

Phase and equation of state of finite density QC₂D at lower temperature

Etsuko Itou,^{a,b,*} Kei Iida,^{c,d} Kotaro Murakami,^{b,e} and Daiki Suenaga^{f,g}

^a*Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*

^b*Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS), RIKEN, Wako, Saitama 351-0198, Japan*

^c*Department of Mathematics and Physics, Kochi University, 2-5-1 Akebono-cho, Kochi 780-8520, Japan*

^d*RIKEN Nishina Center, Wako, Saitama 351-0198, Japan*

^e*Department of Physics, Institute of Science Tokyo, 2-12-1 Ookayama, Meguro, Tokyo 152-8551, Japan*

^f*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya, 464-8602, Japan*

^g*Research Center for Nuclear Physics, Osaka University, Ibaraki 567-0048, Japan*

E-mail: itou@yukawa.kyoto-u.ac.jp

We investigate the phase structure and the equation of state (EoS) for dense two-color QCD at low temperatures, $T = 40$ MeV (32^4 lattice) and $T = 80$ MeV (16^4 lattice). A rich phase structure below the pseudo-critical temperature T_c as a function of quark chemical potential μ has been revealed. By performing $T = 40$ MeV simulations, essentially similar results to the previous ones at $T = 80$ MeV are obtained, but several finer understandings are achieved. Breaking of the conformal bound is also confirmed thanks to smaller statistical errors. This talk is mainly based on Refs. [1, 2]. It also includes related studies and subsequent developments that were not mentioned in the original papers.

The 41st International Symposium on Lattice Field Theory (LATTICE2024)

28 July - 3 August 2024

Liverpool, UK

*Speaker

1. Introduction

The sign problem in three-color QCD at low temperatures and finite densities still remains one of the most challenging issues in lattice Monte Carlo simulations [3]. Meanwhile, multiple groups have actively conducted large-scale first-principles calculations using the (R)HMC algorithm for systems such as two-color QCD with baryon chemical potential and three-color QCD with isospin chemical potential, where the sign problem is absent. Adding explicit symmetry-breaking terms for the U(1) baryon or isospin symmetry (i.e., diquark or pionic source terms) into the finite-density QCD(-like) action has enabled the accumulation of gauge-field configurations in the superfluid phase, which yields various observables that are clearly different from those in the hadron phase and the quark-gluon plasma (QGP) phase.

Most remarkably, first-principles calculations for these QCD-like theories have attracted much attention in the context of breaking the conformal (holographic) bound [4, 5]; the speed of sound in the superfluid phase exceeds the value of relativistic free theory corresponding to $c_s^2/c^2 = 1/3$. The first evidence for such breaking was obtained in dense two-color QCD [1, 2]; shortly afterward, it was independently confirmed in the case of three-color QCD with isospin chemical potential using two different computational approaches [6–8]¹.

In this article, we summarize our results for the phase diagram and equation of state (EoS) of two-color QCD at low temperatures and finite densities [1, 2, 10, 11]. Furthermore, we will compare our results for the speed of sound and EoS with related lattice calculations, effective model studies, and observational data on neutron stars. In particular, we focus on related works that remain to be cited in our previous paper [2].

2. Phase diagram

For the lattice action, we employed the Iwasaki gauge action and naive Wilson fermions. Following the approach described in Refs. [1, 2, 10], we extended the two-color QCD action by including a number operator coupled with the quark chemical potential (μ). In low-temperature and high-density regimes, furthermore, we introduced a diquark source term characterized by a parameter j ; physical quantities were calculated for several finite values of j , and then obtained by taking the $j \rightarrow 0$ limit. Using the above-mentioned action, we performed the RHMC calculations to generate gauge configurations at $\beta = 0.80$, $\kappa = 0.159$ for 32^4 ($T = 40$ MeV), 16^4 ($T = 80$ MeV), and $32^3 \times 8$ ($T = 160$ MeV). The chemical potential (μ) is normalized by the pseudo-scalar meson (corresponding to a pion in QCD) mass at $\mu = 0$, namely μ/m_{PS} , where $m_{PS} \approx 738$ MeV in our simulations [12, 13].

