PROCEEDINGS OF SCIENCE

Helicity states for two- and three-gluon glueballs

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The existence of glueball states remains an elusive subject. Significant effort, both theoretical and experimental, has been devoted to decide the question. In the current work, a constituent approach to Quantum Chromodynamics is developed to get information about the two- and three-gluon parts of the glueball spectrum. The results obtained are shown to be consistent with other resolution methods, such as Lattice QCD, as long as two helicity degrees-of-freedom are considered for the constituent gluons. A systematic procedure for handling helicity states in potential models is developed for two-body systems, and the generalisation to three-body problems is initiated.

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1. Introduction

One of the earliest predictions of Quantum Chromodynamics (QCD) is the existence of color singlet pure-gauge states known as glueballs [1]. However, despite these theoretical predictions, a consensus on their properties and experimental evidence remains elusive. Two-gluon glueball states have been extensively explored both theoretically [2, 3] and experimentally [3, 4]. On the theory side, results from various phenomenological approaches, from functional methods, and lattice QCD have been obtained. Notable experimental efforts, such as those from PANDA, Crystal Barrel or WA102, continue the search for these states. In contrast, three-gluon glueballs have received less attention due to the technical complexity involved. On the theory side, the lattice QCD spectrum should encompass three-gluon levels [5–7], while on the experimental side, the possible observation of an odderon exchange at TOTEM is still debated [8].

This work aims to apply the helicity formalism to describe two- and three-gluon systems within the framework of constituent models. The discussion begins in Section 2, where the concept of glueballs and its meaning in the framework of constituent approaches is reviewed [9]. Section 3 is dedicated to the two-gluon case [10]. After revisiting the formalism for one-body systems, the two-body formalism [11] is introduced and applied to the description of two-gluon glueballs. In Section 4, the three-body helicity formalism [12, 13] is introduced in order to start the generalisation to three-gluon glueball states. Finally, Section 5 concludes by discussing perspectives on obtaining a quantitative spectrum for three-gluon glueballs.

2. Framework

Constituent approaches model hadronic states as colorless bound states composed of constituent quarks, antiquarks, and/or gluons. The requirement of colorlessness determines whether a specific combination of constituent particles can form a hadron. To illustrate this, consider the example of glueballs. Constituent gluons transform under $SU(3)_c$ with the adjoint representation, denoted 8. Consequently, two- and three-gluon bound states respectively transform according to [9]

$$8 \otimes 8 = 1 \oplus \ldots \tag{1}$$

$$8 \otimes 8 \otimes 8 = 1 \oplus \ldots \tag{2}$$

These two relations indicate that bound states composed of two and three constituent gluons, known as two- and three-gluon glueballs, are allowed within constituent approaches.

To extend the analysis and, for instance, determine a glueball energy spectrum, dynamical considerations are required. The implementation of dynamics varies between models, but in many cases, it involves setting up a Hamiltonian, which is then used in a Schrödinger-like equation. The dynamics of the system also depend on the properties attributed to the constituent gluons. Among these, the mass of the constituent gluon remains a topic of debate. Although the QCD gluon is formally massless, it has been shown to acquire a dynamical mass in the non-perturbative regime [14]. This has led some studies to introduce a massive kinetic energy for constituent gluons (typically with a mass around 0.5 GeV) [15, 16]. On the other hand, it has been shown in [10] that helicity degrees-of-freedom of a massless particle must be taken into account for constituent gluons to accurately reproduce the lattice QCD glueball spectrum. Additionally, previous studies in other constituent approaches have demonstrated that modifications in the kinematics of the system can often be absorbed into adjustments of the potential parameters. As a result, in modelling glueballs, the primary challenge lies in the proper application of the helicity formalism, rather than in the specific choice of kinematics. The way this challenge is addressed will be developed in the following Sections.

3. Two-gluon Glueballs

The previous Section emphasized the importance of incorporating helicity degrees-of-freedom for the constituent gluon. In light of this, the current Section begins by summarizing the main definitions and properties of this formalism, progressing from one-body to two-body systems.

