PROCEEDINGS OF SCIENCE

Transverse momentum moments

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The transverse momentum moments (TMMs) are integrals of Transverse Momentum Dependent distributions (TMDs) over transverse momentum, weighted by powers of k_T. In this work, we establish robust relations between TMMs and collinear distributions. Specifically, we prove that the zeroth TMM corresponds to collinear twist-two distributions and derive a conversion factor to express it in the conventional $\overline{\text{MS}}$-scheme. The first and second moments are related to twist-three and twist-four collinear distributions, respectively. We discuss the applications of the zeroth, first, and second TMMs and provide phenomenological results for them based on current TMD extractions. These results open new avenues for the theoretical and phenomenological investigation of three-dimensional and collinear hadron structures.

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1. Introduction

It is well known that 1D and 3D structures are intimately related. Given a 3D density distribution, one can obtain the corresponding 1D distribution by integrating out the appropriate degrees of freedom. Consequently, one would expect a similar relationship between Transverse Momentum Dependent parton distribution functions (TMDs), which describe parton distributions in terms of 3D momentum, and collinear distributions, which describe partons with respect to longitudinal momentum only. Intuitively, 3D TMDs can be reduced to 1D PDFs by "integrating out" the parton's transverse momentum, k_T [1, 2]. However, this relation is not straightforward. In this talk, I present the recent study [3] that unambiguously resolves this problem and defines the relations between the integrals of TMD distributions (so-called Transverse Momentum Moments, or TMMs) and collinear PDFs.

The origin of the problem lies in the fact that TMDs and PDFs are not probability distributions in the strict mathematical sense, but are only interpreted as such. Therefore, a naive relation between TMMs and PDFs does not hold. To establish such a relation, one must return to the operator definition of these functions and consider the transformation of one operator into another. Herewith, the integral over k_T is related to the limit of small transverse distance (small-b), and the integral with k_T^n-weight corresponds to $\sim b^n$ term in the small-b expansion. The small-b expansion of the TMD distribution is well-studied. In particular, the leading $\sim b^0$ term has been derived up to two loops [4, 5] and three loops [6, 7], while the subleading $\sim b^1$ term is known up to one loop [8, 9] (for a review of higher power terms, see ref. [10]). Therefore, the theoretical relation between TMMs and PDFs is known, but it must be translated into a practical form that allows one to determine PDFs from TMDs numerically, without referring to their operator definitions. Such relations are very important in phenomenology because they (i) enable direct and systematic comparison between TMDs and PDFs, (ii) open new possibilities for joint fits of TMDs and PDFs, and (iii) provide novel and unique sources of information about higher-twist collinear distributions.

The key problem in any direct comparison of TMDs and PDFs is scale dependence. As a consequence of the factorization theorem, TMDs depend on two scales: the ultraviolet (UV) renormalization scale μ and the renormalization scale of the rapidity divergence ζ, whereas PDFs depend solely on the UV renormalization scale. The corresponding evolution equations have different analytical structures: a pair of differential Collins-Soper equations diagonal in flavor space, with a nonperturbative Collins-Soper (CS) kernel, versus nondiagonal integrodifferential Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations, or DGLAP-type equations [11], for higher twist distributions. Another issue is the divergence of the integral of TMDs over k_T, which is equivalent to the UV divergence of PDFs and requires regularization. The relation between 3D and 1D structures must be formulated in such a way that the corresponding quantities obtained from 3D densities obey the same evolution equations as their 1D counterparts, while simultaneously exhibiting no dependence on the Collins-Soper kernel. The natural way to resolve both problems is to regularize the integral over k_T and introduce an additional UV scale (this has also been suggested and studied in Refs. [12, 13]). This UV scale must be related to the DGLAP part of the evolution, as it corresponds to the limit of local operator. The left scales of TMD evolution should be fixed in such a way that they do not introduce additional non-perturbative parts. The resulting object is collinear distribution computed in some minimal subtraction scheme (see proof in [3]), in some cases it could be corrected by a factor to the $\overline{MS}$-scheme.