First, we obtained the phase diagram on $T-\mu$ plane as shown in Fig. 1. Here, we utilize three quantities, namely, the magnitude of the Polyakov loop ($\langle |L| \rangle$), the diquark condensate ($\langle qq \rangle$), and the quark number density ($\langle n_q \rangle$) to distinguish among different phases. We use the name of each phase as shown in Table 1, which follows Refs. [10, 17–19].

Let us focus on the two relatively low temperatures, $T = 40$ MeV and $T = 80$ MeV. At both temperatures, the diquark condensate, which is an order parameter of the superfluidity, becomes

¹Also, at the conference, an independent analysis of the breaking of the conformal bound for dense two-color QCD was reported [9].

	Hadronic		Superfluid		QGP
	Hadronic matter ($T > 0$)		BEC	BCS	
$\langle L \rangle$	zero	zero	non-zero		non-zero
$\langle qq \rangle$	zero	zero	non-zero	$(\propto \mu^2)$	zero
$\langle n_q \rangle$	zero	non-zero	non-zero	$\langle n_q \rangle / n_q^{\text{tree}} \approx 1$	non-zero

Table 1: Definition of the phases. To distinguish between the BEC and BCS phases, we use the value of $\langle n_q \rangle$. Meanwhile, $\langle qq \rangle$ is expected to scale as $\propto \mu^2$ by the weak coupling analysis [14–16].

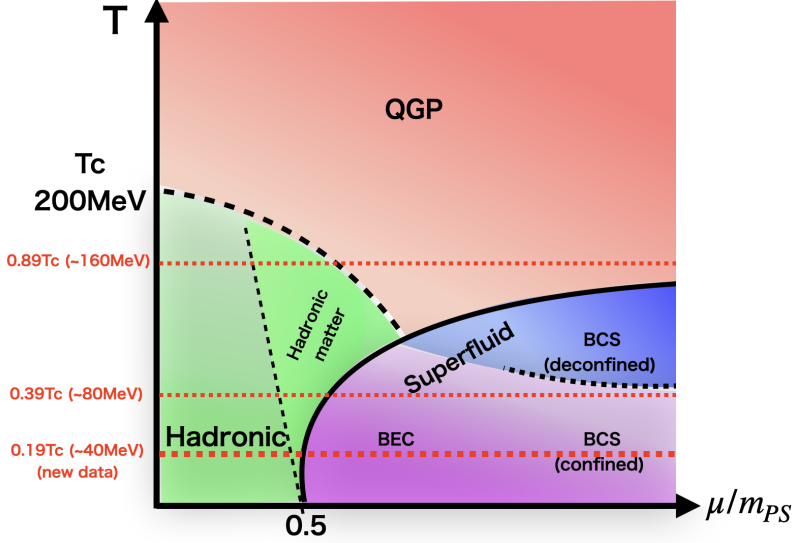


Figure 1: Schematic QC₂D phase diagram. In our previous work [10], we clarified the phase structure at $T = 160$ MeV and 80 MeV, while in our recent work, we addressed what it is like at $T = 40$ MeV.

non-zero beyond $\mu \approx m_{PS}/2$. The chiral perturbation theory (ChPT) gives the critical chemical potential as $\mu_c = m_{PS}/2$ and the scaling behavior around μ_c as

$$\langle qq \rangle = A(\mu - \mu_c)^{1/2}. \quad (1)$$

Our data are almost consistent with these predictions. Indeed, we obtain the critical value at $T = 40$ MeV from the fitting of the data as $\mu/m_{PS} = 0.47$. At $T = 80$ MeV, the “hadronic-matter phase” was clearly observed, where $\langle qq \rangle = 0$ but $\langle n_q \rangle \neq 0$ as defined in Table 1. Interestingly, the study at $T = 40$ MeV revealed that this phase shrinks with respect to μ at lower temperature. It indicates that $\langle n_q \rangle \neq 0$ before the superfluid phase transition is caused by a thermal excitation of diquarks, which correspond to the lightest hadron in the superfluid phase, at finite temperature.

