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Lattice fermions, topological phases and Floquet insulators

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Periodically driven quantum systems defined in continuous time, also known as Floquet systems, share intriguing similarities with static/undriven lattice field theories defined in discrete time. E.g. in the former, periodic driving leads to Brillouin zone in quasi-energy space which is reminiscent of frequency Brillouin zones in the latter. These similarities lead to a natural question, *is there a concrete correspondence between the two systems?* In this work I address this question and demonstrate that there indeed exists a concrete mathematical correspondence between a certain $1+1$ dimensional non-interacting Floquet system and a $1+1$ dimensional lattice Dirac fermion defined on naively discretized time lattice. I also comment on the possibility of extending this type of correspondence to higher dimensional theories.

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1. Introduction

In the action formulation of lattice quantum field theory, it is natural to discretize space and time which lead to Brilliouin zones in momenta and frequency. This means that frequency and momenta are conserved on the lattice modulo 2π divided by the spatial and temporal lattice spacing. In the real world, one can find analogous Brilloun zones in momenta in materials with crystal structure. In fact, the presence of momenta Brillioin zones in both systems sometimes allows the spectra of one being mimicked by the other.

However, the story is somewhat different when it comes to time discretization. In the real world time is never discrete. This leads one to believe that it is impossible to replicate the spectra of a discrete time lattice field theory in the real world where time is continuous. There is an exception to this wisdom. It turns out that, periodically driven quantum systems, can exhibit Brilluoin zones in a quantity known as the quasi-energy. Quasi-energy is analogous to energy and conserved modulo 2π divided by the inverse drive time T. The presence of this Brilloiun zone opens up the question of whether the spectra of a discrete time static lattice theory can be replicated using periodic driving in continuous real time or vice versa. In this note, I show that this is indeed possible in $1 + 1$ space-time dimension. In detail, I build a concrete mathematical correspondence between the two by constructing a one to one map between the spectra of a Floquet system in $1 + 1$ dimension and a discrete time lattice Dirac fermion theory, also in $1 + 1$ dimensions. The mapping requires one to identify the time lattice spacing of the lattice Dirac fermion theory with the drive time of the Floquet system.

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2. Fermion doubling

Before I can present the correspondence between the Floquet system and the discrete time setup, I have to discuss the concept of fermion doubling in lattice field theory, Consider the Dirac equation $(i\gamma^\mu \partial_\mu - m)\phi = 0$. We rewrite this in a form that resembles a Schroedinger equation with a Dirac Hamiltonian H_D,

$$(i\partial_0 - H_D)\phi = 0. \tag{1}$$

Let's now discretize the time direction by replacing ∂_0 by the symmetric finite difference operator $\nabla_{t,t'} = \frac{\delta_{t,t'-1} - \delta_{t,t'+1}}{2T}$ where we have deliberately chosen the time lattice spacing to be T. Now one can Fourier transform Eq. 1 with this naively discretized time derivative, leading to

$$\sin(p_0 T)\phi(p) - H_D \phi(p) = 0. \tag{2}$$

Diagonalizing H_D we see that, if ϕ_e is a specific eigenstate of H_D with eigenvalue ϵ_D, the corresponding time evolution : $e^{i\sin^{-1}(\epsilon_D T)\frac{t}{T}}\phi_e$, solves the above equation 2. At the same time, there is another solution to Eq. 2 which goes as $e^{i\left(\frac{\pi}{T} - \frac{\sin^{-1}(\epsilon_D T)}{T}\right)t}\phi_e$. This second solution is called a doubler and the phenomenon goes by the name of fermion doubling. The main point to remember here is that for every frequency solution to the above equation that goes as one of the eigenvalues of H_D, i.e. ϵ_D, there is another one which goes as $\pi/T - \epsilon_D$.

