PROCEEDINGS OF SCIENCE

Constraints on the Dirac spectrum from chiral symmetry restoration and the fate of $\mathrm{U}(1)_A$ symmetry

Matteo Giordano*

Institute of Physics and Astronomy, ELTE Eötvös Loránd University, Pázmány Péter sétány 1/A, H-1117, Budapest, Hungary

E-mail: giordano@bodri.elte.hu

I discuss chiral symmetry restoration in the chiral limit $m \to 0$ of QCD with two light quark flavours of mass m, focussing on its consequences for scalar and pseudoscalar susceptibilities, and on the resulting constraints on the Dirac spectrum. I show that $\mathrm{U}(1)_A$ symmetry remains broken in the $\mathrm{SU}(2)_A$ symmetric phase if the spectral density $\rho(\lambda; m)$ develops a singular near-zero peak, tending to $O(m^4)/\lambda$ in the chiral limit. Moreover, $\mathrm{SU}(2)_A$ restoration requires that the number of modes in the peak be proportional to the topological susceptibility, indicating that such a peak must be of topological origin.

The 41st International Symposium on Lattice Field Theory (LATTICE2024)
28 July - 3 August 2024
Liverpool, UK

*Speaker

PoS(LATTICE2024)188

1. Introduction

In the two-flavour chiral limit where the up and down quark masses are sent to zero, QCD has a $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry, spontaneously broken to its diagonal component $\mathrm{SU}(2)_V$ at low temperatures, as well as an anomalous $\mathrm{U}(1)_A$ axial symmetry. While it is known that chiral symmetry gets restored at higher temperatures [1], two important and related questions remain open: what is the nature of the transition, and what happens to $\mathrm{U}(1)_A$ in the symmetric phase. In fact, whether $\mathrm{U}(1)_A$ remains broken or gets effectively restored can affect the order/class of the transition [2–6], and is still a matter of debate.

A strategy to gain insight into this issue is to study how the Dirac spectrum responds to chiral symmetry restoration [7–10], an approach recently revived in Ref. [11] on which this contribution is based. Eigenvalues and eigenvectors of the Dirac operator fully encode the dynamics of quarks interacting with the gluon fields, and so their behaviour should reflect the status of the various symmetries. One should then be able to derive constraints on the spectrum following from the restoration of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, which in turn could shed light on the fate of $\mathrm{U}(1)_A$ in the symmetric phase.

The best known relation between chiral symmetry and the Dirac spectrum is certainly the Banks-Casher relation [12] between the chiral condensate, Σ, and the spectral density, $\rho(\lambda; m)$,

$$\rho(\lambda; m) = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \left\langle \sum_{n,\lambda_n\neq 0} \delta(\lambda - \lambda_n) \right\rangle , \tag{1}$$

where m is the fermion mass, λ_n the Dirac eigenvalues in a gauge-field background, and T and V are the temperature and the spatial volume of the system, respectively. In the chiral limit the Banks-Casher relation reads $|\Sigma| = \pi\rho(0^+; 0)$, implying that a finite density of near-zero modes in the chiral limit produces a symmetry-breaking chiral condensate. At low temperatures one then expects a nonzero density of near-zero modes at zero and small quark mass – an expectation supported by ample numerical evidence (e.g., the recent Ref. [13]).[1] One would similarly expect that the density of near-zero modes vanishes at zero and small quark mass in the symmetric phase at high temperature, but numerical results indicate a completely different behaviour – a singular spectral peak near zero at finite mass [17–32]. The fate of this peak in the chiral limit is still unclear: it certainly must vanish in order for chiral symmetry to be restored, but how fast it has to do so, and what this could imply for $\mathrm{U}(1)_A$, are questions not discussed in the literature before Ref. [11].

Even if one could ignore the peak entirely, opposite conclusions about $\mathrm{U}(1)_A$ can be reached depending on one's assumptions. If one assumes that (mass-independent) observables are analytic in m^2 – a reasonable assumption to make in the symmetric phase, then under plausible assumptions on the spectral density one concludes that $\mathrm{U}(1)_A$ must be effectively restored in the symmetric phase [9, 10]. If one assumes instead commutativity of the thermodynamic and chiral limits – an equally reasonable assumption in the symmetric phase, one concludes that most likely $\mathrm{U}(1)_A$ remains effectively broken [33, 34]. Finally, assuming both m^2-analyticity and commutativity of limits, one shows that $\mathrm{U}(1)_A$ can only be broken by a Dirac delta at zero appearing in the spectral

[1] For $m \neq 0$, partially quenched chiral perturbation theory actually predicts a logarithmic divergence in $\rho(\lambda; m)$ at $\lambda = 0$, proportional to $|m| \ln |\lambda|$ [14–16]. However, this divergence is visible only for $|\lambda| \lesssim e^{-1/|m|}$, and disappears in the chiral limit.

density already at finite mass, $\rho(\lambda; m) \sim m^2 \Delta\, \delta(\lambda)$, which seems quite unlikely [35]. It is then important to understand which (if any) of these assumptions follows directly from chiral symmetry restoration.