Another notable finding from the comparison of the two temperatures concerns the scaling behavior of $\langle qq \rangle$ in the BCS phase. The scaling looks nearly linear in μ at $T = 80$ MeV, whereas a quadratic scaling was observed at $T = 40$ MeV. Analytical studies at zero temperature based on the weak-coupling expansion predict $\langle qq \rangle \propto \mu^2$ [14–16]. Our results show that this analytical prediction does not hold at higher temperatures, but become closer to this prediction at lower temperatures. Also, in general, $\langle qq \rangle$ can be expressed approximately as $\Delta(\mu)\mu^2$, where $\Delta(\mu)$ is the

diquark gap. The fact that $\langle qq \rangle$ behaves as μ^2 suggests that the μ -dependence of the diquark gap in the BCS phase is small.

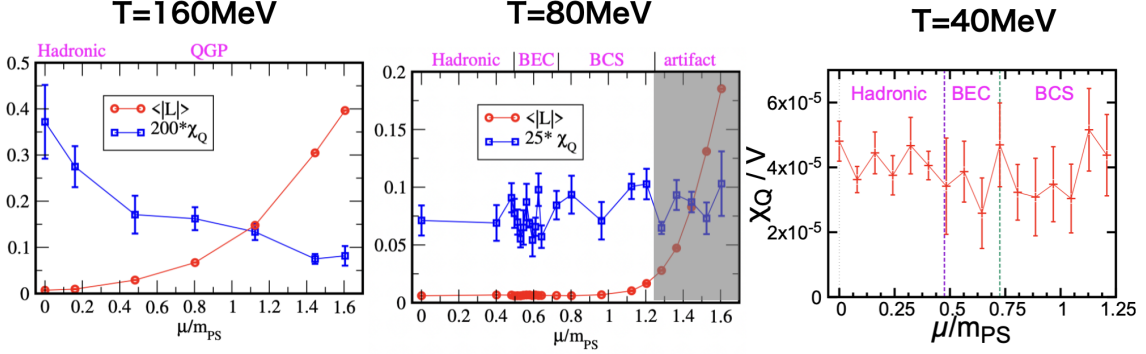


Figure 2: The μ -dependence of the topological susceptibility (red data) at $T = 160$ MeV, 80 MeV, and 40 MeV. At $T = 160$ MeV and 80 MeV, we also show the magnitude of the Polyakov loop (blue data) to see the confining behavior.

After determining the phase diagram, we investigated the μ -dependence of the topological susceptibility in Refs. [2, 10],

$$\chi_Q = \langle Q^2 \rangle - \langle Q \rangle^2. \quad (2)$$

Here, Q denotes the topological charge, which we measured from the gluonic definition,

$$Q(t) = \frac{1}{32\pi^2} \sum_x \text{Tr} \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(x, t) G_{\rho\sigma}^a(x, t), \quad (3)$$

using the gradient flow method. At $T = 160$ MeV, where the hadronic to QGP phase transition occurs as μ increases, χ_Q decreases with μ as shown in the left panel of Fig. 2. On the other hand, at low temperatures of $T = 80$ MeV (the middle panel) and 40 MeV (the right panel), where the hadron-superfluid transition occurs, χ_Q remains almost constant throughout the μ region that covers the hadronic and superfluid phases. We also showed that the confinement remains even in the BCS phase, by studying the Polyakov loop (as shown in the blue data of the middle panel) and also q - \bar{q} potential at $T = 40$ MeV in Ref. [11]. Although the high-density region is naively expected to be approximated by an asymptotically free theory, the macroscopic gluonic dynamics behaves similarly to that in the hadronic phase.

