 3450 model

3450 model in the continuum is a chiral U(1) gauge theory in 1+1 dimensions with 4 Weyl fermions. The chiralities are Left-, Left-, Right-, Right, and the charges are 3, 4, 5, 0. The action for both dynamical U(1) gauge field and 4-Weyl fermions is given as

$$S = \int dt dx \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{Q=3,4} \bar{\psi}_L^Q \gamma^\mu D^Q_\mu \psi_L^Q + \sum_{Q=5,0} \bar{\psi}_R^Q \gamma^\mu D^Q_\mu \psi_R^Q \right], \tag{1}$$

where D_{μ} is the covariant derivative with charge Q gauge interaction. One can see that the theory is free from U(1) gauge anomaly and gravitational anomaly.

4. Domain-wall realization

Corresponding to the continuum theory, we consider 4 species of DW fermions with U(1) charge (3,4,5,0) with Left-, Left-, Right-, Right- Weyl fermions on the physical edge. V_{int} are given by two 6-fermion interactions given here.

$$V_{\rm int} \sim g_1 V_1 + g_2 V_2 + h.c, \tag{2}$$

where

$$V_{1} = \psi^{3} \bar{\psi^{4}} (\partial_{x} \bar{\psi^{4}}) \psi^{5} \psi^{0} (\partial_{x} \psi^{0}), \quad V_{2} = \psi^{3} (\partial_{x} \psi^{3}) \psi^{4} \bar{\psi}^{5} (\partial_{x} \bar{\psi}^{5}) \psi^{0}$$
(3)

As we will see in a moment, it is designed to satisfy the necessary conditions. Assuming that this V_{int} is the correct choice for the moment, let us consider the symmetry H. One finds that with interaction V_{int} , $G = U(1)^4$ symmetry symmetry is broken down to $H = U(1) \times U(1)'$. U(1) and U(1)' have charges given

$$q = (-1, 2, 1, 2), \quad q' = (2, 1, 2, -1)$$
 (4)

and linear combination of the two U(1) contains U(1) charge for the dynamical gauge field, which is $q_{em} = (3, 4, 5, 0) = q + 2q'$.

In this model, the gapping interaction and symmetry H satisfy the necessary condition as shown here.

1. Bulk anomaly is zero.

The bulk anomalies for U(1), U(1)' are zero including mixed anomalies.

$$q_3^2 + q_4^2 - q_5^2 - q_0^2 = 0 (5)$$

$$q_3'^2 + q_4'^2 - q_5'^2 - q_0'^2 = 0 ag{6}$$

$$q_3q'_3 + q_4q'_4 - q_5q'_5 - q_0q'_0 = 0 \tag{7}$$

2. Instanton saturation

And combining V_1 and V_2 one can generate 't Hooft vertex which makes instanton saturation.

$$V_{1}^{\dagger}(V_{2})^{2} \sim \psi^{3}\partial_{x}(\psi^{3})\partial_{x}^{2}(\psi^{3})$$

$$\times \psi_{4}\partial_{x}(\psi_{4})\partial_{x}^{2}(\psi_{4})\partial_{x}^{3}(\psi_{4})$$

$$\times \psi_{5}^{\dagger}\partial_{x}(\psi_{5}^{\dagger})\partial_{x}^{2}(\psi_{5}^{\dagger})\partial_{x}^{3}(\psi_{5}^{\dagger})\partial_{x}^{4}(\psi_{5}^{\dagger})$$

$$(8)$$

Moreover, both semi-classical analysis after bosonization and discussion based on FQHE shows that symmetric gapping does take place with the gapping interaction

5. Discrete symmetries in 3450 model

We now study what is the discrete symmetry allowed by V_{int} in 3-4-5-0 mode l for single flavor. Under general U(1) transformation for the fermion parametrized by α_Q ,

$$\psi^Q \to \exp(i\alpha_O)\psi^Q \quad (Q = 3, 4, 5, 0) \tag{9}$$

V_{int} transforms as

$$V_1 \to \exp(i(\alpha_3 - 2\alpha_4 + \alpha_5 + 2\alpha_0))V_1 \tag{10}$$

$$V_2 \to \exp(i(2\alpha_3 + \alpha_4 - 2\alpha_5 + \alpha_0))V_2 \tag{11}$$

Therefore, invariance of V_{int} requires the condition:

$$\alpha_3 - 2\alpha_4 + \alpha_5 + 2\alpha_0 = 2\pi n,\tag{12}$$

$$2\alpha_3 + \alpha_4 - 2\alpha_5 + \alpha_0 = 2\pi m \tag{13}$$

Here n, m are arbitrary integers. Using $U(1) \times U(1)'$ transformation, you can make two of the α' 's vanish. This gives you the solutions given as

$$\alpha^{(1)} = \frac{2\pi}{3}(1,0,0,1), \quad \alpha^{(2)} = \frac{2\pi}{3}(0,1,-1,0), \quad \alpha^{(3)} = \frac{2\pi}{3}(0,1,0,-1)$$
(14)

$$\alpha^{(4)} = \frac{2\pi}{3}(1, 0, -1, 0), \quad \alpha^{(5)} = \frac{2\pi}{3}(0, 0, 1, 2), \quad \alpha^{(6)} = \frac{2\pi}{3}(2, 1, 0, 0)$$
(15)