3.1 One- and Two-body Helicity Formalism

The helicity formalism [11] provides a complete set of states in which any one-body state can be decomposed. These states, denoted as $|m; p\theta\phi; s\lambda\rangle$, are defined as eigenstates of the mass P^2, spin W^2, four-momentum P_μ, and helicity Λ operators,

$$W^2 |m; p\theta\phi; s\lambda\rangle = -m^2 s(s+1) |m; p\theta\phi; s\lambda\rangle , \quad P^2 |m; p\theta\phi; s\lambda\rangle = m^2 |m; p\theta\phi; s\lambda\rangle , \tag{3}$$

$$P_x |m; p\theta\phi; s\lambda\rangle = p \cos\phi \sin\theta |m; p\theta\phi; s\lambda\rangle , \quad P_z |m; p\theta\phi; s\lambda\rangle = p \cos\theta |m; p\theta\phi; s\lambda\rangle , \tag{4}$$

$$P_y |m; p\theta\phi; s\lambda\rangle = p \sin\phi \sin\theta |m; p\theta\phi; s\lambda\rangle , \quad P_0 |m; p\theta\phi; s\lambda\rangle = \sqrt{m^2 + p^2} |m; p\theta\phi; s\lambda\rangle , \tag{5}$$

$$\Lambda |m; p\theta\phi; s\lambda\rangle = \lambda |m; p\theta\phi; s\lambda\rangle . \tag{6}$$

The above equations describe states for massive particles. For massless particles, the situation is more intricate, and readers can refer to [17] for details. For the current purpose, it is sufficient to note that massive particles have access to any λ in $\{-s, -s+1, ..., +s\}$, whereas massless particles are restricted to $\lambda = \pm s$. The helicity operator represents the projection of spin along the momentum direction, as described by its expression in terms of the three-momentum and spin operators,

$$\Lambda = \frac{\vec{P} \cdot \vec{W}}{|\vec{P}|^2}. \tag{7}$$

With the basics of one-body helicity states established, these can be used to construct a complete set of helicity states for two-body systems [11, 18]. In particular, the study of two-gluon glueballs requires two-body states in their center of mass frame (CoMF), the total energy and the total angular momentum of the two-body state can respectively be interpreted as the mass and the spin of the glueball. Two-body helicity states at rest are constructed from the tensor product of two one-body helicity states with opposite momenta,

$$|p\theta\phi; \lambda_1\lambda_2\rangle = (-1)^{-s_2} |m_1; p\theta\phi; s_1\lambda_1\rangle \otimes |m_2; p(\pi-\theta)(\pi+\phi); s_2\lambda_2\rangle \tag{8}$$

where the phase factor is added to ensure consistency with conventions used in [11]. By construction these states are eigenstates of the mass, the spin, the four-momentum and the helicity operators for both particles. However, since the two-body bound states that model glueballs are expected to have a given total angular-momentum, it is useful to introduce a second complete set of states that are eigenstates of the total-angular momentum operator J^2. These states are obtained by integrating the previous states over the different momentum direction,

$$|p; JM; \lambda_1\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}} \int \mathrm{d}\cos\theta \mathrm{d}\phi \, D^{J*}_{M\,\lambda_1-\lambda_2}(\phi, \theta, 0)\, |p\theta\phi; \lambda_1\lambda_2\rangle\,. \tag{9}$$

These states will be referred to as two-body J-helicity states. One can immediately notice a selection rule at this stage: J must always be greater than or equal to $|\lambda_1 - \lambda_2|$.

3.2 The Two-gluon Glueball Spectrum

The two-body helicity formalism can now be employed to derive a spectrum for two-gluon glueballs. First, the singlet color state in equation (1) is known to possess positive C-parity and to be symmetrical under the exchange of the gluons [9]. Since gluons are bosons, the spin-space part of the state must also be symmetric.

As an initial step, the two-body J-helicity states must be symmetrised and made parity eigenstates. This has been done in [10] resulting in four sets of J^{PC} states,

$$|p; S_+; J^P = (2k)^+\rangle = \frac{1}{\sqrt{2}}(|p; JM; +1+1\rangle + |p; JM; -1-1\rangle), \tag{10}$$

$$|p; S_-; J^P = (2k)^-\rangle = \frac{1}{\sqrt{2}}(|p; JM; +1+1\rangle - |p; JM; -1-1\rangle), \tag{11}$$

$$|p; D_+; J^P = (2k+2)^+\rangle = \frac{1}{\sqrt{2}}(|p; JM; +1-1\rangle + |p; JM; -1+1\rangle), \tag{12}$$

$$|p; D_-; J^P = (2k+3)^+\rangle = \frac{1}{\sqrt{2}}(|p; JM; +1-1\rangle - |p; JM; -1+1\rangle), \tag{13}$$

where $k \in \mathbb{N}$. The symmetry condition restricts the allowed angular momentum values to either even or odd. Additionally, the absence of small J values in the two last sets is a consequence of the selection rule mentioned earlier. This leads to a well-known result, analogous to Yang's theorem for two-photon systems [19], which forbids the existence of $J = 1$ two-gluon glueballs.