3. Equation of state

Next, we consider the EoS at $T = 40$ MeV and $T = 80$ MeV. We calculated the trace anomaly and pressure using the following definitions. As for the pressure, we employed

$$\frac{p}{p_{SB}}(\mu) = \frac{\int_{\mu_0}^{\mu} d\mu' \frac{n_{SB}^{cont.}}{n_q^{latt.}}(\mu')}{\int_{\mu_0}^{\mu} d\mu' n_{SB}^{cont.}(\mu')}, \quad (4)$$

which was originally proposed in Ref. [17]. Here, $p_{SB}(\mu)$, i.e., the pressure value of the non-interacting theory (the Stefan-Boltzmann (SB) limit), was obtained by the numerical integration of

the number density of massless quarks. As for the trace anomaly, we utilized

$$e - 3p = \frac{1}{N_s^3 N_\tau} \left(a \frac{d\beta}{da} \Big|_{\text{LCP}} \left\langle \frac{\partial S}{\partial \beta} \right\rangle_{\text{sub.}} + a \frac{d\kappa}{da} \Big|_{\text{LCP}} \left\langle \frac{\partial S}{\partial \kappa} \right\rangle_{\text{sub.}} + a \frac{\partial j}{\partial a} \Big|_{\text{LCP}} \left\langle \frac{\partial S}{\partial j} \right\rangle_{\text{sub.}} \right), \quad (5)$$

where the nonperturbative β -function was calculated in Ref. [20] as

$$a d\beta/da|_{\beta=0.80, \kappa=0.159} = -0.352, \quad a d\kappa/da|_{\beta=0.80, \kappa=0.159} = 0.0282. \quad (6)$$

The results for the pressure are shown in the left panel of Fig. 3. In the hadronic phase, where $\langle n_q \rangle$ is consistent with zero, the pressure is also zero. Once the superfluid phase transition occurs, the pressure increases monotonically. At $T = 40$ MeV (triangle-blue symbols), the pressure grows sharply in the BEC phase and approaches the SB limit more closely in the high-density region. On

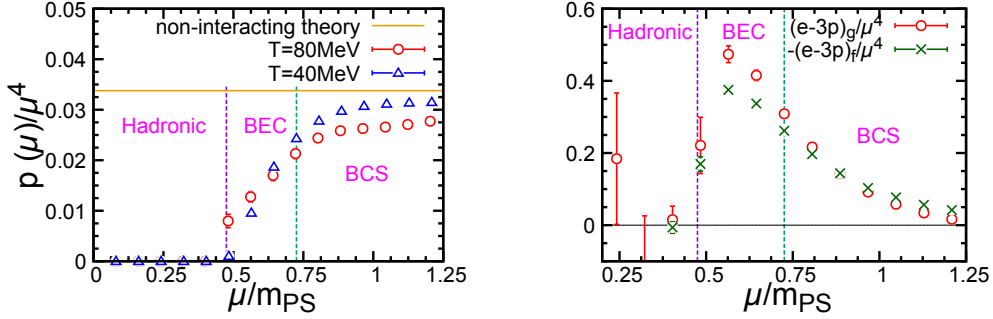


Figure 3: Pressure (left panel) and trace anomaly (right panel) as a function of μ . The right panel depicts the first (circle-red symbol) and second (cross-blue symbol) terms in Eq. (5) at $T = 40$ MeV separately.

the other hand, we plotted the first (gluonic) term and minus the second (fermionic) term of the trace anomaly (5) at $T = 40$ MeV as $\langle e - 3p \rangle_g$ (circle-red symbols) and $-\langle e - 3p \rangle_f$ (cross-green symbols), respectively, in the right panel of Fig. 3. Note that the second term takes negative values, which have the sign flipped in the plot. Furthermore, we neglected the third term in Eq. (5) in our analysis. Thus, the total trace anomaly is given by the circle-red data minus the cross-green data. As can be seen from this plot, the trace anomaly is also zero in the hadronic phase. After the superfluid phase transition, the trace anomaly reaches a maximum value and then decreases. Notably, in the middle of the BCS phase, $-\langle e - 3p \rangle_f$ becomes larger than $\langle e - 3p \rangle_g$, causing the trace anomaly to change from positive to negative.

We would like to discuss the results that were obtained thus far. First, let us consider why the pressure value in the BCS phase at $T = 40$ MeV is larger than at $T = 80$ MeV. One would expect that particles are thermally excited at higher temperatures, resulting in higher pressure. Our results, however, show the opposite. In fact, the spatial volumes differ between these simulations: we performed the simulation on lattices of size 32^4 for $T = 40$ MeV and 16^4 for $T = 80$ MeV. Thus, the data for $T = 80$ MeV may be more severely affected by the finite volume effects. We consider that such effects might lead to a smaller pressure in our data, especially at $T = 80$ MeV. As to how to define the pressure on the lattice in a finite-density regime, furthermore, there is some room for discussion [17].