In this work, we have considered weighted integrals with a momentum cutoff $|k_T| < \mu$, with $\mu \gg \Lambda_{\text{QCD}}$. We have proven that they are equivalent to the corresponding collinear distributions. The critical element of our construction is the selection of scales for TMDs. It must be done such that the nonperturbative Collins-Soper kernel is eliminated as the collinear matrix elements do not depend on it. We identify two cases where this elimination can be achieved:

- The TMD is evaluated using the ζ prescription [14]. In this case, TMMs are defined as

$$\mathcal{M}^{[\Gamma]}_{\nu_1...\nu_n}(x,\mu) \equiv \int^{\mu} d^2\vec{k}_T\, \vec{k}_{T\nu_1}...\vec{k}_{T\nu_n} F^{[\Gamma]}(x,k_T), \tag{1}$$

  where the TMD on the right-hand side is the optimal TMD. It is defined at the point of vanishing Collins-Soper kernel, and thus any TMM does not depend on the Collins-Soper kernel.

- The TMD is evaluated with all involved scales equal to μ, i.e.

$$\mathring{\mathcal{M}}^{[\Gamma]}_{\nu_1...\nu_n}(x,\mu) \equiv \int^{\mu} d^2\vec{k}_T\, \vec{k}_{T\nu_1}...\vec{k}_{T\nu_n} F^{[\Gamma]}(x,k_T;\mu,\mu^2). \tag{2}$$

  In this case, the elimination of the Collins-Soper kernel is not by construction but it happens due to the properties of the OPE and the TMM integral at $n = 0, 1, 2$.

The interpretation of the integral critically depends on the value n. The higher values of n result in a higher-order divergence in μ (generally $\sim \ln\mu$ for $n = 0, 1$ and $\sim \mu^{2[n/2]}$ for higher n's, but this power may vary depending on kind of the TMD distribution). Therefore, for $n > 1$ one needs to make additional subtractions to find out the collinear distributions. The procedure is straightforward, but complicates for higher n. For this reason, we have considered only the lowest-power weights ($n = 0, 1, 2$) as they offer the most interesting phenomenological information:

- The zeroth TMM, also explored in Refs. [12, 13], relates TMDs and collinear parton distribution functions (twist-two PDFs). We provide the next-to-next-to-next-to-leading order (N^{3}LO) expression for the matching between the TMD and MS schemes. The derived relations are verified numerically using unpolarized TMDs from the recent N4LL extraction ART23 [15], demonstrating the consistency of the method.

- The first TMM provides information about the distribution of the average transverse momentum shift of partons [2, 16], and it is expressed via collinear distributions of twist three [17]. Unlike the zeroth TMM, the transformation to the MS scheme is not applicable here due to the loss of information upon integration. However, the first TMM yields important information. As an illustrative example, we consider the extraction of the Sivers function at N^{3}LO [18, 19] and compute the Qiu-Sterman functions, along with the average transverse momentum shifts of quarks in a transversely polarized proton.

- The second moment is related to the average absolute value of transverse momentum and the quadrupole distribution, and it is generally described by collinear operators of twist four. Reduction to the TMD scheme requires a subtraction procedure of power divergences, which we devise for the unpolarized case. Results are substantiated phenomenologically using the data from ART23 [15].

