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(Non)renormalizable noncommutativity in (non)uniform phase

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In this contribution, we review our recent numerical and analytical results on the phase structure of the matrix version of the Grosse-Wulkenhaar model and its renormalizability. The main numerical result is that the curvature of the model shifts the phase transition lines and eliminates the stripe phase associated with UV/IR mixing. This is confirmed analytically in the strong interaction regime up to 4th order in the curvature coupling. Here we further calculate the 6th order effective action and also provide explicit expressions for some useful integrals over the unitary group with integrands containing up to 12 unitary matrices.

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1. Introduction

A plausible strategy for bridging the gap between the seemingly disparate domains of gravity and quantum theory is to modify the short-distance structure of spacetime [1]. One such modification is realized by the introduction of coordinate noncommutativity (NC), which has been initially envisioned to encode both the symmetries of spacetime and the high-energy cutoffs to QFT [2]. However, NC brings about unexpected complications, in particular for the process of renormalization, due to what is known as UV/IR mixing [3]. This phenomenon entangles physics on large and small scales and undermines their clear separation, a separation that is deemed a necessary ingredient of a successful effective field theory [4].

In the last two decades, the Grosse-Wulkenhaar (GW) model [5, 6] has shown great success in solving the generic renormalization problems of NC theories by introducing an additional action term that can be interpreted as coupling with the curvature of the NC background space [7]. We have recently proposed that the nice behavior of the GW model is related to the phase structure of the model and can be explained by the suppression of the NC striped phase [8, 9].

This contribution aims to summarize our numerical and analytical results obtained while working on this topic in recent years and to present some new methods that could help us to explore it more efficiently.

2. GW model & matrix action

Let us first introduce our model and say a few words about its native NC space. We start with the two-dimensional GW model [5]

$$S_{\text{GW}} = \int dx^2 \bigg(\frac{1}{2}\partial^\mu \phi \star \partial_\mu \phi \; + \; \frac{m^2}{2}\phi \star \phi \; + \; \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \; + \\ + \; \frac{\Omega^2}{2}((\theta^{-1})_{\mu\rho}x^\rho \phi) \star ((\theta^{-1})^{\mu\sigma}x_\sigma \phi) \bigg), \quad (2.1)$$

which lives on the Moyal plane equipped with a $\star$-product

$$f \star g = f \, e^{i/2\,\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} \, g \quad (2.2)$$

and NC coordinates

$$[x^\mu, x^\nu]_\star = i\theta\epsilon^{\mu\nu}. \quad (2.3)$$

The first line in (2.1) is just the nonrenormalizable $\lambda\phi^4_\star$ model; thanks to the Ω-term in the second line, the GW-model in two dimensions becomes superrenormalizable [10].

After applying the Weyl transform and promoting the field ϕ into an $N \times N$ Hermitian matrix Φ, the action (2.1) can be rewritten as a matrix model

$$S = N \,\text{tr}\Big(\Phi\mathcal{K}\Phi - g_r R\Phi^2 - g_2\Phi^2 + g_4\Phi^4\Big) \quad (2.4)$$

on a background space spanned[*] by NC coordinates

$$X = \frac{1}{\sqrt{2N}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & +\sqrt{N-1} \\ & & & +\sqrt{N-1} & \end{pmatrix}, \qquad Y = \frac{i}{\sqrt{2N}} \begin{pmatrix} & -\sqrt{1} & & & \\ +\sqrt{1} & & -\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & -\sqrt{N-1} \\ & & & +\sqrt{N-1} & \end{pmatrix}. \tag{2.5}$$

The price of introducing the finite matrix regularization of the NC coordinates is modification of their commutation relations and curving of the initial Moyal space, where the curvature R in fact contains the energy levels of the Ω-term harmonic oscillator

$$R = \frac{15}{2N} - 8\left(X^2 + Y^2\right) \overset{N\gg 1}{\approx} -\frac{16}{N}\,\mathrm{diag}\,(1, 2, \ldots, N)\,. \tag{2.6}$$

However, the infinite matrix size limit recovers the original commutation relations. As usual, the kinetic operator of the model (2.4) is given by a double commutator w.r.t. NC coordinates

$$\mathcal{K}\Phi = [X, [X, \Phi]] + [Y, [Y, \Phi]]\,. \tag{2.7}$$

During the construction of the matrix model, the NC scale θ is absorbed into definitions of the matrices and couplings of the model and set to unity in order to ensure dealing with dimensionless quantities. Finally, we also introduce the unscaled[†] versions of couplings

$$G_2 = Ng_2\,, \qquad G_4 = Ng_4\,, \tag{2.8}$$

as they will appear in the analytical results concerning renormalization.