Additionally, in the thermodynamic limit at zero temperature, there is a relationship between the trace anomaly and the μ -derivative of the pressure²,

$$\frac{d}{d\mu} \left(\frac{p}{\mu^4} \right) = \frac{e - 3p}{\mu^5}. \quad (7)$$

Our data are inconsistent with this equation since the trace anomaly changes its sign in the BCS phase while p/μ^4 increases monotonically. This also suggests that finite volume effects may influence our data at least in the BCS phase.

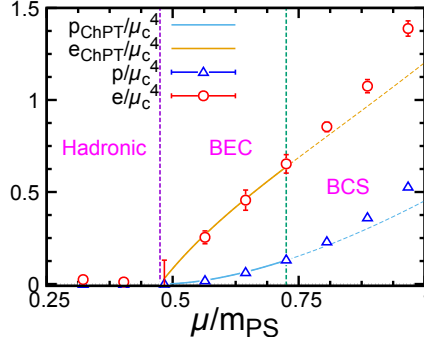


Figure 4: The pressure and internal energy around the BEC phase. The cyan and orange curves represent the fitting functions for p/μ_c^4 and e/μ_c^4 , respectively, whose forms are given by the ChPT theory as shown in Eqs. (8) and (9).

If we focus on the BEC phase, on the other hand, our results for the chiral condensate, diquark condensate, and sound velocity (to be shown later) are consistent with the predictions from ChPT [2]. It would therefore be worthwhile to fit the data for p and e in the BEC phase to the prediction of ChPT [17],

$$p_{\text{ChPT}} = 4N_f F^2 \mu^2 \left(1 - \frac{\mu_c^2}{\mu^2} \right)^2, \quad (8)$$

$$e_{\text{ChPT}} = 4N_f F^2 \mu^2 \left(1 - \frac{\mu_c^2}{\mu^2} \right) \left(1 + 3 \frac{\mu_c^2}{\mu^2} \right), \quad (9)$$

to obtain the pion decay constant (F) in two-color QCD. Figure 4 depicts the μ dependence of p and e both for the data and fitted results. The best fit values were obtained as $F = 51.1(5)$ MeV and $F = 56.7(7)$ MeV from the fits of p/μ_c^4 and e/μ_c^4 , respectively. These values are similar to the earlier result, $F = 60.8(1.6)$ MeV, obtained in Ref. [21] from the fit of the lattice data for the quark number density and the mixing angle between the diquark and chiral condensates around the phase transition point predicted by the ChPT analysis.

Finally, we plotted the data for the speed of sound at $T = 40$ MeV and $T = 80$ MeV in Fig. 5, where we evaluated

$$c_s^2(\mu)/c^2 = \frac{\Delta p(\mu)}{\Delta e(\mu)} = \frac{p(\mu + \Delta\mu) - p(\mu - \Delta\mu)}{e(\mu + \Delta\mu) - e(\mu - \Delta\mu)} \quad (10)$$

²E. I. would like to thank Dr. Yuki Fujimoto for pointing out this discussion.

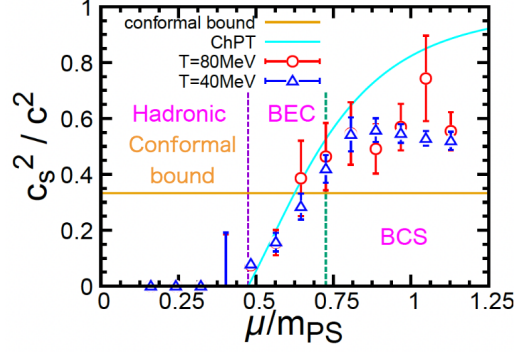


Figure 5: The squared sound velocity at $T = 40$ MeV and $T = 80$ MeV. The cyan curve is the prediction of ChPT given by $c_s^2/c^2 = (1 - \mu_c^4/\mu^4)/(1 + 3\mu_c^4/\mu^4)$. The horizontal line (orange) depicts the conformal bound, $c_s^2/c^2 = 1/3$.

