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Lattice study of RG fixed point based on gradient flow in 3D $O(N)$ sigma model

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We present the lattice simulation of the renormalization group flow in the 3-dimensional $O(N)$ linear sigma model. This model possesses a nontrivial infrared fixed point, called Wilson–Fisher fixed point. Arguing that the parameter space of running coupling constants can be spanned by expectation values of operators evolved by the gradient flow, we exemplify a scaling behavior analysis based on the gradient flow in the large N approximation at criticality. Then, we work out the numerical simulation of the theory with finite N. Depicting the renormalization group flow along the gradient flow, we confirm the existence of the Wilson–Fisher fixed point non-perturbatively.

ArXiv ePrint: 2410.19425

PoS(LATTICE2024)349

The 41st International Symposium on Lattice Field Theory (LATTICE2024)
28 July - 3 August 2024
Liverpool, UK

*Speaker

1. Introduction and summary

The renormalization group (RG) [1, 2] has been instrumental in understanding scale-dependent phenomena and phase transitions at criticality. A significant step of its transformation may be provided by the coarse-graining such as spin blocking or integrating over higher momentum modes. Similarly, in perturbative RG technique in quantum field theory, the change of the renormalization scale gives rise to effectively energy-dependent coupling constants, or the running couplings being subject to an RG equation. Outside the perturbative regime, we expect that coupling constants flow along RG and some of them hopefully possess critical values at RG-invariant fixed points [3–5]. Attractive issues in every quantum system *sub specie aeternitatis*, as discussed in Ref. [6], are concentrated on clarifying it as universal scaling law near a nontrivial fixed point.

For half a century, physicists and mathematicians have studied the RG in nature, e.g., based on perturbative calculations, ϵ expansion, large N approximation, and so on. See Ref. [7] from the physical point of view. Despite remarkable achievements, there has been no available and effective formulation of the non-perturbative RG in any gauge theory. The functional definition of RG requires a momentum cutoff function and then is usually incompatible with gauge symmetry. For attempts to make a smooth momentum cutoff compatible with gauge symmetry, see Refs. [8, 9].

To put this program into practice, it would be a wonder [10] that the RG flow can be identified with the so-called gradient flow [11–14]. The gradient flow is a kind of diffusion equation evolving fields $\{\mathcal{A}_i\}$ along the fictitious time t,

$$\partial_t \mathcal{A}_i(t,x) = \partial_\mu^2 \mathcal{A}_i(t,x) + \dots, \tag{1}$$

the leading formal solution $\mathcal{A}_i(t,x) \sim t^{-D/2} \int d^D y\, e^{-(x-y)^2/4t} \mathcal{A}_i(0,y)$ appearing similar to the coarse-graining if t is identified with the renormalization scale. We note that this can be constructed in a gauge-covariant manner. There are many studies on non-perturbative formulation and nontrivial fixed points of RG based on the gradient flow [15–36] (see also Refs. [37–42]).

In particular, one finds that a dimensionless flowed operator $O_i(t,x)$ associated with the gauge coupling g_i provides a renormalization scheme such that a running coupling $g_i^2(\mu)_{\mathrm{GF}}$ is non-perturbatively defined as

$$g_i^2(\mu)_{\mathrm{GF}} \equiv \langle O_i(t) \rangle_{\sqrt{8t}=1/\mu} \sim g_i^2 + O(g_i^4) \tag{2}$$

with the renormalization scale μ identical to $1/\sqrt{8t}$. Furthermore, gearing the flow time to the finite physical box size in lattice gauge theory, we can compute numerically the running coupling through a sophisticated finite-size scaling analysis [43, 44]. Based on this method, for instance, the size of the strong coupling α_s in QCD was determined with high accuracy [45].