2. Chiral symmetry restoration in the scalar and pseudoscalar sector

In a local quantum field theory, restoration of a symmetry in the limit in which one removes all sources of explicit breaking requires that correlators of local operators that are related by a symmetry transformation become equal. The same is expected to apply to susceptibilities, i.e., to the (suitably normalised) spacetime integrals of connected correlation functions, since in the symmetric phase one generally does not find massless excitations and the correlation length of the system is finite. This applies in a general setting, and I will refer to it as "level 1" restoration. In the specific case of chiral symmetry in gauge theories, since gauge fields are unaffected by chiral transformations, it is reasonable to assume that the conditions above extend also to correlators involving generic functionals of the gauge fields only, including nonlocal ones (such as the spectral density), on top of operators built out of fermion fields. I will refer to this additional assumption as "level 2" restoration.

I now use these basic symmetry-restoration conditions to characterise the restoration of chiral symmetry in the scalar and pseudoscalar sector of a gauge theory with two light (eventually massless) fermions of equal mass, and a rather arbitrary content in gauge fields and additional massive fermions. To have both a mathematically sound framework and a good notion of chiral symmetry, I put the theory on the lattice using Ginsparg-Wilson fermions [36]. A Ginsparg-Wilson Dirac operator obeys the relation $\{D, \gamma_5\} = 2D\gamma_5 RD$, with R local. Ginsparg-Wilson fermions have an exact $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry [37], that allows one to define suitable fermion bilinears with simple transformation properties under chiral transformations. I restrict here to scalar and pseudoscalar bilinears, since the corresponding susceptibilities can be expressed in terms of Dirac eigenvalues only, thus providing the sought-after constraints on the spectrum. The relevant bilinears are

$$\begin{aligned} S &\equiv \bar{\psi}(1 - DR)\psi\,, & P &\equiv \bar{\psi}(1 - DR)\gamma_5\psi\,, \\ \vec{P} &\equiv \bar{\psi}(1 - DR)\vec{\sigma}\gamma_5\psi\,, & \vec{S} &\equiv \bar{\psi}(1 - DR)\vec{\sigma}\psi\,, \end{aligned} \tag{2}$$

where $\psi, \bar{\psi}$ denote the light fermion fields, and carry spacetime (including Dirac), colour, and flavour indices, suppressed here and in the following for notational simplicity. Scalar and pseudoscalar bilinears form irreducible multiplets $O_V \equiv (S, i\vec{P})$ and $O_W \equiv (iP, -\vec{S})$ under chiral transformations of the light fermion fields, transforming as

$$O_{V,W} \to \mathcal{R}^T O_{V,W}\,, \qquad \mathcal{R} \in \mathrm{SO}(4)\,, \tag{3}$$

i.e., as four-dimensional vectors under a rotation.

Chiral symmetry restoration for scalar and pseudoscalar susceptibilities is conveniently expressed in terms of the corresponding generating function, $\mathcal{W} \equiv \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \ln \mathcal{Z}$, where the partition function $\mathcal{Z}$ is defined in the usual way by including suitable source terms in the action,

$$\begin{aligned} \mathcal{Z}(V, W; m) &\equiv \int DUD\psi D\bar{\psi}\, e^{-S(U) - \bar{\psi}D_m(U)\psi - K(\psi,\bar{\psi},U;V,W)}\,, \\ K(\psi, \bar{\psi}, U; V, W) &\equiv j_S S + i\vec{j}_P \cdot \vec{P} + i j_P P - \vec{j}_S \cdot \vec{S} = V \cdot O_V + W \cdot O_W\,, \end{aligned} \tag{4}$$

where $D_m \equiv D + m(1 - DR)$ with m the light fermion mass, the action S includes the discretised Yang-Mills action for the gauge fields U and the contribution of massive fermion fields after these have been integrated out, and finally $V \equiv (j_S, \vec{j}_P)$ and $W \equiv (j_P, \vec{j}_S)$. Since the generating function with rotated sources, $\mathcal{W}(\mathcal{R}V, \mathcal{R}W; m)$, generates the chirally transformed susceptibilities [see Eq. (3)], the request of symmetry restoration reduces to asking that

$$\lim_{m\to 0} [\mathcal{W}(\mathcal{R}V, \mathcal{R}W; m) - \mathcal{W}(V, W; m)] = 0\,, \tag{5}$$

i.e., that the generating function $\mathcal{W}$ becomes an SO(4)-invariant function of the sources in the chiral limit.