at fixed temperature. The prediction of the ChPT, which was shown as cyan-curve, is that c_s^2/c^2 approaches 1 with increasing density. Our data are consistent with this prediction in the low-density region of the BEC phase. In the high-density region of the BEC phase, however, the values become smaller than those predicted by ChPT. While this behavior remains in the BCS phase, the conformal bound (shown as orange-line), known as $c_s^2/c^2 = 1/3$, is nevertheless clearly exceeded. Before our study [1], for instance, a number of finite temperature studies at $\mu = 0$ had existed, but no first-principles calculation in any QCD-like theory had demonstrated a violation of this conformal bound. It is a characteristic property of low-temperature and high-density QCD-like theories, which could help to support earlier suggestions of such a violation from neutron star phenomenology and effective model analyses [22–28].

4. Summary and related works

We investigated the phase structure and the EoS for dense two-color QCD at low temperatures: $T = 40$ MeV (32^4 lattice) and $T = 80$ MeV (16^4 lattice). From a comparison of various observables between the two temperatures, we found that the hadronic-matter phase shrinks as the temperature decreases and that the diquark condensate shows more of a quadratic scaling tendency in the BCS phase, which is predicted by the weak coupling expansion. Furthermore, careful analyses confirm that the topological susceptibility is independent of μ even in a high-density regime. We also compared the data obtained at the two temperatures for the pressure, internal energy, and sound velocity as a function of μ . The pressure increases around the hadronic-superfluid phase transition more rapidly at the lower temperature, while the temperature dependence of the speed of sound is invisible. Breaking of the conformal bound was also confirmed thanks to sufficiently small statistical errors at $T = 40$ MeV. It is interesting to note that recent lattice results for three-color QCD with isospin chemical potential μ_I confirm that the speed of sound agrees with ChPT in a relatively small μ_I regime of the pionic BEC phase and subsequently violates the conformal bound at higher isospin densities [6–8]. As for the upper bound of the speed of sound, a recent paper [29] proposed a new value of the bound from a hydrodynamics analysis, which gives $c_s^2/c^2 < 0.781$. Our data also satisfy this bound.

It is worth mentioning several recent analytical studies that consider the origin of the large value of the speed of sound and the negativity of the trace anomaly. It has been pointed out that the presence of a pairing gap tends to increase the speed of sound compared with the perturbative analysis [28, 30, 31]. According to a recent study by Fujimoto [32], the pressure in three-color isospin QCD approaches the SB limit from above due to the gap. In contrast, in dense two-color QCD where corrections from the gap are relatively small, the pressure is only slightly modified from perturbation theory, and hence the pressure approaches the SB limit from below. Such predictions look consistent with the high-density lattice results for three-color isospin QCD (Ref. [8]) and for dense two-color QCD shown here. Furthermore, Fukushima and Minato have conducted studies on the negativity of the trace anomaly for dense two-color QCD and three-color isospin QCD, as well as two-flavor color-superconducting (2SC) matter, using a unified treatment of the perturbative gap and the correction due to instanton effects, and thereby they have revealed differences among these systems [33]. In particular, 2SC matter shows a qualitatively different behavior for the speed of sound and the trace anomaly as a function of μ from the others. On the other hand, the constraints from neutron star observational data, which can be effectively regarded as those from real finite-density QCD, suggest that the conformal bound is violated [34, 35]. In conjunction with the results of these analytical and phenomenological studies, it would be fascinating to delve into the similarities and differences among dense two-color QCD and three-color isospin systems, let alone, three-color QCD matter in our Universe.