Suppose that there exists a one-to-one mapping of the parameter space of *all* coupling constants into the space spanned by the set $\{\langle O_i(t) \rangle\}$. Reference [18] addressed the analytical illustration of this RG flow in the following theories, in which a 2-dimensional parameter space $(\langle O_1(t) \rangle, \langle O_2(t) \rangle)$ plays an important role: the two-loop approximation of the 4-dimensional many-flavor gauge theory and the large-N limit of the 3-dimensional (3D) $O(N)$ linear sigma model. Then, we can confirm whether (a combination of) $\langle O_i(t) \rangle$ is relevant or irrelevant around an infrared fixed point in the limit $t \to \infty$ by way of illustration. (See also Ref. [17])

In this paper, we reconsider one example given in Ref. [18], the 3D $O(N)$ linear sigma model, which possesses the Wilson–Fisher fixed point [46] in the infrared limit. First, we review the flowed scalar theory and the construction of flowed operators $\{O_i\}_{i=1,2}$ under the large N approximation following Ref. [18], and then compute the critical exponent of the relevant parameter. Next, we simulate numerically this model with finite N based on the lattice regularization. As a completely non-perturbative approach, we finally depict the RG flow in our parameter space $(\langle O_1(t)\rangle, \langle O_2(t)\rangle)$ and observe the Wilson–Fisher fixed point.

In future, we hope to get a better understanding of RG in gauge theory via the gradient flow. As already mentioned, this new approach to RG is manifestly gauge invariant. We can simply apply our method to lattice simulations of gauge theory.

2. Infrared criticality of $O(N)$ sigma model on 3D continuum spacetime

2.1 Gradient flow for scalar fields and its relation with Wilsonian RG

The 3D $O(N)$ linear sigma model is defined by the following Euclidean action

$$S_{\mathrm{E}} = \int d^3x \left\{\frac{1}{2}\left[\partial_\mu \phi_i(x)\right]^2 + \frac{1}{2}m_0^2\phi_i^2(x) + \frac{1}{8N}\lambda_0\left[\phi_i(x)^2\right]^2\right\}, \tag{3}$$

where $i = 1, \dots, N$. For the scalar fields $\{\phi_i\}$, we introduce the flow equation [47]

$$\partial_t\varphi_i(t,x) = \partial_\mu^2\varphi_i(t,x), \qquad \varphi_i(t=0,x) = \phi_i(x). \tag{4}$$

It is proved in perturbation theory that, using the renormalized coupling and the wave function renormalization, any composite operator of $\varphi_i(t,x)$ is automatically a finite renormalized operator. The correlation functions of $\varphi_i(t,x)$ can be computed by substituting

$$\varphi_i(t,x) = \int d^3y \int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} e^{-tp^2} \phi_i(x). \tag{5}$$

The ringed field variable $\mathring{\varphi}_i(t,x)$ defined by

$$\mathring{\varphi}_i(t,x) \equiv \sqrt{\frac{N}{2(2\pi)^{3/2}t^{1/2}\langle\varphi_j(t,x)^2\rangle}}\varphi_i(t,x) \overset{t\to 0}{\to} \varphi_i(t,x) + O(1/N) \tag{6}$$

is free from the multiplicative renormalization factor.

Assuming the translational invariance for one-point functions, we see the scaling relation for the flowed operators constructed by φ_i under $x \mapsto e^{\xi}x$ [18]

$$\left\langle O_i(e^{2\xi}t)\right\rangle_{\{g_j\}} = \langle O_i(t)\rangle_{\{g_j(\xi)\}}, \tag{7}$$

up to a nontrivial operator mixing. The coupling constants, $\{g_i\}$, run along the RG flow parameterized by ξ as $\{g_i(\xi)\}$. In general, on the assumption that the mapping as

$$g_i(\xi) \mapsto \langle O_i(t)\rangle = \mathcal{R}_i[\{g_j(\xi)\}] \tag{8}$$

is one-to-one, the set of one-point functions $\{\langle O_i(t)\rangle\}$ can be regarded as a set of running couplings non-perturbatively defined.

2.2 Large N solution and the critical exponent

The solution of the model is well known at the $1/N$ expansion through the use of the auxiliary field method. The physical mass scale M is given by the mass-gap equation

$$M^2 + \frac{\lambda_0}{8\pi}M = m_0^2 + \frac{1}{4\pi^2}\lambda_0\Lambda, \tag{9}$$

where Λ is the momentum cutoff, and the renormalized coupling at the renormalization scale μ is

$$\frac{\lambda}{\mu} = \frac{\lambda_0}{\mu}\left(1 + \frac{\sqrt{3}}{96}\frac{\lambda_0}{\mu}\right)^{-1}. \tag{10}$$