For example, the first solution give Z_3 symmetry

e.g.
$$\alpha^{(1)}$$
: $\psi_3 \to \omega_3 \psi_3, \ \psi_0 \to \omega_3 \psi_0$ (16)

$$\psi_{4,5}$$
 invariant. $(\omega_3 = \exp(2\pi i/3))$ (17)

However, it turns out that these discrete symmetries are included in contiinuous U(1)x U(1) symmetry.

e.g.
$$\frac{2\pi}{3}(q+q') = \frac{2\pi}{3}(1,3,3,1) = \alpha^{(1)} \pmod{2\pi\mathbb{Z}}$$
 (18)

So there is no 0-form discrete symmetry for a single flavor.

5.1 Discrete symmetry for multi-flavor case

Next we study the discrete symmetry for multi-flavor case. In this case V_{int} can be given by this. Then the symmetry reduces to $U(1) \times U(1) \times S_{NF}$, where S_{NF} is the permutation group. Therefore multi-flavor system has 0-form discrete symmetry.

6. 't Hooft anomaly

Let us now compute 't Hooft anomaly. For simplicity let us consider 2-flavor case. The symmetry is $U(1) \times U(1) \times S_2$. After gauging these symmetries, we have three gauge fields :

One is dynamical U(1) gauge field denoted by *a*. Another is U(1) 't Hooft gauge field denoted by *A*. The last one is the Z_2 gauge field, since S_2 is isomorphic to Z_2 . This is described by a set of 1-form and 0-form gauge fields B1 and B0 satisfying this constraint.

$$2B^{(1)} = dB^{(0)} \tag{19}$$

And the field satisfies

$$\frac{1}{2\pi} \int_{\Sigma_1} dB^{(0)} \in \mathbb{Z} \longrightarrow \frac{1}{2\pi} \int_{\Sigma_1} B^{(1)} \in \frac{1}{2}\mathbb{Z}$$
(20)

Denoting the two flavor fermion with charge Q as $\psi_Q = \begin{pmatrix} \psi_Q^1 \\ \psi_Q^2 \\ \psi_Q^2 \end{pmatrix}$, the symmetry transformation is given as

$$U(1)_Q \times U(1)q \qquad : \ \psi_Q \to \exp(\beta Q \theta_Q + iq_Q \theta_q)\psi_Q \tag{21}$$

$$: \psi_Q \to \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix} \psi_Q \tag{22}$$

(23)

By changing the basis as $\tilde{\psi}_Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \psi$, the transformation becomes

 S_2

$$U(1)_Q \times U(1)q \quad : \tilde{\psi}_Q \to \exp(iQ\theta_Q + iq_Q\theta_q)\tilde{\psi}_Q \tag{24}$$

$$S_2 : \tilde{\psi}_Q \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{\psi}_Q$$
 (25)

(26)

so that only the lower component transforms non-trivially under Z_2 . We employ the Stora-Zumino procedure and compute the 't Hooft anomaly 4 dimensional SPT action is given as

$$S_{SPT} = \frac{2\pi}{2!(2\pi)^2} \int_{M_4} \left(\sum_{Q=3,4} tr(\mathcal{F}_Q^2) - \sum_{Q=5,0} tr(\mathcal{F}_Q^2) \right)$$
(27)

where

$$\mathcal{F}_{Q} = \begin{pmatrix} Qf + q_{Q}F & 0\\ 0 & Qf + q_{Q}F + dB^{(1)} \end{pmatrix}$$
(28)

Using the discent equation the 3-dimensional topological action is

$$S_{3dim} = \frac{2\pi}{2!(2\pi)^2} \int_{M_3} 4B^{(1)}(f+F)$$
(29)

where anomalies which do not involve discrete gauge field and only involving continuous U(1) symmetries vanish as already shown before. From this we can obtain the mixed anomaly involving the discrete symmetry as

$$\mathcal{A} = \frac{2\pi}{2!(2\pi)^2} \int_{M_2} 4\frac{2\pi}{2} (f+F) = 2\pi \frac{1}{2\pi} \int_{M_2} (f+F) \in 2\pi\mathbb{Z}$$
(30)

Thus we find that 't Hooft anomalies involving discrete symmetry cancel for mixed-anomaly, formally cancel for self-anomaly (except for mathematical subtleties). This gives further consistency check from the t' Hooft anomaly for discrete symmetry that the symmetric gapping with V_{int} can work. A more mathematically rigorous analysis is in progress.