Acknowledgments

We would like to thank Drs. Y. Fujimoto, K. Fukushima, S. Hands, T. Hatsuda, T. Kojo, Y. Namekawa, and N. Yamamoto for helpful comments. The work of K. I. is supported by JSPS KAKENHI with Grant Numbers 18H05406 and 23K25864. The work of E. I. is supported by JST PRESTO Grant Number JPMJPR2113, JSPS Grant-in-Aid for Transformative Research Areas (A) JP21H05190, JST Grant Number JPMJPF2221, JPMJCR24I3, and also supported by Program for Promoting Researches on the Supercomputer “Fugaku” (Simulation for basic science: from fundamental laws of particles to creation of nuclei) and (Simulation for basic science: approaching the new quantum era), and Joint Institute for Computational Fundamental Science (JICFuS), Grant Number JPMXP1020230411. The work of E. I. is supported also by Center for Gravitational Physics and Quantum Information (CGPQI) at YITP. E. I. and D. S. are supported by JSPS KAKENHI with Grant Number 23H05439. K. M. is supported in part by Grants-in-Aid for JSPS Fellows (Nos. JP22J14889, JP22KJ1870) and by JSPS KAKENHI with Grant No. 22H04917. The work of D. S. is also supported by JSPS KAKENHI with Grant Number 23K03377. The numerical simulation is supported by the HPCI-JHPCN System Research Project (Project ID: jh220021) and HOKUSAI in RIKEN.

References

- [1] K. Iida and E. Itou, *Velocity of sound beyond the high-density relativistic limit from lattice simulation of dense two-color QCD*, *PTEP* **2022** (2022) 111B01 [2207.01253].

- [2] K. Iida, E. Itou, K. Murakami and D. Suenaga, *Lattice study on finite density QC_2D towards zero temperature*, *JHEP* **10** (2024) 022 [2405.20566].
- [3] K. Nagata, *Finite-density lattice QCD and sign problem: Current status and open problems*, *Prog. Part. Nucl. Phys.* **127** (2022) 103991 [2108.12423].
- [4] A. Cherman, T.D. Cohen and A. Nellore, *A Bound on the speed of sound from holography*, *Phys. Rev. D* **80** (2009) 066003 [0905.0903].
- [5] P.M. Hohler and M.A. Stephanov, *Holography and the speed of sound at high temperatures*, *Phys. Rev. D* **80** (2009) 066002 [0905.0900].
- [6] B.B. Brandt, F. Cuteri and G. Endrodi, *Equation of state and speed of sound of isospin-asymmetric QCD on the lattice*, *JHEP* **07** (2023) 055 [2212.14016].
- [7] R. Abbott, W. Detmold, F. Romero-López, Z. Davoudi, M. Illa, A. Parreño et al., *Lattice quantum chromodynamics at large isospin density: 6144 pions in a box*, [2307.15014](#).
- [8] R. Abbott, W. Detmold, M. Illa, A. Parreño, R.J. Perry, F. Romero-López et al., *QCD constraints on isospin-dense matter and the nuclear equation of state*, [2406.09273](#).
- [9] S. Hands, S. Kim, D. Lawlor, A. Lee-Mitchell and J.-I. Skullerud, *Dense QC_2D . What's up with that?!?*, [2412.15872](#).
- [10] K. Iida, E. Itou and T.-G. Lee, *Two-colour QCD phases and the topology at low temperature and high density*, *JHEP* **01** (2020) 181 [1910.07872].
- [11] K. Ishiguro, K. Iida and E. Itou, *Flux tube profiles in two-color QCD at low temperature and high density*, *PoS LATTICE2021* (2022) 063 [2111.13067].
- [12] K. Murakami, D. Suenaga, K. Iida and E. Itou, *Measurement of hadron masses in 2-color finite density QCD*, *PoS LATTICE2022* (2023) 154 [2211.13472].
- [13] K. Murakami, E. Itou and K. Iida, *Chemical potential (in)dependence of hadron scatterings in the hadronic phase of QCD-like theories and its applications*, [2309.08143](#).
- [14] T. Schäfer, *Patterns of symmetry breaking in QCD at high baryon density*, *Nucl. Phys. B* **575** (2000) 269 [hep-ph/9909574].
- [15] M. Hanada and N. Yamamoto, *Universality of Phases in QCD and QCD-like Theories*, *JHEP* **02** (2012) 138 [1103.5480].
- [16] T. Kanazawa, T. Wettig and N. Yamamoto, *Banks-Casher-type relation for the BCS gap at high density*, *Eur. Phys. J. A* **49** (2013) 88 [1211.5332].
- [17] S. Hands, S. Kim and J.-I. Skullerud, *Deconfinement in dense 2-color QCD*, *Eur. Phys. J. C* **48** (2006) 193 [hep-lat/0604004].