We have the RG equations

$$\beta\left(\frac{\lambda}{\mu}\right) \equiv \left(\mu\frac{\partial}{\partial\mu}\right)_0 \frac{\lambda}{\mu} = -\frac{\lambda}{\mu} + \frac{\sqrt{3}}{96}\left(\frac{\lambda}{\mu}\right)^2, \qquad \left(\mu\frac{\partial}{\partial\mu}\right)_0 \frac{M}{\mu} = -\frac{M}{\mu}. \tag{11}$$

The fixed points, that is, zeros of the beta functions (11), are given at $(\frac{\lambda_*}{\mu}, \frac{M_*}{\mu}) = (0,0)$ and $(\frac{\lambda_*}{\mu}, \frac{M_*}{\mu}) = (\frac{96}{\sqrt{3}}, 0)$. The critical exponents correspond to the slopes of the beta function β' near the fixed points:

- At $(\frac{\lambda_*}{\mu}, \frac{M_*}{\mu}) = (0,0)$, λ is relevant as $\beta'(\frac{\lambda_*}{\mu}) = -1$ and M is also relevant as -1 (Gaussian fixed point).
- At $(\frac{\lambda_*}{\mu}, \frac{M_*}{\mu}) = (\frac{96}{\sqrt{3}}, 0)$, λ is irrelevant as $\beta'(\frac{\lambda_*}{\mu}) = +1$ but M is relevant as -1 (Wilson–Fisher fixed point).

Instead of the running coupling λ and the mass M, we define corresponding dimensionless operators as follows:

$$O_1(t,x) \equiv -\frac{4(2\pi)^3}{N}t\left[\mathring{\varphi}(t,x)^2\right]^2 + N + 2, \tag{12}$$

$$O_2(t,x) \equiv \frac{16\pi}{N}t^{3/2}\left[\partial_\mu\mathring{\varphi}_i(t,x)\right]^2 - \frac{1}{(2\pi)^{1/2}}. \tag{13}$$

From the analytical computation at the large N approximation, the asymptotic behaviors are given by

$$\langle O_1(t)\rangle \overset{t\to 0}{\to} K\lambda_0 t^{1/2} \tag{14}$$

$$\overset{t\to\infty}{\to} \begin{cases} K'\frac{\lambda_0}{M}\left(1+\frac{1}{16\pi}\frac{\lambda_0}{M}\right)^{-1}\frac{1}{M^3t^{3/2}} & \text{for } M/\lambda_0 > 0, \\ K_* & \text{for } M/\lambda_0 \to 0, \end{cases} \tag{15}$$

$$\langle O_2(t)\rangle \overset{t\to 0}{\to} Mt^{1/2} \tag{16}$$

$$\overset{t\to\infty}{\to} \left(\frac{2}{\pi}\right)^{1/2} - \frac{3}{(8\pi)^{1/2}}\frac{1}{M^2t}, \tag{17}$$

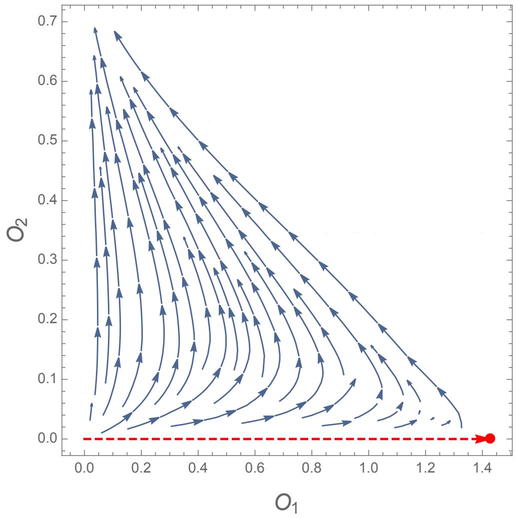

Figure 1: The RG flow of $(\langle O_1(t)\rangle, \langle O_2(t)\rangle)$ depicted in Ref. [18]

where

$$K \cong 0.289432, \qquad K' = \frac{1}{(4\pi)^{3/2}}, \qquad K_* \cong 1.4259. \tag{18}$$

Note that $\langle O_2(t)\rangle \equiv 0$ in the limit $M \to 0$. We see the RG flow of the parameter space of $\langle O_1(t)\rangle$ and $\langle O_2(t)\rangle$ arrowed along t in Fig. 1 [18]. The infrared Wilson–Fisher fixed point as depicted by the red point is indicated at $(\langle O_1(t)\rangle, \langle O_2(t)\rangle) = (K_*, 0)$. This discussion completes the observation given in Ref. [18].