7. Summary

Lattice 3450 model is expected to realize 3450 chiral gauge theory in the continuum by symmetric gapping. If bulk anomaly exist, it contradicts with symmetric gapping because anomaly inflow requires edge modes. We discovered that there is a discrete symmetry for lattice 3450 model with multiple flavors. We examined 't Hooft anomalies including discrete symmetry via Stora-Zumino procedure and found that they seem to vanish. This gives further evidence that symmetric gapping works for lattice 3450 model. We would like to apply this kind of analysis to other interesting models.

Acknowledgments

The work of H.W. was supported in part by JSPS KAKENHI Grant-in-Aid for JSPS fellows Grant Number 24KJ1603. The work of T.O. was supported in part by JSPS KAKENHI Grant Number 23K03387. The work of T.Y. was supported in part by JST SPRING, Grant Number JPMJSP2138.

References

- [1] J. Wang and X.-G. Wen, *Nonperturbative regularization of (1+1)-dimensional anomaly-free chiral fermions and bosons: On the equivalence of anomaly matching conditions and boundary gapping rules, Phys. Rev. B* **107** (2023) 014311 [1307.7480].
- [2] M. Luscher, Abelian chiral gauge theories on the lattice with exact gauge invariance, Nucl. Phys. B **549** (1999) 295 [hep-lat/9811032].
- [3] M. Luscher, Chiral gauge theories revisited, Subnucl. Ser. 38 (2002) 41 [hep-th/0102028].
- [4] D.B. Kaplan, A Method for simulating chiral fermions on the lattice, Phys. Lett. B 288 (1992) 342 [hep-lat/9206013].

- [5] L. Fidkowski and A. Kitaev, The effects of interactions on the topological classification of free fermion systems, Phys. Rev. B 81 (2010) 134509 [0904.2197].
- [6] J. Wang and Y.-Z. You, Symmetric Mass Generation, Symmetry 14 (2022) 1475
 [2204.14271].
- [7] Z. Wan and J. Wang, Higher anomalies, higher symmetries, and cobordisms I: classification of higher-symmetry-protected topological states and their boundary fermionic/bosonic anomalies via a generalized cobordism theory, Ann. Math. Sci. Appl. 4 (2019) 107
 [1812.11967].
- [8] J. Wang, CT or P problem and symmetric gapped fermion solution, Phys. Rev. D 106 (2022) 125007 [2207.14813].
- [9] C. Chen, J. Giedt and E. Poppitz, *On the decoupling of mirror fermions*, *JHEP* **04** (2013) 131 [1211.6947].
- [10] Y. You, Y. BenTov and C. Xu, *Interacting Topological Superconductors and possible Origin* of 16n Chiral Fermions in the Standard Model, 1402.4151.
- [11] Y. Kikukawa, Why is the mission impossible? Decoupling the mirror Ginsparg-Wilson fermions in the lattice models for two-dimensional abelian chiral gauge theories, PTEP 2019 (2019) 073B02 [1710.11101].
- [12] Y. Kikukawa, On the gauge invariant path-integral measure for the overlap Weyl fermions in 16 of SO(10), PTEP 2019 (2019) 113B03 [1710.11618].
- [13] J. Wang and X.-G. Wen, A Solution to the 1+1D Gauged Chiral Fermion Problem, Phys. Rev. D 99 (2018) 111501 [1807.05998].
- [14] M. Zeng, Z. Zhu, J. Wang and Y.-Z. You, Symmetric Mass Generation in the 1+1 Dimensional Chiral Fermion 3-4-5-0 Model, Phys. Rev. Lett. 128 (2022) 185301 [2202.12355].
- [15] D.-C. Lu, M. Zeng, J. Wang and Y.-Z. You, Fermi surface symmetric mass generation, Phys. Rev. B 107 (2023) 195133 [2210.16304].
- [16] D.-C. Lu, M. Li, Z.-Y. Zeng, W. Hou, J. Wang, F. Yang et al., Superconductivity from Doping Symmetric Mass Generation Insulators: Application to La₃Ni₂O₇ under Pressure, 2308.11195.
- [17] L.-X. Xu, Staggered Fermions with Chiral Anomaly Cancellation, 2501.10837.