The symmetries of the theory constrain the functional form of $\mathcal{W}$, that can depend only on certain combinations of the sources. Since $\mathcal{Z}$ depends only on $j_S + m$, and since the exactly massless theory in a finite volume is chirally symmetric, one has that $\mathcal{W}$ depends only on SO(4) invariants built out of W and $\tilde{V} \equiv (j_S + m, \vec{j}_P)$,

$$\begin{aligned}\mathcal{W}(V, W; m) &= \hat{\mathcal{W}}(\tilde{V}^2, W^2, 2\tilde{V}\cdot W) = \hat{\mathcal{W}}(m^2 + u, w, \tilde{u}) \\ &= \sum_{n_u, n_w, n_{\tilde{u}} \geq 0} \frac{u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}}}{n_u! n_w! n_{\tilde{u}}!} \mathcal{A}_{n_u, n_w, n_{\tilde{u}}}(m^2)\,,\end{aligned} \tag{6}$$

where

$$\mathcal{A}_{n_u, n_w, n_{\tilde{u}}}(m^2) \equiv \partial_u^{n_u} \partial_w^{n_w} \partial_{\tilde{u}}^{n_{\tilde{u}}} \hat{\mathcal{W}}(m^2 + u, w, \tilde{u})|_{u=w=\tilde{u}=0}\,, \tag{7}$$

with

$$u \equiv 2m j_S + V^2\,, \qquad w \equiv W^2\,, \qquad \tilde{u} \equiv 2(m j_P + V\cdot W)\,, \tag{8}$$

and $\partial_x \equiv \partial/\partial x$. For the considerations to follow, it is important to notice that $\mathcal{W}$ can be treated as a formal power series in the sources, and so in practice as a polynomial of arbitrarily high order in V and W, allowing one to freely exchange derivatives with respect to the sources with other operations, including the limit $m \to 0$.

The coefficients $\mathcal{A}_{n_u, n_w, n_{\tilde{u}}}$ allow for a full characterisation of chiral symmetry restoration in the scalar and pseudoscalar sector. In fact, a necessary and sufficient condition for symmetry restoration in this sector [in the sense of Eq. (5)] is that these coefficients remain finite (meaning they do not diverge) in the chiral limit [11]. Sufficiency is rather obvious: if $\mathcal{A}_{n_u, n_w, n_{\tilde{u}}}$ are finite in the chiral limit, one can simply drop the m-dependent terms in u and $\tilde{u}$ as $m \to 0$, and SO(4) invariance of $\mathcal{W}$ becomes manifest. To prove necessity,[2] one first notices that the symmetry restoration condition Eq. (5) combined with the functional form Eq. (6) implies for arbitrary $\vec{\alpha} = \alpha\hat{\alpha}$, $\hat{\alpha}^2 = 1$, that

$$\begin{aligned}0 &= \lim_{m\to 0} \partial_\alpha \hat{\mathcal{W}}(m^2 + 2mx(\alpha) + V^2, W^2, 2(my(\alpha) + V\cdot W)) \\ &= \lim_{m\to 0} 2m \left(\dot{x}(\alpha)\partial_{V^2} + \dot{y}(\alpha)\partial_{2V\cdot W}\right) \hat{\mathcal{W}}(m^2 + 2mx(\alpha) + V^2, W^2, 2(my(\alpha) + V\cdot W))\,,\end{aligned} \tag{9}$$

where $x(\alpha) \equiv \cos(\alpha) j_S + \sin(\alpha)\hat{\alpha}\cdot\vec{j}_P$ and $y(\alpha) \equiv \cos(\alpha) j_P + \sin(\alpha)\hat{\alpha}\cdot\vec{j}_S$. Setting $\alpha = 0$, one finds

$$\lim_{m\to 0} m\,(\hat{\alpha}\cdot\vec{j}_P \partial_u + \hat{\alpha}\cdot\vec{j}_S \partial_{\tilde{u}})\,\hat{\mathcal{W}}(m^2 + u(m), w, \tilde{u}(m)) = 0\,, \tag{10}$$

[2] The following is an alternative proof to that given in Ref. [11].

where the dependence of u and $\tilde{u}$ on m is made explicit. Since $\hat{\mathcal{W}}$ does not depend explicitly on $\vec{j}_{P,S}$, applying one or the other of the differential operators $\hat{\alpha}\cdot\left(\partial_{\vec{j}_P} - 2(\vec{j}_P\partial_u + \vec{j}_S\partial_{\tilde{u}})\right)$ and $\hat{\alpha}\cdot\left(\partial_{\vec{j}_S} - 2(\vec{j}_S\partial_w + \vec{j}_P\partial_{\tilde{u}})\right)$ one finds

$$\lim_{m\to 0} m\partial_u \hat{\mathcal{W}}(m^2+u(m),w,\tilde{u}(m)) = \lim_{m\to 0} m\partial_{\tilde{u}} \hat{\mathcal{W}}(m^2+u(m),w,\tilde{u}(m)) = 0\,. \tag{11}$$