From now on, let us compute the critical exponent based on the RG equation with regard to $\langle O_1(t)\rangle$ and $\langle O_2(t)\rangle$. We first note that the theory around the Gaussian fixed point at $(0,0)$ possesses the small enough dimensionless parameter λ_0/M while that around the Wilson–Fisher fixed point at $(K_*, 0)$ does the large one. Then, we observe the critical behavior at the Gaussian fixed point in the $\lambda_0 \to 0$ limit

$$\langle O_1(t)\rangle_{\mathrm{G}} \propto \lambda_0 t^{1/2}, \qquad \langle O_2(t)\rangle_{\mathrm{G}} \propto M t^{1/2}, \tag{19}$$

and therefore we find the same critical exponents as Eq. (11)

$$\left(t\frac{d}{dt}\right)\langle O_i(t)\rangle_{\mathrm{G}} = \frac{1}{2}\langle O_i(t)\rangle_{\mathrm{G}} + O(\langle O_i(t)\rangle_{\mathrm{G}}^2). \tag{20}$$

Here note that the mass dimension of t is -2 while λ is 1. On the other hand, at the Wilson–Fisher fixed point, we see the irrelevant behavior in the $M \to 0$ limit

$$\langle O_1(t)\rangle_{\mathrm{WF}} - K_* \propto \lambda_0^{-1/2} t^{-1/2}, \tag{21}$$

and the relevant one in the $\lambda_0 \to \infty$ limit

$$\langle O_1(t)\rangle_{\mathrm{WF}} - K_* \propto M t^{1/2}, \qquad \langle O_2(t)\rangle_{\mathrm{WF}} \propto M t^{1/2}. \tag{22}$$

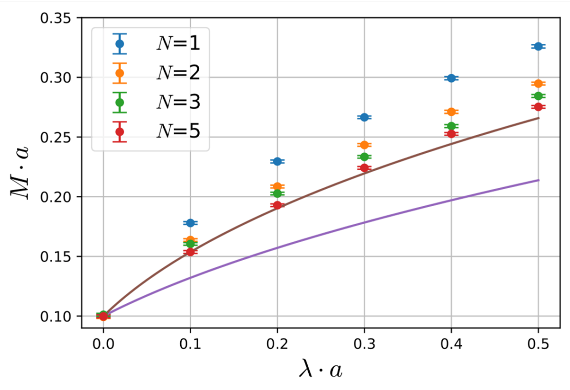

Figure 2: Effective mass, M, as a function of the coupling λ. Dependence to N is also shown. Two curves are given by the gap equation in the large N approximation in Eq. (9) with $\Lambda = \pi/2a$ (above orange) and $\Lambda = \pi/a$ (below purple). Despite the different regularizations, the lattice approach seems to be consistent with the large N solution.

Redefining $\langle O_2(t)\rangle$ as an appropriate linear combination of $(\langle O_1(t)\rangle - K_*)$ and $\langle O_2(t)\rangle$, that is, diagonalizing Eqs. (21) and (22), one finds the RG equations as expected

$$\left(t\frac{d}{dt}\right)\langle O_1(t)\rangle_{\mathrm{WF}} = -\frac{1}{2}\langle O_1(t)\rangle_{\mathrm{WF}} + O(\langle O_1(t)\rangle_{\mathrm{WF}}^2), \tag{23}$$

$$\left(t\frac{d}{dt}\right)\langle O_2(t)\rangle_{\mathrm{WF}} = \frac{1}{2}\langle O_2(t)\rangle_{\mathrm{WF}} + O(\langle O_2(t)\rangle_{\mathrm{WF}}^2). \tag{24}$$

3. Lattice simulation of RG flow in 3D $O(N)$ sigma model

In this section, we attempt numerical simulations for the finite-N $O(N)$ sigmal model. By using the simple symmetric difference instead of the derivative, the discretized lattice action includes the tunable parameters, $m_0 a$ and $\lambda_0 a$. We utilized the overrelaxed heatbath method for configuration generation. From the computation of the two-point function of ϕ, as usual, we see the coupling dependence of the effective mass in Fig. 2 for $N = 1, 2, 3, 5$, and also the result of the large N gap equation.