3. The Floquet insulator model

Now, consider the following $1+1$ dimensional lattice model defined on $2N$ spatial lattice sites where the time evolution operator is given by

$$U(t) = \begin{cases} e^{-iH_0 t} & \text{for } 0 < t < t_0 \\ e^{-iH_1(t-t_0)} e^{-iH_0 t_0} & \text{for } t_0 \le t < t_0 + t_1 \end{cases}, \tag{3}$$

with

$$\begin{aligned} H_0 &= 2\sum_{j=0}^{N-1} (a^\dagger_{2j} a_{2j+1} + \text{H.c.}) \\ H_1 &= 2\sum_{j=0}^{N-1} (a^\dagger_{2j+1} a_{2j+2} + \text{H.c.}), \end{aligned} \tag{4}$$

where a_i is a fermion annihilation operator on site $i = 0, \ldots, 2N-1$. Clearly the time evolution operator here is describing a model with periodic driving. One could consider this model with both periodic or open boundary condition.

The individual Hamiltonian H_0 and H_1 shown in Eq . (3), (4) correspond to two different parameter regimes of a class of models known as the SSH model [1], [2],. The Hamiltonian of a generic SSH model is given by

$$H_{\text{SSH}} = \frac{u}{2} H_1 + \frac{v}{2} H_0 \tag{5}$$

where u and v are some arbitrary constants. Under periodic boundary condition the model has spectra given by

$$E_{\text{SSH}}(k) = \pm\sqrt{u^2 + v^2 + 2uv\cos(2k)}, \tag{6}$$

where $0 \le k < \pi$ is the crystal momentum and spatial lattice spacing has been set to 1. The dispersion relation of the SSH model describes a lattice Dirac fermion with mass $u - v$ when expanded above $k = \pi/2$. Interestingly, massive Dirac fermion in any number of space-time dimension can be in a topological phase depending the sign of the mass. According to the convention we pick here, the SSH model is in a topological phase when $m > 0$ and is in a trivial phase when $m < 0$. Thus, H_0 represents a trivial phase and H_1 a topological phase. Interestingly, with periodic boundary condition spectra there is no way to tell the difference between $m = |m_0|$ and $m = -|m_0|$. However, with open boundary condition (OBC), the $m > 0$ regime or the topological regime hosts massless (zero energy) edge states whereas the $m < 0$ regime hosts nonce. So, the edge spectra of the two phases differ under OBC.

The Floquet spectra is extracted from the time evolution operator of the driven system sampled at integer multiples of the drive time. In this case the Floquet evolution operator is given by

$$U_{\text{F}} = U(T) = e^{-iH_1 t_1} e^{-iH_0 t_0} \equiv e^{-iH_{\text{F}} T}, \tag{7}$$

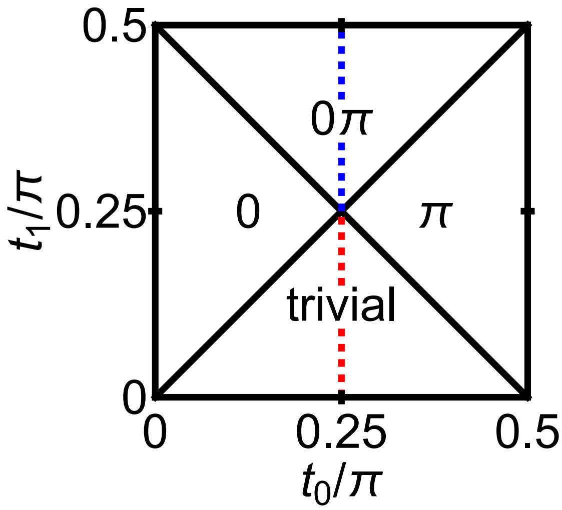

Figure 1: Phase diagram of the Floquet model (7). The phases are labeled by the presence or absence of zero and π modes localized to boundaries. This work focuses on the vertical dashed line at $t_0 = \frac{\pi}{4}$, which passes through the trivial and 0π phases.

where $T = t_0 + t_1$ is the driving period and $H_F = \frac{i}{T} \ln U_F$. H_F is the Floquet or stroboscopic Hamiltonian. The eigenvalues of the stroboscopic Hamiltonian are known as the quasi-energy and denoted as ϵ. They are conserved modulo $2\pi/T$. The Floquet quasi-energy exhibits periodicity set by $2\pi/T$ which is indicative of the presence of a Brillouin zone in quasi-energy. The quasi-energy spectra of this model can be used to investigate the different possible topological phases this model can be in. In fact the spectra reveals that the Floquet system can be in four different topological phases as a function of t_0 and t_1 as shown in Fig 1. They are labeled trivial, 0, π, and 0π. The quasi-energy gap with PBC closes on the phase boundaries of these topological phases. One of the ways to confirm that these four regions of the parameter space indeed describes different topological phases is to consider the Floquet spectra with OBC. Under OBC, the trivial phase does not exhibit any edge state, the 0 phase exhibits zero quasi-energy edge states, the π phase exhibits edge states with quasi-energy π/T and the 0π phase exhibits edge states with both quasi-energy 0 and π/T.