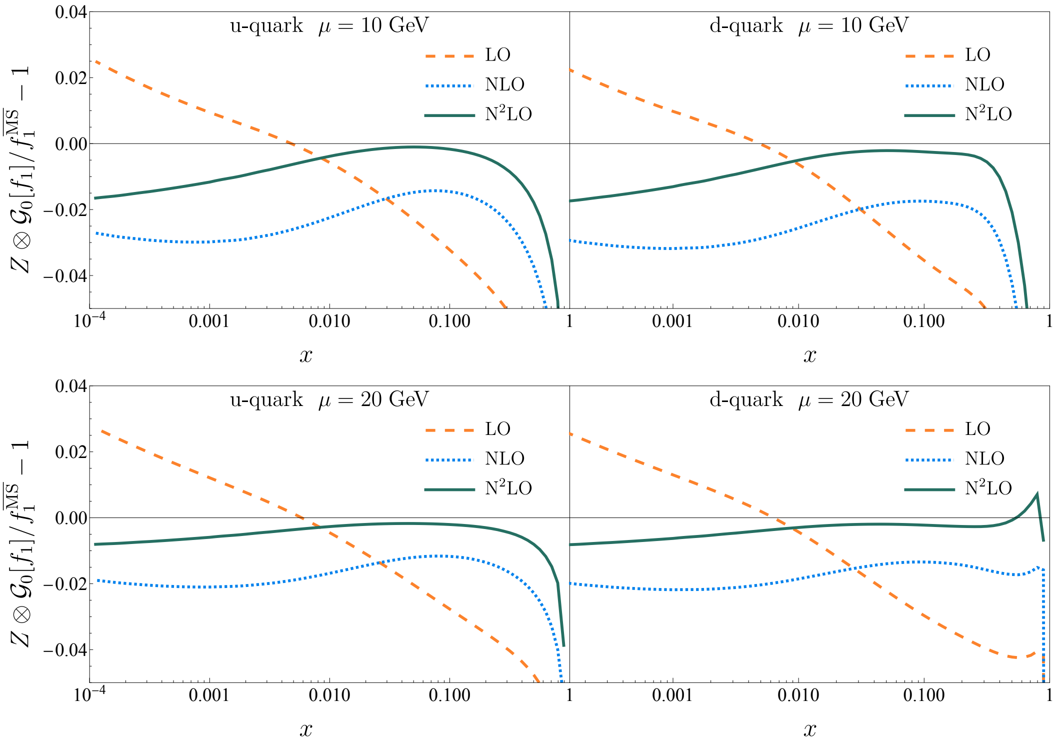

Figure 1: Comparison of unpolarized PDF for u and d quarks determined from the unpolarized TMD (central values of ART23 extraction [15]), as a function of x at fixed $\mu = 10$ (the upper row) and 20 GeV (the bottom row). The plots show the deviation from the $\overline{\text{MS}}$ value which was used in the fit of TMD (extraction MSHT20 [22]). Dashed orange lines, dotted blue lines, and solid green lines correspond to the LO, NLO, and NNLO order of factor $Z^{\overline{\text{MS}}/\text{TMD}}$.

In the rest of the text, I highlight the main phenomenological results which were obtained in the work [3], and in some subsequent works.

2. The moments of unpolarized TMD distribution

The unpolarized TMD distribution $f_1(x, k_T)$ is the most studied TMD distribution. The information about it can be extracted from the Drell-Yan process and thus provided by a large amount of collider experiments. The most resent and detailed determinations are ART23 [15], SV19 [20], and MAP22 [21]. These determinations are done at N^{3}LO and N^{2}LO perturbative accuracy. The integral of $f_1(x, k_T)$ leads to the unpolarized PDF in the "TMD-scheme". The $\overline{MS}$-expression can be obtained by convolution with the finite-renormalization constant. The resulting expression in the ζ-prescription (1) reads

$$f_1(x, \mu) = \int_x^1 \frac{dy}{y} \left(Z^{(\text{TMD}/\overline{MS})}(y)\right)^{-1} \int^{\mu} d^2\vec{k}_T f_1\left(\frac{x}{y}, k_T\right), \tag{3}$$

where Z is the finite renormalization constant

$$\left(Z^{(\text{TMD}/\overline{MS})}(x)\right)^{-1} = 1 - \frac{\alpha_s}{4\pi} C_F \left(1 - x + \frac{\pi^2}{6}\delta(1-x)\right) + O(\alpha_s^2). \tag{4}$$