Now, we again consider the flowed operators, $O_1(t)$ and $O_2(t)$, via the 4th order Runge–Kutta method for gradient flow. To look forward to the critical behavior, tuning $m_0 a$ so as to make M smaller, we show the t-$\langle O_i(t)\rangle$ plots with fixed $\lambda_0 a = 5.0$ and $N = 1$ in Fig. 3 (the lattice size is taken to 128^3). This figure appears to imply the existence of the plateau, that is, t-independent critical couplings for large enough t. To see this behavior explicitly, Fig. 4 gives the t flow of $\langle O_1\rangle$ (horizontal axis) and $\langle O_2\rangle$ (vertical axis) simultaneously. This is just the figure of the RG flow for the effective couplings, $\langle O_1\rangle$ and $\langle O_2\rangle$. The left below side means the UV region (small t), while the left above or the right below side is the IR region (large t). In between, the gray curve tends to stop for large t, which indicates the existence of the Wilson–Fisher fixed point around there. The RG trajectories flowing to the left are in the symmetric phase, while those flowing to the right are in the broken phase. This is our main result.

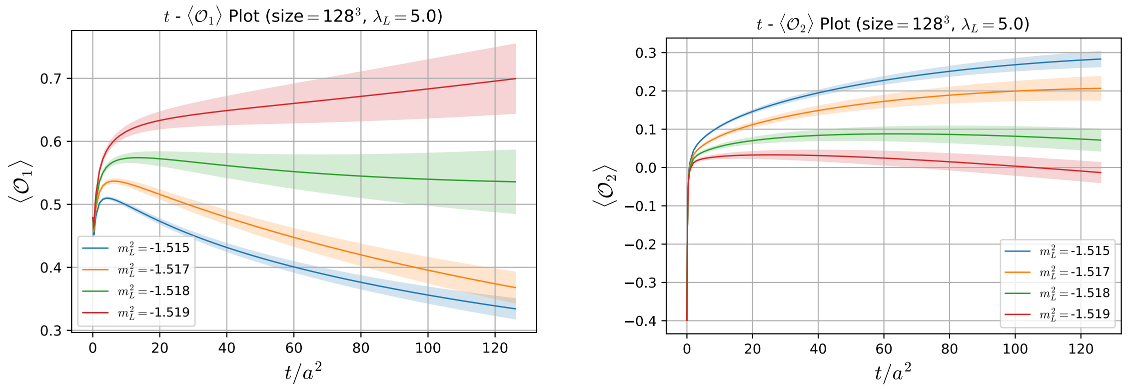

Figure 3: $\langle O_1(t)\rangle$ (left) or $\langle O_2(t)\rangle$ (right) plots as functions of gradient flow time t. For a fixed $\lambda_L = \lambda_0 a$, various values of $m_L = m_0 a$ are shown. We see the plateau between $m_L^2 = -1.519$ and -1.518.

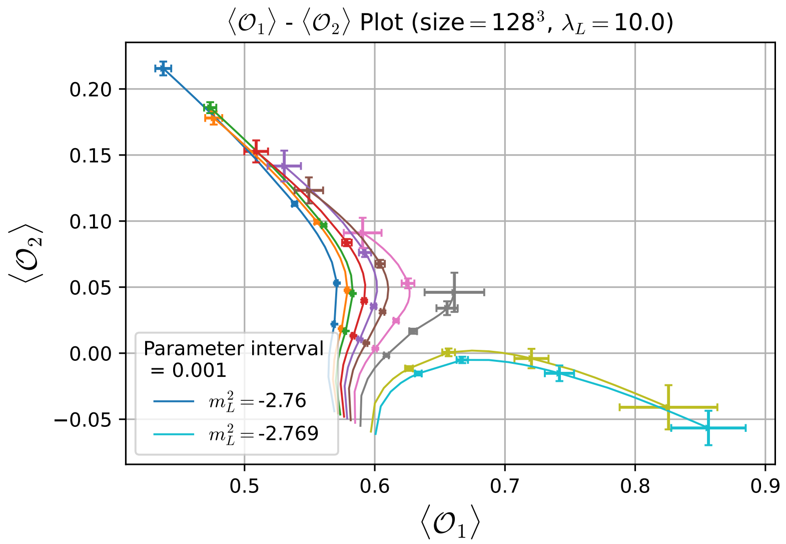

Figure 4: The RG flow of $(\langle O_1(t)\rangle, \langle O_2(t)\rangle)$ along gradient flow

Here, from Fig. 4, one may ask the following questions: (i) why the RG trajectories are likely to be crossing each other at large t (near IR); (ii) where is the Gaussian fixed point because the point of each trajectory at $t = 0$ (UV) is different.