Upon closer inspection, one finds an intriguing feature of the quasi-energy spectra for $t_0 = \pi/4$. This is a vertical line, colored red and blue and shown in dashes, in Fig. 1 phase diagram, along which the quasi-energy eigenvalues are "π paired" under both OBC and PBC. The phrase π-pairing refers to the feature that for every quasi-energy eigenvalue ϵ, there is another one with eigenvalue $\pi/T - \epsilon$. Moreover, the PBC eigenvalues on $t_0 = \pi/4$ line are reflection symmetric across the horizontal line $t_1 = \frac{\pi}{4}$. This is to say that with PBC, $\epsilon\big|_{t_0=\frac{\pi}{4},\eta} = \epsilon\big|_{t_0=\frac{\pi}{4},-\eta}$ where $\eta \equiv t_1 - \frac{\pi}{4}$.

This suggests that the part of the Floquet phase diagram along $t_0 = \pi/4$ may be mappable to a lattice fermion theory with naively discretized time direction such that the spectra exhibits π-pairing or fermion doubling.

At this point it's useful to notice the pattern of π pairing with PBC as illustrated by Fig 2. As is shown in the Fig.2 the blue and the orange bands are π paired whereas the eigenvalues within the blue or the orange bands are not π paired with each other. We first aim to construct a lattice

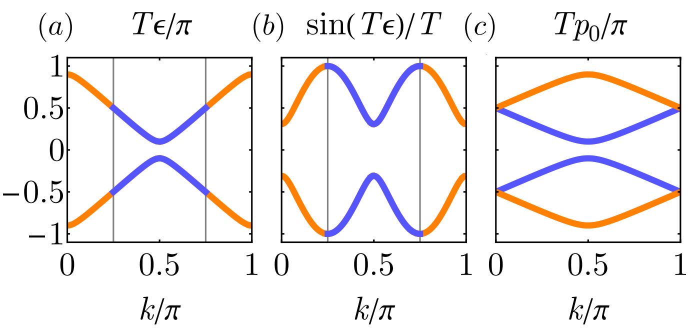

Figure 2: On the leftmost panel we show PBC quasi-energy eigenvalues as a function of crystal momenta for $t_0 = \pi/4$ for some representative value of t_1. The middle panel shows the sine of the eigenvalues. The rightmost panel shows the frequency solutions of the discrete time Schroedinger equation where one uses the sine of the blue band from the middle panel as eigenvalues of the target static Hamiltonian.

fermion Hamiltonian H_s which when fed to the frequency space version of the naively discretized Schroedinger/Dirac equation

$$i\nabla_0\psi = H_s\psi, \tag{8}$$

i.e.

$$\sin(p_0T)\psi(p) = (TH_s)\psi(p) \tag{9}$$

produces frequency solutions that match the PBC Floquet eigenvalues one to one. Diagonalizing H_s we can get the PBC eigenvalues of the target lattice Hamiltonian H_s which we call ϵ_s. It is now easy to see that in order to reproduce the Floquet quasi-energy eigenvalues, we have to demand that the eigenvalues ϵ_s match onto the sine of the Floquet quasi-energy eigenvalues. Since the blue and the orange bands are $\pi-$ paired, taking the sine of the quasi-energy eigenvalues produces a double degeneracy as shown in the middle panel of the Fig. 2. In order to construct the map to a discrete time theory, we need to retain only one set of the eigenvalues discarding the corresponding degenerate set. E.g. we could work with just the sine of the blue band, i.e. $\sin(T\epsilon(k))$ for $\frac{3\pi}{4} > k \geq \frac{\pi}{4}$. Solving,

$$\sin(p_0T) = \sin(T\epsilon(k)) \tag{10}$$

for $\frac{3\pi}{4} > k \geq \frac{\pi}{4}$ we now produce the blue band solution with $p_0 = \epsilon(k)$ for $\frac{3\pi}{4} > k \geq \frac{\pi}{4}$ and their π pairs, $p_0 = \frac{\pi}{T} - \epsilon(k) = \epsilon(k + \frac{\pi}{2})$ which reproduces the orange band quasi-energy eigenvalues.

