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Generalized boost transformations in finite volumes and application to Hamiltonian methods

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The extraction of physical observables from lattice energy spectra using finite volume quantization conditions is one of the key methods in the study of hadron physics. Due to the expensive and limited lattice configurations, there is significant interest in extracting as much information as possible from these datasets. One effective approach involves varying the total momentum within the finite volume, which allows for the identification of additional finite volume energy levels in moving systems. However, the finite volume quantization conditions applicable to moving systems require careful consideration of momentum transformations between different reference frames, and there exists several different method to perform the three-momentum transformation. This work systematically presents a general scheme for three-momentum transformations in a finite volume, and this scheme is able to generate two existed transformations in literature. In addition, we propose a new transformation method that circumvents reliance on the total energy during the transformation process, which is crucial for employing the Hamiltonian Effective Field Theory (HEFT) approach to extract scattering amplitudes. At last, we also demonstrate the consistence between our method with others through numerical comparisons, employing a phenomenological $\pi\pi$ scattering example.

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1. Introduction

The quantization conditions applicable to two-particle systems were established by Lüscher in the 1990s [1, 2] and have since undergone significant development; for a comprehensive review, please refer to the literature [3]. These quantization conditions successfully connect the finite volume energy spectrum of two-hadron systems to their scattering matrices in infinite space. With advancements in computational technology, extensive research has been conducted on systems such as $\pi\pi$ rescattering, [4–9]. The lattice calculations for the finite volume energy spectrum of two-body systems are based on lattice configurations, which can be quite costly. Hence, it is imperative to extract as much finite volume energy spectrum information as possible from a given configuration.

In the study of two-particle scattering involving decaying particles, moving systems not only provide access to additional two-body energy levels, but they also make the investigation of decay processes more feasible compared to stationary systems. For instance, in lattice studies of ρ meson scattering with $\pi\pi$, literature [4] reports only three energy points in the rest frame. Translating these into the relationship between scattering phase shifts and total energy yields insufficient data to ascertain the resonance pattern of the ρ meson. However, considering systems with different total momenta produces a variety of finite volume energy levels. Utilizing the quantization conditions allows for the reconstruction of a complete phase shift image, showing a significant transition from 0 degrees to 180 degrees. Thus, the application of finite volume quantization conditions to moving systems is crucial for the analysis of lattice data.

There are two distinct forms of quantization conditions for two-body moving systems, respectively derived from Rummukainen and Gottlieb (RG) [10] and Kim, Sachrajda, and Sharpe (KSS) [11]. They employ different methodologies to achieve three-momentum transformations across different reference frames. However, both transformation schemes are heavily dependent on the system's total energy (i.e., the energy level of interest), necessitating the solution of equations to obtain finite volume energy levels. An alternative approach based on Hamiltonian Effective Field Theory (HEFT) [12–16] directly derives finite volume energy levels by solving eigenvalue problems of the finite volume Hamiltonian. Nevertheless, in moving systems, this Hamiltonian does not inherently include the sought energy levels, specifically the total energy. Therefore, the existing three-momentum transformation schemes are not suitable for HEFT.

In this paper, we introduce a new transformation method [17] based on a systematic description of finite volume three-momentum transformation schemes, which is independent of total energy and relies solely on the momenta of the two-particle system. This makes it compatible with the HEFT approach. Additionally, we find that this method is also beneficial for three-body systems, ensuring that the velocities of any particles do not exceed the speed of light in any spectator reference [18].

2. Finite volume quantization condition for two-body system

In this section, we first introduce the quantization condition for two-body system based on the Bethe-Salpeter equation (BSE). As shown in Fig. 1, we provide three diagrams to show how to connect the T-matrices in the finite and infinite volume. Fig. 1(a) and (b) show the BSE in the infinite and finite volumes, $T = V + VG_2T$ and $T^L = V + VG_2^BT^L$ respectively. Here G_2 and G_2^B are the corresponding operator of the propagators. Then we can derive $T^L = V + V\left(G_2 + G_2^B - G_2\right)T^L =$