The (total) mass derivative of the generating function at fixed values of the sources becomes then

$$\begin{aligned}\lim_{m\to 0}\partial_m \hat{\mathcal{W}}(m^2+u(m),w,\tilde{u}(m)) &= 2\lim_{m\to 0}\left((j_S+m)\partial_u + j_P\partial_{\tilde{u}}\right)\hat{\mathcal{W}}(m^2+u(m),w,\tilde{u}(m))\\ &= 2\lim_{m\to 0}\left(j_S\partial_u + j_P\partial_{\tilde{u}}\right)\hat{\mathcal{W}}(m^2+u(m),w,\tilde{u}(m))\,,\end{aligned}\tag{12}$$

and so at $j_S = j_P = 0$

$$\begin{aligned}0 &= \lim_{m\to 0}\partial_m \hat{\mathcal{W}}(m^2+\vec{j}_P^{\,2},\vec{j}_S^{\,2},2\vec{j}_P\cdot\vec{j}_S)\\ &= \sum_{n_u,n_w,n_{\tilde{u}}\geq 0}\frac{(\vec{j}_P^{\,2})^{n_u}(\vec{j}_S^{\,2})^{n_w}(2\vec{j}_P\cdot\vec{j}_S)^{n_{\tilde{u}}}}{n_u!n_w!n_{\tilde{u}}!}\lim_{m\to 0}\partial_m\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2)\,.\end{aligned}\tag{13}$$

This readily implies that $\lim_{m\to 0}\partial_m\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2) = 0$ for each coefficient separately, and so that

$$\lim_{m\to 0}\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2) = \mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m_0^2) + \lim_{m\to 0}\int_{m_0}^{m} d\bar{m}\,\partial_{\bar{m}}\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(\bar{m}^2) \tag{14}$$

is finite, $\forall n_{u,w,\tilde{u}}$.[3] A corollary is that these coefficients must be infinitely differentiable in m^2 ("m^2-differentiable", for short): in fact, since from Eq. (7)

$$\begin{aligned}\partial_{m^2}^k\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2) &= \partial_{m^2}^k\partial_u^{n_u}\partial_w^{n_w}\partial_{\tilde{u}}^{n_{\tilde{u}}}\hat{\mathcal{W}}(m^2+u,w,\tilde{u})|_{u=w=\tilde{u}=0}\\ &= \partial_u^{n_u+k}\partial_w^{n_w}\partial_{\tilde{u}}^{n_{\tilde{u}}}\hat{\mathcal{W}}(m^2+u,w,\tilde{u})|_{u=w=\tilde{u}=0} = \mathcal{A}_{n_u+k,n_w,n_{\tilde{u}}}(m^2)\,,\end{aligned}\tag{15}$$

finiteness of all $\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}$ as $m\to 0$ means also finiteness of all their m^2-derivatives in that limit.

So far only the assumption of "level 1" restoration was used. Assuming "level 2" restoration, the argument above extends straightforwardly to susceptibilities involving also generic gauge-field functionals, and so it follows in particular that the spectral density must be infinitely differentiable in m^2.[4] The m^2-differentiability assumed in Refs. [8–10] is then not just an assumption,[5] but a necessary consequence of symmetry restoration.

3. Constraints on the Dirac spectrum

At this point one needs to work out the coefficients $\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}$ explicitly and impose their finiteness in the chiral limit. Here I restrict to the case $R=\frac{1}{2}$ and $\gamma_5 D\gamma_5 = D^\dagger$ (which includes domain-wall [38] and overlap fermions [39]), for which $1-D$ is a unitary operator. Since $\mathcal{W}$

[3] Here it is implicitly assumed that $\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}$ is finite and differentiable in m for $m\neq 0$.

[4] This result can be obtained using only local operators, if one assumes that chiral symmetry is restored also in an extended theory with partially quenched fermions fields.

[5] Refs. [8–10] assume analyticity in m^2, but while the existence of the m^2-derivatives at zero is proved, the radius of convergence of the expansion may be zero, and terms vanishing with all their derivatives as $m\to 0$ are also allowed.

depends only on $\tilde{u}^2$ thanks to CP symmetry, the lowest-order coefficients are $\mathcal{A}_{1,0,0}$, $\mathcal{A}_{0,1,0}$, and $\mathcal{A}_{0,0,2}$, that read

$$\begin{aligned}
\mathcal{A}_{1,0,0} &= \frac{\chi_\pi}{2} = -\frac{1}{2}\lim_{\mathrm{V}\to\infty}\frac{\langle P_1^2\rangle}{\mathrm{V/T}} &&= \frac{n_0}{m^2} + 2\int_0^2 d\lambda\, \frac{h(\lambda)\rho(\lambda;m)}{\lambda^2+m^2h(\lambda)}\,,\\
\mathcal{A}_{0,1,0} &= \frac{\chi_\delta}{2} = \frac{1}{2}\lim_{\mathrm{V}\to\infty}\frac{\langle {S_1}^2\rangle}{\mathrm{V/T}} &&= -\frac{n_0}{m^2} + 2\int_0^2 d\lambda\, \frac{h(\lambda)[\lambda^2-m^2h(\lambda)]\rho(\lambda;m)}{\left[\lambda^2+m^2h(\lambda)\right]^2}\,,\\
\mathcal{A}_{0,0,2} &= -\frac{1}{4}\lim_{\mathrm{V}\to\infty}\frac{\langle P_1S_1P_2S_2\rangle}{\mathrm{V/T}} &&= \frac{n_0-\chi_t}{m^4} + 2\int_0^2 d\lambda\, \frac{h(\lambda)^2\rho(\lambda;m)}{\left[\lambda^2+m^2h(\lambda)\right]^2}\,,
\end{aligned} \tag{16}$$