Each of these four sets can be used to construct two-gluon bound states $|\Psi; S_\pm/D_\pm; J^P\rangle$ by integrating on the remaining continuous degree-of-freedom,

$$|\Psi; S_\pm/D_\pm; J^P\rangle = \int \frac{p^2 \mathrm{d}p}{4w_1(p)w_2(p)}\, \Psi(p)\, |p; S_\pm/D_\pm; J^P\rangle\,, \tag{14}$$

where $w_i(p) = \sqrt{m_i^2 + p^2}$ and where $\Psi(p)$ is the helicity-momentum wave function of the glueball, normalised such that

$$\int \frac{p^2 \mathrm{d}p}{4w_1(p)w_2(p)}\, |\Psi(p)|^2 = 1. \tag{15}$$

To derive a glueball spectrum, one can set up a Hamiltonian and, for instance, choose a trial helicity momentum wave function to apply the variational theorem. Considering the same Hamiltonian as in reference [10] and using the following trial wave-function,

$$\Psi_a(p) = A\, p\, e^{-ap^2}, \tag{16}$$

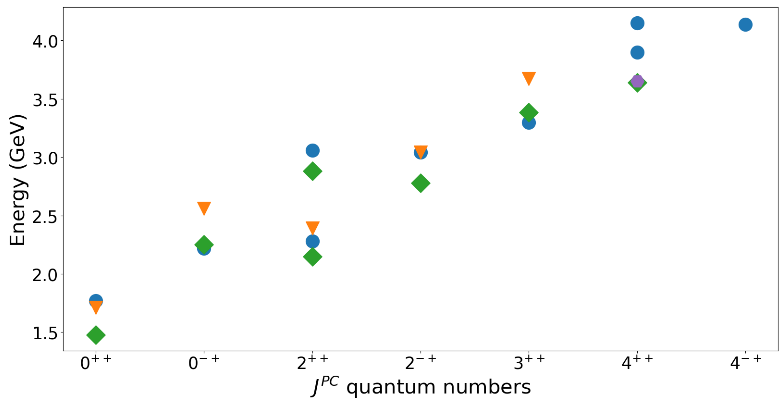

Figure 1: Comparison of two-gluon glueball spectra. Upper bounds obtained with a single Gaussian trial state (blue circles) are compared to lattice QCD results from [7] (orange triangles), [6] (green diamonds) and [5] (purple hexagon).

a mass spectrum, represented by the blue circles in Figure 1, is obtained. These results are consistent with different lattice QCD calculations. The computation of Hamilonian matrix elements is based on expanding the two-body J-helicity states in the LS basis [11, 18] and utilising the matrix element evaluation method in momentum space developed in [20].

4. Three-gluon Glueballs

As already mentioned, two-gluon systems can only produce glueballs with positive charge conjugation. To access the negative charge conjugation part of the spectrum, one must consider glueballs composed of at least three-gluon [9]. This involves working with the helicity formalism for three-body systems. There are two main ways to define three-body helicity states.

4.1 Three-body Helicity States

One approach to generalise the helicity formalism to three-body systems is suggested by Wick [12]. Three-body states with a given total angular momentum are constructed through two successive two-body couplings. First, particles 1 and 2 are coupled in their CoMF to form a two-body J-helicity state,

$$|p'_{12}; j_{12}\lambda_{12}; \lambda'_1\lambda'_2\rangle = \sqrt{\frac{2j_{12}+1}{4\pi}} \int \mathrm{d}\cos\theta' \mathrm{d}\phi' \, D^{j_{12}*}_{\lambda_{12}\,\lambda'_1-\lambda'_2}(\phi',\theta',0)\,|p'_{12}\theta'\phi';\lambda'_1\lambda'_2\rangle \tag{17}$$

where $|p'_{12}\theta'\phi';\lambda'_1\lambda'_2\rangle = (-1)^{-s_2}\,|m_1; p'_{12}\theta'\phi'; s_1\lambda_1\rangle \otimes |m_2; p'_{12}(\pi-\theta')(\pi+\phi'); s_2\lambda'_2\rangle$. The primes indicates variables defined in the CoMF of particle 1 and 2. In the second step, this intermediate two-body state is treated as a particle in its own which is coupled with the third one,

$$\begin{aligned}|p_3; JM; j_{12}\lambda_{12}\lambda_3; p'_{12}\lambda'_1\lambda'_2\rangle \\ = \sqrt{\frac{2J+1}{4\pi}} \int \mathrm{d}\cos\theta \mathrm{d}\phi \, D^{J*}_{M\,\lambda_{12}-\lambda_3}(\phi,\theta,0)\,|p_3\theta\phi; j_{12}\lambda_{12}\lambda_3; p'_{12}\lambda'_1\lambda'_2\rangle\end{aligned} \tag{18}$$

where

$$|p\theta\phi; j_{12}\lambda_{12}\lambda_3; p'\lambda'_1\lambda'_2\rangle = (-1)^{-s_3}\, |p_3\theta\phi; j_{12}\lambda_{12}; p'_{12}\lambda'_1\lambda'_2\rangle \otimes |m_3; p_3(\pi-\theta)(\pi+\phi); s_3\lambda'_3\rangle \tag{19}$$