- [18] V.V. Braguta, E.M. Ilgenfritz, A.Y. Kotov, A.V. Molochkov and A.A. Nikolaev, *Study of the phase diagram of dense two-color QCD within lattice simulation*, *Phys. Rev. D* **94** (2016) 114510 [1605.04090].
- [19] T. Boz, P. Giudice, S. Hands and J.-I. Skullerud, *Dense two-color QCD towards continuum and chiral limits*, *Phys. Rev. D* **101** (2020) 074506 [1912.10975].
- [20] K. Iida, E. Itou and T.-G. Lee, *Relative scale setting for two-color QCD with $N_f=2$ Wilson fermions*, *PTEP* **2021** (2021) 013B05 [2008.06322].
- [21] N. Astrakhantsev, V.V. Braguta, E.M. Ilgenfritz, A.Y. Kotov and A.A. Nikolaev, *Lattice study of thermodynamic properties of dense QC_2D* , *Phys. Rev. D* **102** (2020) 074507 [2007.07640].
- [22] K. Masuda, T. Hatsuda and T. Takatsuka, *Hadron–quark crossover and massive hybrid stars*, *PTEP* **2013** (2013) 073D01 [1212.6803].
- [23] G. Baym, T. Hatsuda, T. Kojo, P.D. Powell, Y. Song and T. Takatsuka, *From hadrons to quarks in neutron stars: a review*, *Rept. Prog. Phys.* **81** (2018) 056902 [1707.04966].
- [24] L. McLerran and S. Reddy, *Quarkyonic Matter and Neutron Stars*, *Phys. Rev. Lett.* **122** (2019) 122701 [1811.12503].
- [25] Y. Fujimoto and K. Fukushima, *Equation of state of cold and dense QCD matter in resummed perturbation theory*, *Phys. Rev. D* **105** (2022) 014025 [2011.10891].
- [26] T. Kojo, *Stiffening of matter in quark-hadron continuity*, *Phys. Rev. D* **104** (2021) 074005 [2106.06687].
- [27] T. Kojo and D. Suenaga, *Peaks of sound velocity in two color dense QCD: Quark saturation effects and semishort range correlations*, *Phys. Rev. D* **105** (2022) 076001 [2110.02100].
- [28] J. Braun, A. Geißel and B. Schallmo, *Speed of sound in dense strong-interaction matter*, 2206.06328.
- [29] M. Hippert, J. Noronha and P. Romatschke, *Upper Bound on the Speed of Sound in Nuclear Matter from Transport*, 2402.14085.
- [30] T. Kojo, P.D. Powell, Y. Song and G. Baym, *Phenomenological QCD equation of state for massive neutron stars*, *Phys. Rev. D* **91** (2015) 045003 [1412.1108].
- [31] M. Leonhardt, M. Pospiech, B. Schallmo, J. Braun, C. Drischler, K. Hebeler et al., *Symmetric nuclear matter from the strong interaction*, *Phys. Rev. Lett.* **125** (2020) 142502 [1907.05814].
- [32] Y. Fujimoto, *Interplay between the weak-coupling results and the lattice data in dense QCD*, 2408.12514.

- [33] K. Fukushima and S. Minato, *Speed of sound and trace anomaly in a unified treatment of the two-color diquark superfluid, the pion-condensed high-isospin matter, and the 2SC quark matter*, [2411.03781](#).
- [34] S. Altiparmak, C. Ecker and L. Rezzolla, *On the Sound Speed in Neutron Stars*, *Astrophys. J. Lett.* **939** (2022) L34 [[2203.14974](#)].
- [35] E. Annala, T. Gorda, J. Hirvonen, O. Komoltsev, A. Kurkela, J. Nättilä et al., *Strongly interacting matter exhibits deconfined behavior in massive neutron stars*, *Nature Commun.* **14** (2023) 8451 [[2303.11356](#)].