where

$$h(\lambda) \equiv 1-\frac{\lambda^2}{4}\,, \qquad n_0 \equiv \lim_{\mathrm{V}\to\infty}\frac{\langle N_+ + N_-\rangle}{\mathrm{V/T}}\,, \qquad \chi_t \equiv \lim_{\mathrm{V}\to\infty}\frac{\langle (N_+ - N_-)^2\rangle}{\mathrm{V/T}}\,, \tag{17}$$

with $N_\pm$ the number of exact zero modes of D of chirality ± 1. Here ρ is defined by Eq. (1) with $\lambda_n \equiv 2\sin\frac{\varphi_n}{2}$, with $\mu_n = 1 - e^{-i\varphi_n}$ the eigenvalues of D with positive imaginary part. Since $n_0 = 0$ and $|\chi_\delta| \le \chi_\pi$, to lowest order the constraints boil down to asking for finiteness of the pion susceptibility,

$$\lim_{m\to 0}\frac{\chi_\pi}{4} = \lim_{m\to 0}\int_0^2 d\lambda\, \frac{h(\lambda)\rho(\lambda;m)}{\lambda^2+m^2h(\lambda)} < \infty\,, \tag{18}$$

and that $\frac{\chi_\pi-\chi_\delta}{4} - \frac{\chi_t}{m^2} = O(m^2)$, so that the $\mathrm{U}(1)_A$ order parameter Δ equals the chiral limit of the topological susceptibility divided by m^2,

$$\Delta \equiv \lim_{m\to 0}\frac{\chi_\pi-\chi_\delta}{4} = \lim_{m\to 0}\int_0^2 d\lambda\, \frac{2m^2h(\lambda)^2\rho(\lambda;m)}{\left[\lambda^2+m^2h(\lambda)\right]^2} = \lim_{m\to 0}\frac{\chi_t}{m^2} < \infty\,. \tag{19}$$

These constraints are not new, but what this approach shows is that they are *all* the direct constraints that one has to impose on ρ and χ_t coming from the scalar and pseudoscalar susceptibilities: in fact, constraints coming from imposing finiteness of higher-order coefficients $\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}$ in the chiral limit involve higher-point eigenvalue correlators.

At this stage $\mathrm{U}(1)_A$ is compatible with chiral symmetry restoration. However, for the constraints above to be practically useful one needs to make further assumptions on ρ. If ρ admits a power expansion in λ near zero, $\rho(\lambda;m) = \sum_n \rho_n(m^2)\lambda^n$, possibly supplemented by a non-analytic power law with positive exponent, $\rho(\lambda;m) \simeq C(m)\lambda^\alpha$, $\alpha > 0$, then (using also the m^2-differentiability of ρ required by "level 2" restoration) these constraints imply that $\mathrm{U}(1)_A$ must be effectively restored in the symmetric phase [11], confirming previous studies [9, 10]. One wonders then if there is any simple way at all in which $\mathrm{U}(1)_A$ can remain broken in the chirally symmetric phase. Barring the implausible Dirac delta mentioned at the end of Section 1, this necessarily requires dropping the assumption that the thermodynamic and chiral limits commute.

4. $\mathrm{U}(1)_A$ breaking by a singular spectral peak

The only *simple* possibility left is that of a singular peak in the spectral density. As pointed out in Section 1, restrictions from chiral symmetry restoration on the behaviour of such a peak in the chiral limit, and their consequences for $\mathrm{U}(1)_A$, had not been considered before Ref. [11]. Assume then that the spectral density is dominated near zero by a power-law term, $\rho(\lambda;m) \simeq \rho_{\text{peak}}(\lambda;m) = C(m)\lambda^{\alpha(m)}$, with a possibly mass-dependent exponent $\alpha(m)$ allowed to take also negative values. Without loss of generality one can restrict to $|\alpha(m)| < 1$ for $m \neq 0$ and $\alpha(0) \neq 1$, since $\alpha(m) \leq -1$ at nonzero m is simply unacceptable, and since $\alpha(0) = 1$ is uninteresting as it cannot lead to $\mathrm{U}(1)_A$ breaking.[6] Imposing "level 1" symmetry restoration for the lowest-order scalar and pseudoscalar susceptibilities one finds the requirements

$$C(m) = \frac{\cos\left(\alpha(m)\frac{\pi}{2}\right)}{(1-\alpha(0))\frac{\pi}{2}}|m|^{1-\alpha(0)}\hat{C}(m)\,, \qquad |\hat{C}(0)| < \infty\,. \tag{20}$$

For the $\mathrm{U}(1)_A$ order parameter one finds $\Delta = \hat{C}(0)$, and so $\mathrm{U}(1)_A$ is broken if $\hat{C}(0) \neq 0$. This tells us how fast the peak has at least to vanish for symmetry restoration: any slower than in Eq. (20) and chiral symmetry remains broken; any faster [i.e., $\hat{C}(0) = 0$] and also $\mathrm{U}(1)_A$ gets restored.