3. Phase transitions & renormalization

The finite matrices not only provide a way to regularize the model, but can also be easily simulated on a computer, allowing us to perform nonperturbative calculations. Having defined our matrix action, we can now probe the relevant observables $\mathcal{O}$ by well-defined matrix path integrals

$$\langle \mathcal{O} \rangle = \frac{\int [d\Phi]\,\mathcal{O}\,e^{-S}}{\int [d\Phi]\,e^{-S}}\,, \qquad \mathrm{Var}\,\mathcal{O} = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2. \tag{3.9}$$

With their help, we can in principle deduce the positions of the phase transition lines, which, as advertised in the introduction, appear to signal the (non)renormalizability of the model. We will now discuss the details of this connection.

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*Actually, the background space is 3-dim with coordinates $\sqrt{N}X$, $\sqrt{N}Y$, but we are here interested in their rescaled versions and a subspace where the third coordinate is set to 0 in a weak limit of infinite matrix size, which reproduces the GW-model. We are here also using the rescaled version of the curvature R. More details in [7].

†The unscaled parameters contain the factor N from the action (2.4).

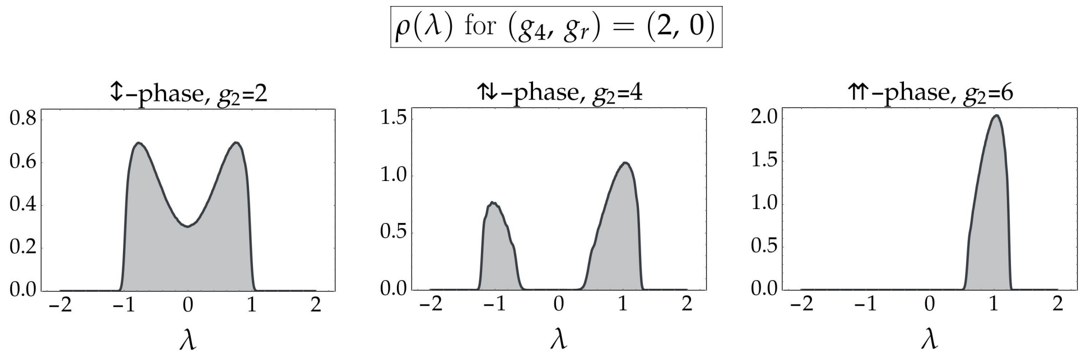

Figure 1: Eigenvalue distribution $\rho(\lambda)$ for the matrix GW model in different phases ($N = 24$).

In order to understand the phase structure of the model, we must first classify its vacuum solutions. Applying the saddle point method to the matrix action (2.4) leads to the following equation of motion

$$2\mathcal{K}\Phi - g_r\,(R\Phi + \Phi R) + 2\Phi\left(2g_4\Phi^2 - g_2\mathbb{1}\right) = 0\,, \tag{3.10}$$

whose solutions corresponding to the kinetic ($\mathcal{K}$), curvature (R), and pure-potential[‡] (g_2, g_4) parts are given, respectively, by

$$\Phi_{\upuparrows} = \frac{\operatorname{tr}\Phi}{N}\mathbb{1}\,, \qquad \Phi_{\updownarrow} = \mathbb{0}\,, \qquad \Phi^2_{\uparrow\downarrow} = \frac{g_2}{2g_4}\mathbb{1}\,. \tag{3.11}$$

The first two solutions are the ordered ($\upuparrows$) and the disordered ($\updownarrow$) vacuums that are also present in commutative models. The third one—the so-called stripe/matrix vacuum ($\uparrow\downarrow$)—is proportional to nontrivial square roots of the identity matrix $\mathbb{1}$ and is therefore the pure consequence of NC. The stripe phase thus contains both positive and negative eigenvalues, which causes it to vary throughout space and break the translational symmetry [11–13]. The field eigenvalue distribution in each of these phases is shown in Figure 1. As their shape suggests, they are also called 1-cut symmetric ($\updownarrow$), 2-cut ($\uparrow\downarrow$), and 1-cut asymmetric phase ($\upuparrows$).