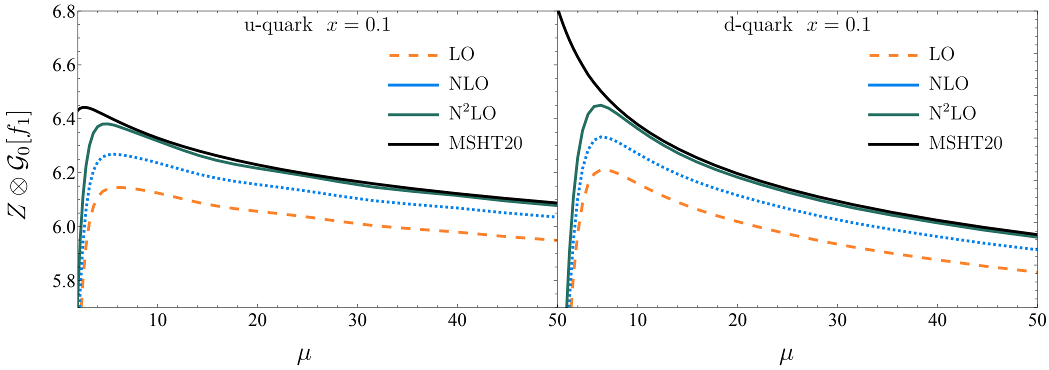

Figure 2: Comparison of unpolarized PDF for u and d quarks determined from the unpolarized TMD (extraction ART23 [15]), as a function of μ at fixed $x = 0.1$. The plot shows the deviation from the $\overline{\text{MS}}$ value which was used in the fit of TMD (extraction MSHT20 [22]). Different lines correspond to different orders of correction factor $Z^{\overline{\text{MS}}/\text{TMD}}$.

The higher order terms (up to term $\sim \alpha_s^3$) can be found in ref.[3]. In the Collins-Soper scheme (2) the expression receives different factor Z, which however, coincides with (4) at one-loop.

In figure 1, we demonstrate the comparison of result of application of this formula with corresponding unpolarized PDF. The comparison is done with MSHT20 collinear PDF [22], because it was used in the definition of ART23. The comparison shows the very good convergence of TMM to PDF (apart of the region of large-x, which is possibly due to the contribution of threshold logarithms). The figure 2 demonstrates the μ-dependence of determined PDF (which is governed by the DGLAP equation). The agreement is amazing. The deviation starts for $\mu < 5 - 7$GeV, which indicates the typical scale of new non-twist-two contributions.

The second moment of the unpolarized distribution brings us the information about the twist-4 collinear distribution. To reach it, one should subtract the asymptotic $\sim \mu^2$ term. It leads to the definition

$$f_{\text{tw-4}}(x,\mu) = \int^{\mu} d^2\vec{k}_T \vec{k}_T^2 f_1(x,k_T) - \text{AS}[f_1(x,\mu)], \tag{5}$$

where the subtraction term contains only the twist-two distribution and known coefficient function. This term is known up to N^{3}LO. The essential check of the consistency of the approach is the demonstration that the resulting function has logarithmic behavior in μ, as it is expected in a minimal subtraction scheme. The numerical check of this fact is provided in ref.[3] (see fig.5).

The twist-four distributions do not have any simple interpretation. However, following the interpretation of unpolarized TMD distribution as a momentum distribution, we can identify

$$\langle \vec{k}_T^2 \rangle_q = \int dx\, f_{q,\text{tw-4}}(x,\mu), \tag{6}$$

as an average transverse momentum carried by a quark q. This object is divergent, due to the fact that the total number of quarks explodes at $x \to 0$. However, the number of valence quarks is preserved and thus the expression

$$\langle \vec{k}_T^2 \rangle_{\text{val}.q} = \int dx \left(f_{q,\text{tw-4}}(x,\mu) - f_{\bar{q},\text{tw-4}}(x,\mu) \right), \tag{7}$$

should remains finite. There is not and *could not be* a formal proof of this statement. However, it could be considered as a limitation for a model of TMD distribution, such that the resulting function has simpler physical interpretation.