At this stage there is no restriction on $\alpha(0)$, with any value in the range $[-1, 1)$ allowed, and so no restriction on how one can break $\mathrm{U}(1)_A$. However, if we impose "level 2" restoration (or, more directly, that the spectral density be m^2-differentiable), then the possibilities are drastically reduced: the only acceptable possibility is that $\alpha(0) = -1$, with $\alpha(m)$ and $\hat{C}(m)$ m^2-differentiable [11]. The only simple way to break $\mathrm{U}(1)_A$ in the restored phase is then for the spectral density to develop a singular near-zero peak, $\rho \simeq \rho_{\text{peak}}$ for $\lambda \simeq 0$, behaving as follows in the chiral limit,

$$\rho_{\text{peak}}(\lambda;m) \underset{m\to 0}{\to} \left[\Delta + O(m^2)\right]\frac{m^2}{2}\frac{\gamma(m)}{\lambda^{1-\gamma(m)}}\,, \qquad \gamma(m) > 0\,,\ \Delta \neq 0\,, \tag{21}$$

with $\gamma(m) = O(m^2)$, so that $\rho_{\text{peak}} \sim O(m^4)/\lambda$. Chiral symmetry restoration requires that the resulting density of peak modes, n_{peak}, equals the topological susceptibility to leading order in m,

$$\lim_{m\to 0}\frac{n_{\text{peak}}}{m^2} = \lim_{m\to 0}\frac{2}{m^2}\int_0^2 d\lambda\,\rho_{\text{peak}}(\lambda;m) = \Delta = \lim_{m\to 0}\frac{\chi_t}{m^2}\,, \tag{22}$$

indicating a strong connection with the topological features of gauge field configurations.

The behaviour Eq. (21) may seem at first an edge case like the Dirac delta mentioned above, to be likewise dismissed. There are, however, good reasons not to do so. First of all, there is some numerical evidence for the peak, and not for the delta. Secondly, and perhaps more importantly, there is a concrete model that produces a spectral density with these features, and so a concrete physical mechanism that can lead to Eq. (21). This model is the weakly interacting, dilute instanton gas model of Ref. [40]. In the chiral limit of this model the topological objects organise into instanton-anti-instanton molecules, plus a free-gas component of total density $n_{\text{inst}} = \chi_t \propto m^2$, entirely responsible for the global topological properties. In this model one indeed finds a singular spectral peak originating in the zero modes associated with isolated instantons and anti-instantons, with a mass-dependent power. Furthermore, this power is very likely tending to -1 in the chiral limit. In fact, this is the behaviour observed in the limit of large disorder in an analogous condensed-matter model [41], with the same symmetry features as the instanton model of Ref. [40] and producing a similar near-zero singular peak in the spectrum; in the instanton model this limit corresponds

[6] In this case chiral symmetry restoration requires that $C(m) = \frac{\hat{C}(m)}{\ln(2/|m|)}$ with $|\hat{C}(0)| < \infty$, which leads to $\Delta = 0$.

(counterintuitively, at first sight) to the free-gas density χ_t going to zero – i.e., to the chiral limit. Finally, in the instanton model the density of peak modes equals the density n_{inst} of free topological objects, and so Eq. (22) is satisfied.

The premise of an instanton gas-like behaviour for the topology of gauge field configurations may seem rather *ad hoc*, but it is actually a necessary condition for chiral symmetry restoration that the distribution of the topological charge in the chiral limit be identical to that of an ideal instanton gas of vanishingly small density χ_t [42]. This result, obtained assuming m^2-analyticity of the free energy in the presence of a θ term, can be recovered and put on a firmer basis using the framework discussed here. It should be stressed that neither the analysis of Ref. [42] nor the model of Ref. [40] require that the relevant topological degrees of freedom be standard instantons (or, more precisely, calorons): all that is needed is the presence of non-interacting topological objects carrying unit topological charge, therefore supporting an exact zero mode when isolated; for the applicability of the model of Ref. [40] these zero modes have to be exponentially localised.

At the very least, then, the behaviour of the spectral density shown in Eq. (21) is physically plausible. Whether it is the actual behaviour found in QCD in the chiral limit is, of course, a very different question, one that lacking deeper analytical insight should be studied by means of numerical simulations. If this turned out to be indeed the behaviour of the spectral density in the chiral limit, though, it is hard to imagine a more natural explanation than the presence of an ideal instanton gas-like component in the topological content of typical gauge configurations.