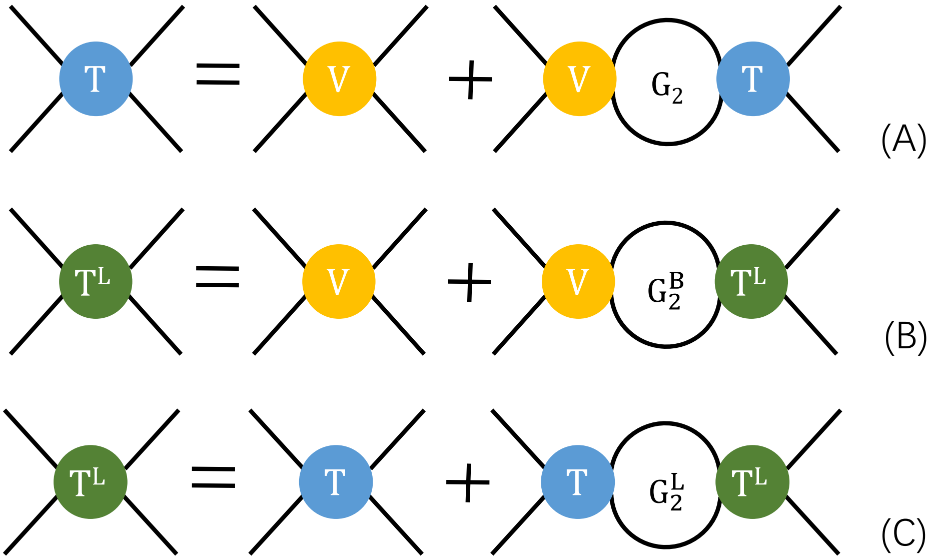

Figure 1: Diagram (A) indicates the usual Bethe-Salpeter equation in the continuum. Diagram (B) shows corresponding form in the finite volume. Diagram (C) represents the relationship between T and T^L with $G_2^L = G_2^B - G_2$.

$V + V\left(G_2 + G_2^L\right) T^L$ to obtain $T - T^L = -(1 - VG_2)^{-1} V G_2^L T^L$. By using $T = (1 - VG_2)^{-1} V$, we have $T^L = T + TG_2^L T^L$, which graphically is given by Fig.1(C). An similar proof can be found in Ref. [1].

Base on the relationship between T and T^L, it is straightforward to obtain the following equation at the rest frame,

$$T^L(p_f^*, p_i^*; P^*) = T(p_f^*, p_i^*; P^*) + \int \frac{dk_0^*}{2\pi}\left(\frac{1}{L^3}\sum_{\mathbf{k}^*} - \int \frac{d^3k^*}{(2\pi)^3}\right) \times T(p_f^*, k^*; P^*) G_2(k^*, P^*) T^L(k^*, p_i^*; P^*)\,. \tag{1}$$

Then if the total momentum of the system is non-vanishing in the box, the momentum in the rest frame is not integer momentum, while the momentum modes of the field quanta can be more naturally expressed in relation to the lattice rest frame; we denote these momenta with a superscript r, for example, $\mathbf{k}^r$. Thus, $\sum_{\mathbf{k}^*}$ should change to $\sum_{\mathbf{k}^r}$. Then we also will introduce a transformation between $\mathbf{k}^*$ and $\mathbf{k}^r$, as well as a Jacobian factor in the sum and integral,

$$\int \frac{d^3k^*}{(2\pi)^3} \to \int \frac{d^3k^r}{(2\pi)^3}\mathcal{J}^r, \quad \frac{1}{L^3}\sum_{\mathbf{k}^*} \to \frac{1}{L^3}\sum_{\mathbf{k}^r}\mathcal{J}^r. \tag{2}$$

Now, Eq. (3) can be written as,

$$T^{r,L}(p_f^*, p_i^*; P^*) = T(p_f^*, p_i^*; P^*) + \int \frac{dk_0^*}{2\pi}\left(\frac{1}{L^3}\sum_{\mathbf{k}^r} - \int \frac{d^3k^r}{(2\pi)^3}\right) \times \mathcal{J}^r T(p_f^*, k^*; P^*) G_2(k^*, P^*) T^{r,L}(k^*, p_i^*; P^*)\,. \tag{3}$$

After integrating the k_0^*, we will reduce such integral equation to the three dimensional form,

$$T^{r,L}(\mathbf{p}_f^*, \mathbf{p}_i^*; E^*) = T(\mathbf{p}_f^*, \mathbf{p}_i^*; E^*) + i \left(\frac{1}{L^3} \sum_{\mathbf{k}^r} - \int \frac{d^3k^r}{(2\pi)^3} \right) \times \mathcal{J}^r \frac{T(\mathbf{p}_f^*, \mathbf{k}^*; E^*)}{4\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)} \frac{T^{r,L}(\mathbf{k}^*, \mathbf{p}_i^*; E^*)}{E^* - \omega_1(\mathbf{k}^*) - \omega_2(\mathbf{k}^*) + i\epsilon}, \quad (4)$$

where $\omega_i(q) = \sqrt{m_i^2 + q^2}$. In fact, the energy levels in the finite volume are the poles of the $T^{r,L}$. By applying the Poisson summation formula and the partial wave expansion, we will obtain the quantization condition as follows, ($E^*(q) = \omega_1(q) + \omega_2(q)$)

$$0 = \det\left([\cot\delta(q)] + [M(q;\mathbf{P})]\right), \quad (5)$$

$$[\cot\delta(q)]_{lm,l'm'} = \cot\delta_l(q)\delta_{l,l'}\delta_{m,m'}, \quad (6)$$