Above, $|p_3\theta\phi; j_{12}\lambda_{12}; p'_{12}\lambda'_1\lambda'_2\rangle$ is treated as a one-body helicity state for a composite particle of spin j_{12} and mass

$$m_{12} = \sqrt{m_1^2 + p'^2_{12}} + \sqrt{m_2^2 + p'^2_{12}} \tag{20}$$

whose internal degrees-of-freedom are described by the quantum numbers, p'_{12}, λ'_1 and λ'_2.

A second method for defining for three-body helicity states with total angular momentum is proposed by Berman and Jacob [13]. This approach begins with tensor products of three one-body helicity states,

$$\begin{aligned} |\alpha\beta\gamma; w_1w_2w_3; \lambda_1\lambda_2\lambda_3\rangle = U(R(\alpha,\beta,\gamma))\big[\, |p_1\pi/2\,\phi_1; s_1\lambda_1\rangle \otimes |p_2\pi/2\,\phi_2; s_2\lambda_2\rangle \\ \otimes |p_3\pi/2\,\phi_3; s_3\lambda_3\rangle \,\big]. \end{aligned} \tag{21}$$

Above, $U(R(\alpha,\beta,\gamma))$ denotes the unitary operator for a rotation by Euler angles α,β,γ. The momentum $p_i = \sqrt{w_i^2 - m_i^2}$, and the angles ϕ_i are defined as follows,

$$\phi_1 = \varphi_{13} - \pi/2, \quad \phi_2 = \varphi_{13} + \varphi_{12} - \pi/2, \quad \phi_3 = 3\pi/2 \quad \text{where } \cos\varphi_{ij} = \frac{p_k^2 - p_i^2 - p_j^2}{2p_ip_j}. \tag{22}$$

In definition (21), the momenta of the three one-body helicity states enclosed within square brackets lie in the xy plane. This plane is then tilted by the rotation $R(\alpha,\beta,\gamma)$, thereby resulting in a generic three-body helicity-state at rest. These states do not have a definite total angular momentum, but this can be remedied by integrating over the α,β,γ angles,

$$|JM\mu; w_1w_2w_3; \lambda_1\lambda_2\lambda_3\rangle = \sqrt{\frac{2J+1}{8\pi^2}} \int \mathrm{d}\alpha\mathrm{d}\cos\beta\mathrm{d}\gamma\, D^{J*}_{M\mu}(\alpha,\beta,\gamma)\, |\alpha\beta\gamma; w_1w_2w_3; \lambda_1\lambda_2\lambda_3\rangle \tag{23}$$

This definition bears a striking resemblance to the construction of two-body J-helicity state.

4.2 Perspectives on the Study of Three-gluon Glueballs

One might anticipate applying the successful approach used for two-gluon glueball to the case of the three-gluon glueballs. To begin, J-helicity states have to be symmetrised. This is more easily accomplished using Berman's definition, where the three-particles are treated on an equal footing. Trial bound states are then constructed by integrating these symmetrised states over their remaining continuous degrees-of-freedom, namely w_1, w_2 and w_3. Finally, Hamiltonian matrix elements have to be computed to get upper bounds of the three-gluon glueballs masses. However, evaluating two-body potential matrix elements proves simpler with Wick's definition, due to its intermediate two-body coupling.

As a result, determinating a three-gluon glueball spectrum involves in three key steps. First, symmetrising Berman's states. Second, developing a transformation formula between Berman's and Wick's definition. Third, constructing trial states and computing matrix-elements on them, using the results of the first two steps. This work is currently ongoing in our unit.

5. Conclusion

Incorporating helicity degrees-of-freedom for constituent gluons enables to reproduce results from lattice QCD, particularly for the two-gluon sector of the glueball spectrum [10]. Working within the helicity formalism naturally leads to the emergence of selection rules, and even a simple Hamiltonian can quantitatively capture the masses of two-gluon glueballs.

Extending these calculations to three-gluon glueballs would give access to the negative charge conjugation sector of the glueball spectrum [9]. However, this extension is not straigthforward, as the different definitions for three-body helicity states [12, 13] introduce distinct challenge and advantages. Research to address these complexities is ongoing in our unit.

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