The first issue happens when the lattice size is too small; it is a finite-size effect. To see this, we compare the $\langle O_1(t)\rangle$-$\langle O_2(t)\rangle$ plot for different lattice sizes. In Fig. 5, for $\lambda_0 a = 10.0$, the left panel is devoted to the larger lattice size 128^3, while the right panel is to the smaller lattice size 64^3. Each trajectory then suffers from more severe intersections if the lattice size is smaller. In fact, for such a large flow time, $\langle O_i(t)\rangle$ is oversmeared such that the diffusion length is comparable to the lattice size.

For the second point, that is, the UV behavior, we can simply say that the system is not taken to the continuum limit, and then is just a finite lattice model. Actually, for different values of λ, Fig. 6 shows that small λ (and small M) makes $\langle O_1\rangle$ small. For the continuum limit, we expect

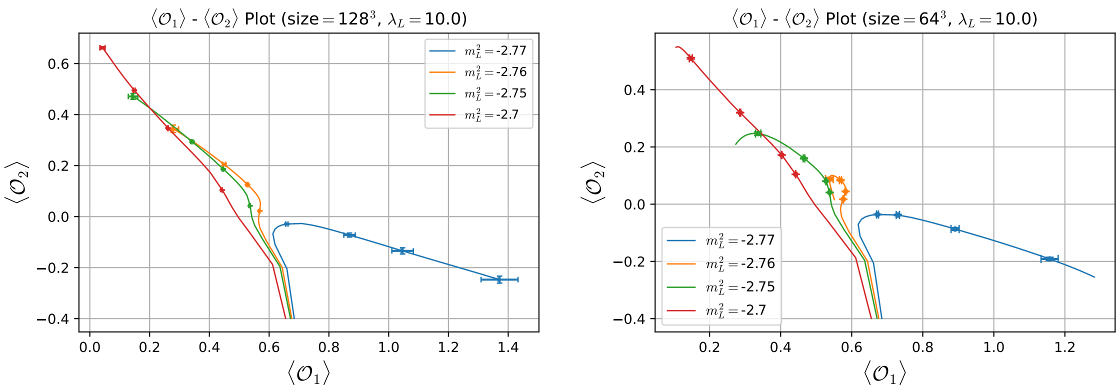

Figure 5: Finite size effect

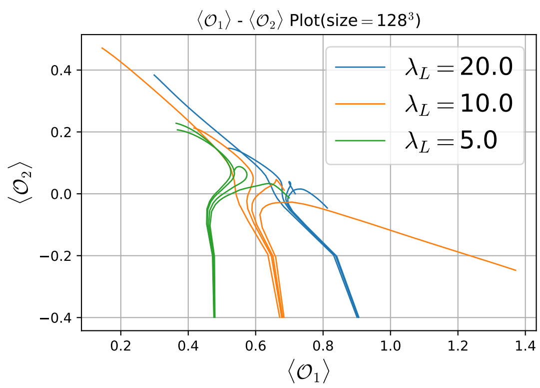

Figure 6: Lattice model details at UV

the Gaussian fixed point with λ and M near zero. Also, the critical behavior for each lattice model should be the same near the Wilson–Fisher fixed point. We can consider the following two situations:

- For small λ, the system is close to the Gaussian and far from the Wilson–Fisher fixed point. We need to have a sufficiently larger lattice size and try to reduce numerical errors for solving the gradient flow.
- For large λ, the system is far from the Gaussian but hopefully close to the Wilson–Fisher fixed point. More computational costs when generating configurations is predictable for strongly coupled theories.

Acknowledgements

This work was partially supported by Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research Grant Numbers JP19H05598, JP22K03619, JP23H04507, JP24K07049 (M.K.), JP21J30003, JP22KJ2096 (O.M.), and JP23K03418 (H.S.). O.M. acknowledges the RIKEN Special Postdoctoral Researcher Program.

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