$$[M(q;\mathbf{P})]_{lm,l'm'} = \frac{1}{q}\left(\frac{1}{L^3}\sum_{\mathbf{k}} - \mathcal{P}\int \frac{d^3k^r}{(2\pi)^3} \right) \frac{32\pi^2 E^*(q)\mathcal{J}^r}{4\omega_1(\mathbf{k}^*)\,\omega_2(\mathbf{k}^*)} \frac{Y_{lm}(\hat{\mathbf{k}}^*)Y^*_{l'm'}(\hat{\mathbf{k}}^*)\left(\frac{|\mathbf{k}^*|}{q}\right)^{l+l'}}{E^*(q) - (\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*))}. \quad (7)$$

The detailed derivation can be found in Ref. [17]. Please note $\mathbf{k}^*$ in $M(q;\mathbf{P})$ is the function of $\mathbf{k}^r$.

3. The three-momentum transformation

Now the key problem is to specify the exact three-momentum transformation $\mathbf{k}^* \to \mathbf{k}^r$. There should have some freedom because of lacking the energy components for these three-momenta. We need to introduce two variables a^* and b^* for the energy parts of $\mathbf{P}^*$ and $\mathbf{k}^*$, respectively, which have several different choices, for example,

$$a^* = E^*(q) \ \text{ or } \ \omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*), \qquad b^* = \omega_1(q) \ \text{ or } \ \omega_1(\mathbf{k}^*). \quad (8)$$

Then, we can explicitly write down the relationship between $\mathbf{k}^*$ and $\mathbf{k}^r$,

$$\mathbf{k}^r = (k_\parallel^r, \mathbf{k}_\perp^r) = (\gamma\beta b^* + \gamma k_\parallel^*, \mathbf{k}_\perp^*) \equiv \mathcal{A}\,\mathbf{k}_\parallel^* + \mathcal{B}\,\mathbf{P} + \mathbf{k}_\perp^*, \quad (9)$$

$$\beta = \frac{|\mathbf{P}|}{\sqrt{a^{*2} + \mathbf{P}^2}}, \qquad \mathcal{A} = \gamma = \frac{\sqrt{a^{*2} + \mathbf{P}^2}}{a^*}, \qquad \mathcal{B} = \frac{b^*}{a^*}. \quad (10)$$

By the different choices of a^* and b^*, we will have three different transformation at least. Two of them are exact same as the forms developed in Refs. [11] and [10], named as **KSS** and **RG**, respectively. The another new one is named as **LWLY**.

- r=**LWLY**,

$$a^* = \omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*), \quad b^* = \omega_1(\mathbf{k}^*), \quad \mathcal{J}^r = \frac{\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)} \frac{\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)}{\omega_1(\mathbf{k}^r)\omega_2(\mathbf{P} - \mathbf{k}^r)},$$

$$\mathbf{k}^r = \frac{\sqrt{(\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*))^2 + \mathbf{P}^2}}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}\mathbf{k}_\parallel^* + \frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}\mathbf{P} + \mathbf{k}_\perp^*,$$

$$\mathbf{k}^* = \frac{\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)}{\sqrt{(\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r))^2 - \mathbf{P}^2}}\mathbf{k}_\parallel^r - \frac{\omega_1(\mathbf{k}^r)}{\sqrt{(\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r))^2 - \mathbf{P}^2}}\mathbf{P} + \mathbf{k}_\perp^r.$$

- r=**KSS**,

$$a^* = E^*(q)\,, \quad b^* = \omega_1(\mathbf{k}^*)\,, \quad \mathcal{J}^r = \frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k}^r)}$$

$$\mathbf{k}^r = \frac{E(q)}{E^*(q)}\mathbf{k}^*_{\parallel} + \frac{\omega_1(\mathbf{k}^*)}{E^*(q)}\mathbf{P} + \mathbf{k}^*_{\perp}\,, \quad \mathbf{k}^* = \frac{E(q)}{E^*(q)}\mathbf{k}^r_{\parallel} - \frac{\omega_1(\mathbf{k}^r)}{E^*(q)}\mathbf{P} + \mathbf{k}^r_{\perp}\,.$$

- r=**RG**,

$$a^* = E^*(q)\,, \quad b^* = \frac{E^*(q)}{2} + \frac{m_1^2 - m_2^2}{2E^*(q)} = \omega_1(q)\,, \quad \mathcal{J}^r = \frac{E^*(q)}{E(q)}\,,$$

$$\mathbf{k}^r = \frac{E(q)}{E^*(q)}\mathbf{k}^*_{\parallel} + \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{E^{*2}(q)}\right)\mathbf{P} + \mathbf{k}^*_{\perp}\,, \quad \mathbf{k}^* = \frac{E^*(q)}{E(q)}\left(\mathbf{k}^r_{\parallel} - \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{E^{*2}(q)}\right)\mathbf{P}\right) + \mathbf{k}^r_{\perp}\,.$$

We will find that in r =**KSS**, the first particle is always on-shell while in r =**RG**, the arrangement of energies follows the masses of the two particles which are both off-shell. Regardless, both transformations are dependent on energy E^*. On the contrary, in r =**LWLY**, the transformation is independent of energy, which ensures that the potential energy in the moving frame is also independent of energy. Consequently, the eigenvectors of the Hamiltonian constructed in the discrete momentum space are complete and orthogonal.