Remarkably, we find that $\sin(T\epsilon(k))$ for $\frac{3\pi}{4} > k \geq \frac{\pi}{4}$ can be fit to the eigenvalues of a massive Dirac Hamiltonian itself, e.g. an SSH Hamiltonian. Only the point $t_0 = t_1 = \pi/4$ maps to a massless Dirac Hamiltonian. The details of this fit is given in [3]. The fit for the PBC eigenvalues work with a Dirac Hamiltonian with a positive mass as well as a negative mass of the same magnitude. This is expected given that the PBC eigenvalues of a Dirac Hamiltonian does not care about the sign of the Dirac mass. Given that the PBC quasi-energy eigenvalues satisfy $\epsilon\big|_{t_0=\frac{\pi}{4},\eta} = \epsilon\big|_{t_0=\frac{\pi}{4},-\eta}$, we have two possible mapping choices for $\eta = \pm|\eta|$, e.g. let's say that the sine of the PBC quasi-energy eigenvalues for a specific $\eta = \eta_0$ with $\eta_0 > 0$ can be fit to a Dirac Hamiltonian of mass $+m_0$ or $-m_0$

with $m_0 > 0$, then the same is true of the sine of the quasi-energy eigenvalues for $\eta = -\eta_0$. If we now want to reproduce the OBC quasi-energy spectra as well, our choices get restricted. We see from the phase diagram, that, for $t_1 > \pi/4$, along the $t_0 = \pi/4$ line one finds both 0 and π quasi-energy edge states whereas for $t_1 < \pi/4$ there aren't any. Therefore, the correct choice of map for $t_1 > \pi/4$, i.e. some $\eta = \eta_0 > 0$, is a positive mass Dirac fermion of mass $m = m_0$ with $m_0 > 0$. Similarly, for $t_1 < \pi/4$, the correct choice for the corresponding $\eta = -\eta_0$ would be, a negative mass Dirac fermion with $m = -m_0$. Since the positive mass Dirac Hamiltonian has zero energy edge states under OBC, the corresponding discrete time theory with naive time discretization has both zero and π frequency edge states matching onto the spectra for quasi-energies for $t_1 > \pi/4$. Similarly, the negative mass Dirac Hamiltonian has no edge states of zero energy and therefore the corresponding discrete time theory has neither zero frequency nor π frequency edge states, matching onto the OBC quasi-energy spectra for $t_1 < \pi/4$. This is the essence of the mapping described in [3]. Of course, the correspondence described in this talk pertains to the $t_0 = \pi/4$ line with a $\pi-$ paired spectra. However, there has been follow up work which extend this mapping away from $t_0 = \pi/4$ [4].

4. Summary and future directions:

Periodically driven systems observed at integer multiples of drive time are often referred to as discrete time systems in a vague sense. We made this notion concrete for a certain $1 + 1$ D Floquet system by demonstrating that its spectra can indeed be reproduced from a discrete time theory with a static Hamiltonian. This mapping has been extended to the cases where there is no $\pi-$ pairing. There has also been work on the possibility of extending the map to higher dimensions. For reference see [5]. There remain several exciting unanswered questions:

1. How general is this mapping? Under what conditions do such correspondence hold?
2. Does a similar map exist between interacting Floquet system and discrete time static lattice theories?
3. How would one identify which interacting theories on the Floquet side can be mapped to which interacting theory on the lattice side?
4. Does the correspondence amount to a duality relation between the two sides? If so, how would one formulate such a duality?

Beyond exploring the nature of the correspondence between discrete time static theories and Floquet systems, it may be worthwhile considering whether this correspondence can be made useful for the quantum simulation of certain types of fermion theories.

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