5. Conclusions

In this contribution I have shown that the m^2-differentiability of scalar and pseudoscalar susceptibilities, often assumed in studies of the chirally symmetric phase in the two-flavour chiral limit of a gauge theory, is actually a necessary condition for chiral symmetry restoration. More precisely, a necessary and sufficient condition for symmetry restoration at the level of scalar and pseudoscalar susceptibilities is the m^2-differentiability of the coefficients $\mathcal{A}_{n_u,n_w,n_{\tilde{u}}}(m^2)$, Eq. (7). This, in turn, implies that susceptibilities involving an even number of S and P are m^2-differentiable, and those involving an odd number of S and P are m times an m^2-differentiable function.

Effective breaking of $\mathrm{U}(1)_A$ is compatible in principle with chiral symmetry restoration, and can be achieved in practice if the spectral density shows a characteristic singular behaviour near zero. Further peculiar features of the spectrum, not discussed in detail here, are required by second-order constraints, namely a two-point eigenvalue correlator singular at the origin, and that near-zero modes be not localised [11]. All these features can be obtained naturally if the local topology of typical gauge configurations includes a contribution behaving like an ideal instanton gas, of vanishing density in the chiral limit. This provides a concrete (and plausible) physical mechanism for their realisation.

It would be interesting to extend this study to other sectors of the theory and to a larger number of massless flavours; it seems more pressing to verify the scenario discussed above by means of numerical simulations.

PoS(LATTICE2024)188

References

[1] HotQCD collaboration, *Chiral Phase Transition Temperature in* $(2+1)$*-Flavor QCD*, *Phys. Rev. Lett.* **123** (2019) 062002 [1903.04801].

[2] R.D. Pisarski and F. Wilczek, *Remarks on the Chiral Phase Transition in Chromodynamics*, *Phys. Rev. D* **29** (1984) 338.

[3] A. Pelissetto and E. Vicari, *Relevance of the axial anomaly at the finite-temperature chiral transition in QCD*, *Phys. Rev. D* **88** (2013) 105018 [1309.5446].

[4] J. Bernhardt and C.S. Fischer, *QCD phase transitions in the light quark chiral limit*, *Phys. Rev. D* **108** (2023) 114018 [2309.06737].

[5] R.D. Pisarski and F. Rennecke, *Conjectures about the Chiral Phase Transition in QCD from Anomalous Multi-Instanton Interactions*, *Phys. Rev. Lett.* **132** (2024) 251903 [2401.06130].

[6] G. Fejős and T. Hatsuda, *Order of the* $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ *chiral transition via the functional renormalization group*, *Phys. Rev. D* **110** (2024) 016021 [2404.00554].

[7] T.D. Cohen, *The High temperature phase of QCD and* $\mathrm{U}(1)_A$ *symmetry*, *Phys. Rev. D* **54** (1996) R1867 [hep-ph/9601216].

[8] T.D. Cohen, *The spectral density of the Dirac operator above* T_c, in *APCTP Workshop on Astro-Hadron Physics: Properties of Hadrons in Matter* (1997) 100 [nucl-th/9801061].

[9] S. Aoki, H. Fukaya and Y. Taniguchi, *Chiral symmetry restoration, eigenvalue density of Dirac operator and axial* $\mathrm{U}(1)$ *anomaly at finite temperature*, *Phys. Rev. D* **86** (2012) 114512 [1209.2061].

[10] T. Kanazawa and N. Yamamoto, $\mathrm{U}(1)$ *axial symmetry and Dirac spectra in QCD at high temperature*, *J. High Energy Phys.* **01** (2016) 141 [1508.02416].

[11] M. Giordano, *Constraints on the Dirac spectrum from chiral symmetry restoration*, *Phys. Rev. D* **110** (2024) L091504 [2404.03546].

[12] T. Banks and A. Casher, *Chiral symmetry breaking in confining theories*, *Nucl. Phys. B* **169** (1980) 103.

[13] C. Bonanno, F. D'Angelo and M. D'Elia, *The chiral condensate of* $N_f = 2+1$ *QCD from the spectrum of the staggered Dirac operator*, *J. High Energy Phys.* **11** (2023) 013 [2308.01303].

[14] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, *From chiral random matrix theory to chiral perturbation theory*, *Nucl. Phys. B* **540** (1999) 317 [hep-th/9806110].

[15] P.H. Damgaard and H. Fukaya, *The Chiral Condensate in a Finite Volume*, *J. High Energy Phys.* **01** (2009) 052 [0812.2797].

[16] L. Giusti and M. Lüscher, *Chiral symmetry breaking and the Banks-Casher relation in lattice QCD with Wilson quarks*, *J. High Energy Phys.* **03** (2009) 013 [0812.3638].

[17] R.G. Edwards, U.M. Heller, J.E. Kiskis and R. Narayanan, *Chiral condensate in the deconfined phase of quenched gauge theories*, *Phys. Rev. D* **61** (2000) 074504 [hep-lat/9910041].

[18] G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, H. Matsufuru et al., *Finite temperature study of the axial* U(1) *symmetry on the lattice with overlap fermion formulation*, *Phys. Rev. D* **87** (2013) 114514 [1304.6145].