4. The application of LWLY in $\pi\pi$ s-wave scattering in the finite volume

In this section, we will give an example to show how to use r =**LWLY** in the HEFT. We will have a potential of s-wave $\pi\pi$ scattering as a s-channel bare σ exchange. Then we only need to know the coupling between σ and $\pi\pi$, and the form of such coupling is as follows,

$$g(q) = \frac{g_{\pi\pi}}{\sqrt{m_\pi}} \frac{1}{1 + (c \times k)^2}, \tag{11}$$

where $g = 0.647$ and $c = 0.440$ fm. In addition, we also need a bare mass of σ as $m_\sigma = 948.96$ MeV. In the HEFT, we will have a Hamiltonian matrxi in the finite volume with total momentum $\vec{P}$ as follows,

$$H = \begin{pmatrix} m_\sigma & g^L(\vec{k}_1, \vec{P}) & g^L(\vec{k}_2, \vec{P}) & \cdots \\ g^L(\vec{k}_1, \vec{P}) & \sqrt{(\omega_\pi(\vec{k}_1) + \omega_\pi(\vec{P} - \vec{k}_1))^2 - \vec{P}^2} & 0 & \cdots \\ g^L(\vec{k}_2, \vec{P}) & 0 & \sqrt{(\omega_\pi(\vec{k}_2) + \omega_\pi(\vec{P} - \vec{k}_2))^2 - \vec{P}^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{12}$$

where, $g^L(\vec{k}_1, \vec{P}) = \left(\frac{2\pi}{L}\right)^{\frac{3}{2}} \frac{g(|\vec{k}^*|)}{\sqrt{4\pi}} \mathcal{J}^{\mathbf{LWLY}}$.

In Fig. 2, we show the finite energy levels vs lattice size based on the same potential model but with different approach. The scheme **KSS/RG** use the phase shift generated by the potential model as input to obtain energy levels by solving the eigenvalue equation of the quantization conditions as shown in Eq. (5), while the scheme **LWLY** directly computes the eigenvalues of the finite volume Hamiltonian matrix as shown in Eq. (12) by using the same potential model. It can be seen that their results are very consistent at large lattice size.

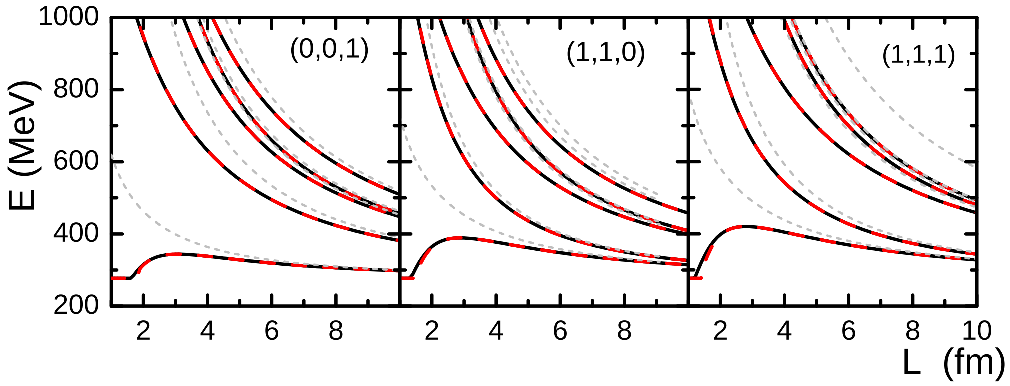

Figure 2: Spectra for systems with total momentums $\frac{2\pi}{L}(0,0,1)$, $\frac{2\pi}{L}(1,1,0)$ and $\frac{2\pi}{L}(1,1,1)$ solved in the scheme **KSS/RG** (red dashed) and the scheme **LWLY** (black solid) with the pure S-wave phase shift. Gray short dotted lines represent non-interacting energies.

5. Summary

In this proceeding, we introduce the three-momentum transformation in the finite volume. Our generalized boost transformation can reproduce previous work, and also provide a new one. The new one transformation parameters are all independent on the total energy in the system, which is suitable for using the HEFT to compute the energy levels of finite volume. At last, we also make an example to show our method is consistent with previous ones.

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