The general structure of the phase diagram is shown[§] in Figure 2, which was obtained[¶] by Hamiltonian Monte Carlo simulations [15, 16] and is quite similar to the phase structure on other fuzzy spaces, for example the fuzzy sphere [17]. The right-hand plot illustrates a result that is important for the renormalization of the GW model: When the curvature term is included in the action, the phases of the model shift towards higher values of the mass parameter g_2, relative to the model without curvature. The proxy for this shift is the shift in the position of the triple point of the model δg_2^{tp}, which we numerically found [8] to be proportional to the curvature parameter g_r. Since the renormalization shift of the GW model mass parameter is [18]

$$\delta m^2_{\mathrm{ren}} = \frac{\lambda}{12\pi(1+\Omega^2)}\log\frac{\Lambda^2\theta}{\Omega}\,, \tag{3.12}$$

[‡]We are interested in the $g_2 > 0$ regime with the spontaneous symmetry breaking.

[§]Figures 2 and 3 are the adapted versions of our figures from [8] and [9].

[¶]For inspection of the GW model without the kinetic term, we also used the eigenvalue-flipping algorithm [14].

and the matrix size serves as a cutoff $\Lambda^2 \propto N$ [5], by using the qualitative identification between $m \leftrightarrow G_2$ and $\Omega \leftrightarrow g_r$ [8], we find the renormalization shift of the matrix GW model to be [8]

$$\delta G_2^{\mathrm{ren}} \sim -\log N\,. \tag{3.13}$$

This means that the bare G_2 has to shift by

$$\left|\delta G_2^{\mathrm{ren}}\right| \sim \log N \tag{3.14}$$

to compensate for the quantum corrections. Since this shift is smaller than that of the triple point

$$\delta G_2^{\mathrm{tp}} = N\delta g_2^{\mathrm{tp}} \sim N g_r\,, \tag{3.15}$$

which is the lowest-most-g_2 point the stripe phase (Figure 2), we conclude that the bare mass of the GW model cannot lie in the stripe phase associated with the UV/IR mixing, but instead in the disordered phase with the trivial vacuum. The same is true for the renormalized version of the $\lambda\phi^4_\star$ model[‖] that is obtained [5, 10] as a limit of a series of the GW models with vanishing curvature coupling

$$\Omega \sim \frac{1}{\log N} \to 0\,. \tag{3.16}$$

A detailed review of this matter can be found in [19]. In contrast to these renormalizable models, the above does not hold for the original $\lambda\phi^4_\star$ without the curvature term, because its triple point remains locked to the origin in the large N limit [8] as in the $g_r = 0$ case in Figure 2.

Let us here also mention that modifications to the kinetic term can also induce shifts in the position of the triple point of the model [20] and are also capable of resolving the UV/IR mixing problems [21].

4. Analytical results & RTNI package

Since the conclusions about the connection between the renormalizability of the GW model and the suppression of the stripe phase critically depend on the numerically obtained phase-diagram shift, it would be desirable to also confirm the existence of this shift analytically. The first steps in this direction have been undertaken in [9], where the $O(g_r^4)$ effective action and the disordered-to-stripe transition line have been found for the GW model without the kinetic term.

Although, as can be seen in Figure 3, the transition lines in the strong coupling regime show excellent agreement even with crude approximations to the effective action, the position of the starting point (i.e. the triple point) needs more careful consideration. In fact, we need to compare the results from several different orders of approximations to demonstrate the convincing convergence of the transition-line turning points in the perturbative regime. To that end, we will at least require results from the $O(g_r^6)$ effective action, which will be derived in this section.

Let us now outline the general analytical procedure and present some new results. We saw in Figure 1 that the phases depend only on the distribution of the field eigenvalues, which means

[‖] We will denote this model by $\lambda\phi^4_{\mathrm{GW}}$.