[19] V. Dick, F. Karsch, E. Laermann, S. Mukherjee and S. Sharma, *Microscopic origin of $U_A(1)$ symmetry violation in the high temperature phase of QCD*, *Phys. Rev. D* **91** (2015) 094504 [1502.06190].

[20] A. Alexandru and I. Horváth, *Phases of SU(3) Gauge Theories with Fundamental Quarks via Dirac Spectral Density*, *Phys. Rev. D* **92** (2015) 045038 [1502.07732].

[21] A. Tomiya, G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko et al., *Evidence of effective axial U(1) symmetry restoration at high temperature QCD*, *Phys. Rev. D* **96** (2017) 034509 [1612.01908].

[22] T.G. Kovács and R.Á. Vig, *Localization transition in SU(3) gauge theory*, *Phys. Rev. D* **97** (2018) 014502 [1706.03562].

[23] A. Alexandru and I. Horváth, *Possible New Phase of Thermal QCD*, *Phys. Rev. D* **100** (2019) 094507 [1906.08047].

[24] H.T. Ding, S.T. Li, S. Mukherjee, A. Tomiya, X.D. Wang and Y. Zhang, *Correlated Dirac Eigenvalues and Axial Anomaly in Chiral Symmetric QCD*, *Phys. Rev. Lett.* **126** (2021) 082001 [2010.14836].

[25] JLQCD collaboration, *Study of the axial $U(1)$ anomaly at high temperature with lattice chiral fermions*, *Phys. Rev. D* **103** (2021) 074506 [2011.01499].

[26] R.Á. Vig and T.G. Kovács, *Ideal topological gas in the high temperature phase of SU(3) gauge theory*, *Phys. Rev. D* **103** (2021) 114510 [2101.01498].

[27] O. Kaczmarek, L. Mazur and S. Sharma, *Eigenvalue spectra of QCD and the fate of $U_A(1)$ breaking towards the chiral limit*, *Phys. Rev. D* **104** (2021) 094518 [2102.06136].

[28] A. Alexandru and I. Horváth, *Unusual Features of QCD Low-Energy Modes in the Infrared Phase*, *Phys. Rev. Lett.* **127** (2021) 052303 [2103.05607].

[29] A. Alexandru and I. Horváth, *Anderson metal-to-critical transition in QCD*, *Phys. Lett. B* **833** (2022) 137370 [2110.04833].

[30] O. Kaczmarek, R. Shanker and S. Sharma, *Eigenvalues of the QCD Dirac matrix with improved staggered quarks in the continuum limit*, *Phys. Rev. D* **108** (2023) 094501 [2301.11610].

[31] X.-L. Meng, P. Sun, A. Alexandru, I. Horváth, K.-F. Liu, G. Wang et al., *Separation of Infrared and Bulk in Thermal QCD*, 2305.09459.

[32] A. Alexandru, C. Bonanno, M. D'Elia and I. Horváth, *Dirac spectral density in $N_f = 2+1$ QCD at $T = 230$ MeV*, *Phys. Rev. D* **110** (2024) 074515 [2404.12298].

[33] N.J. Evans, S.D.H. Hsu and M. Schwetz, *Topological charge and $U(1)_A$ symmetry in the high temperature phase of QCD*, *Phys. Lett. B* **375** (1996) 262 [hep-ph/9601361].

[34] S.H. Lee and T. Hatsuda, $\mathrm{U_A}(1)$ *symmetry restoration in QCD with N_f flavors*, *Phys. Rev. D* **54** (1996) R1871 [hep-ph/9601373].

[35] V. Azcoiti, *Spectral density of the Dirac-Ginsparg-Wilson operator, chiral $U(1)_A$ anomaly, and analyticity in the high temperature phase of QCD*, *Phys. Rev. D* **107** (2023) 114516 [2304.14725].

[36] P.H. Ginsparg and K.G. Wilson, *A Remnant of Chiral Symmetry on the Lattice*, *Phys. Rev. D* **25** (1982) 2649.

[37] M. Lüscher, *Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation*, *Phys. Lett. B* **428** (1998) 342 [hep-lat/9802011].

[38] D.B. Kaplan, *A method for simulating chiral fermions on the lattice*, *Phys. Lett. B* **288** (1992) 342 [hep-lat/9206013].

[39] H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett. B* **417** (1998) 141 [hep-lat/9707022].

[40] T.G. Kovács, *Fate of Chiral Symmetries in the Quark-Gluon Plasma from an Instanton-Based Random Matrix Model of QCD*, *Phys. Rev. Lett.* **132** (2024) 131902 [2311.04208].

[41] S.N. Evangelou and D.E. Katsanos, *Spectral statistics in chiral-orthogonal disordered systems*, *Jour. Phys. A: Math. Gen.* **36** (2003) 3237 [cond-mat/0206089].

[42] T. Kanazawa and N. Yamamoto, *Quasi-instantons in QCD with chiral symmetry restoration*, *Phys. Rev. D* **91** (2015) 105015 [1410.3614].

PoS(LATTICE2024)188