Practically, the requirement of finiteness of average valence momentum (7) is similar to the number-sum-rule restriction for collinear PDFs. I.e. it must be imposed in the model for fitting, and consists in the requirement that small-x asymptotic for TMDs are the same for quark and the anti-quark contributions. It is a new requirement, and it was imposed by global fits so far (although we recommend to use it in the future extractions). The corresponding update of ART23 will be published soon. In this update (alongside of other modifications) we fix the parameters that are dominant at small-x, to be the same. The resulting value for the average transverse momentum squared for the valence quark $(u+d)/2$ at $\mu = 10$GeV is

$$\langle \vec{k}_T^2 \rangle_{\text{val.q}} = 1.45 \pm 0.55\text{GeV}^2. \tag{8}$$

This value agrees with typical expectations. Importantly, that with the presented methodology it has a definite operator definition, and thus reproducible by other means.

3. Twist-three distributions

Another exciting possibility granted by TMMs is a controllable access to twist-three collinear distributions. The first TMMs are known integrals of twist-three distributions, and the present state of knowledge of TMDs is often better alternative than a "direct" measurement. Particularly, one finds that

$$\int^{\mu} d^2\vec{k} \frac{\vec{k}_T^2}{2M^2} f_{1T}^{\perp}(x, \vec{k}_T) = \frac{\pi}{2} T(-x, 0, x; \mu), \qquad \int^{\mu} d^2\vec{k} \frac{\vec{k}_T^2}{2M^2} h_1^{\perp}(x, \vec{k}_T) = -\frac{\pi}{2} E(-x, 0, x; \mu), \tag{9}$$

where $f_{1T}^{\perp}$ and $h_1^{\perp}$ are Sivers and Boer-Mulders functions (defined in Drell-Yan configuration), and T and E are the twist-three collinear distributions (see e.g.[23]). Both distribution are defined in a "TMD-scheme", which is a minimal subtraction scheme that differ from $\overline{\text{MS}}$-scheme at order α_s^2. Using these relation we are able to extract distribution T and E. The example of the determination of T is shown in fig.3. The determination of E is given in ref.[24], which is the first example of determination of this function from experimental data. The worm-gear TMD distributions are also sensitive to the twist-three functions, but they also contain admixture of twist-two distribution (aka Wandzura-Wilczek part).

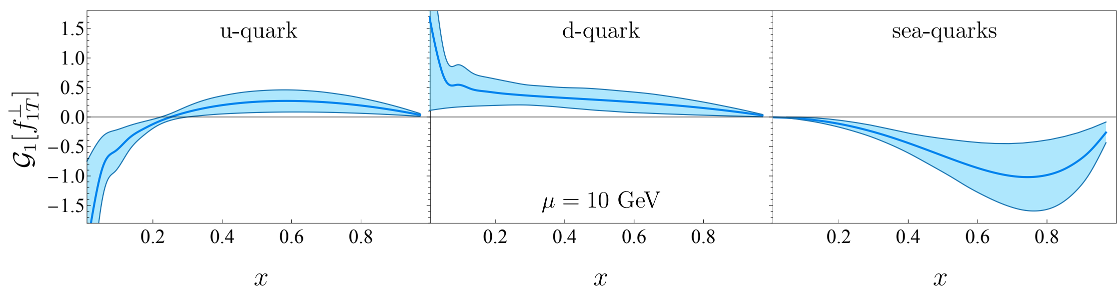

Figure 3: The first TMM for the Sivers function (extraction [18, 19]) for different flavors, computed at $\mu = 10$ GeV.

4. Conclusion

The relationship between TMD distributions and collinear distributions is governed by the operator product expansion (OPE). In ref.[3], we established a practical form of the OPE inversion, which enables the unambiguous determination of collinear counterparts of TMDs from their numerical values. This requires the consideration of integrals of TMDs with respect to transverse momentum, known as transverse momentum moments (TMMs). We derived relations for the zeroth, first, and second TMMs, corresponding to collinear PDFs of twist-two, twist-three, and twist-four, respectively. These relations open a new avenue in the study of hadron properties, as they allow for the systematic determination of various integral properties of partons, such as average transverse momentum and the size of interference between partons. Additionally, TMMs enable a precise cross-check of TMD phenomenological extractions with those based on collinear factorization theorems. These properties will be further explored in future works.

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