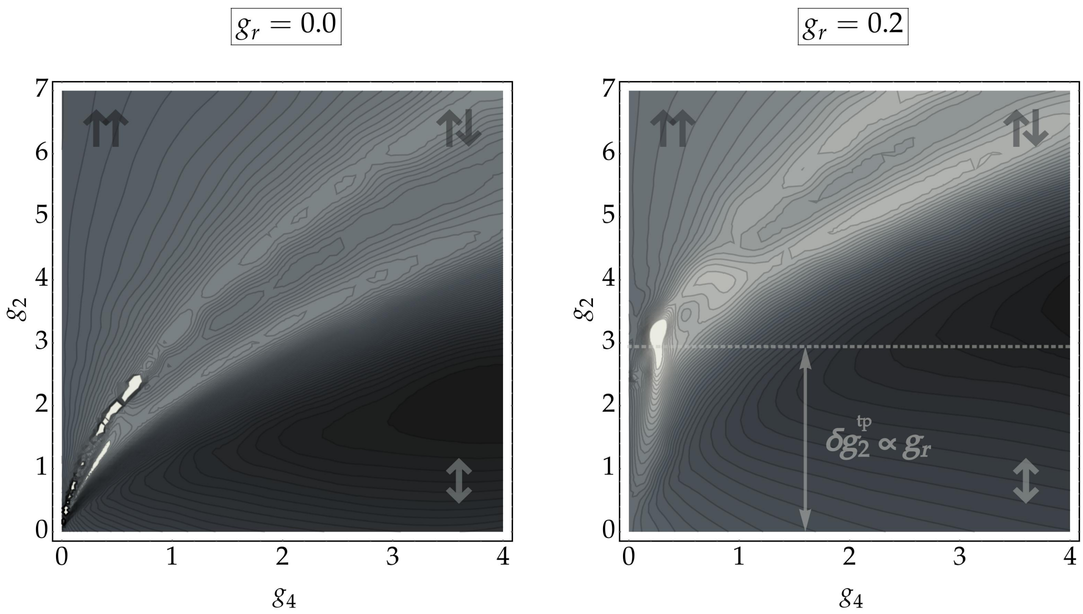

Figure 2: Phase diagrams of the $N = 24$ matrix GW model with (right) and without (left) the curvature term. The darker areas in the plots have lower values and the lighter areas have higher values of the specific heat. The bright stripes therefore represent the transition lines between the phases, which are indicated by arrows in the corners of the plots. Note that the relative shift of the triple point and the transition lines in the right-hand plot is $\delta g_2^{\rm tp} \approx 16 g_r = 16 \cdot 0.2 = 3.2$.

that we must first obtain the effective action by integrating out non-eigenvalue degrees of freedom. Unfortunately, when we decompose our field Φ into the eigenvalues Λ and the unitary part U

$$S = N\,\mathrm{tr}\Big((U\Lambda U^\dagger)\mathcal{K}(U\Lambda U^\dagger) - g_r R U\Lambda^2 U^\dagger - g_2\Lambda^2 + g_4\Lambda^4\Big)\,, \tag{4.17}$$

we are faced with rather complicated integrals over the unitary group. Nevertheless, the shift of the triple point is the sole result of the curvature term so, as a first step, we can try to simplify our analysis and disregard the kinetic term for the time being. We expect this to reproduce the magnitude of the triple point shift if not the exact position of the triple point. This expectation is also backed up by numerical simulations, during which the eigenvalue distribution in the vicinity of the triple point revealed a field configuration

$$\Phi_R^2 = \frac{g_2\mathbb{1} + g_r R}{2g_4}\,. \tag{4.18}$$

This configuration is the solution of the equations of motion (3.10) without the kinetic term and its domain of existence sets a lower bound on the position of the triple point:

$$g_2^{\rm tp} \geq 16 g_r\,. \tag{4.19}$$

On the other hand, replacement of the curvature entries by its minimal/maximal eigenvalue, gives the upper bound on the triple point position [22]

$$g_2^{\rm tp} \leq 16 g_r\,. \tag{4.20}$$

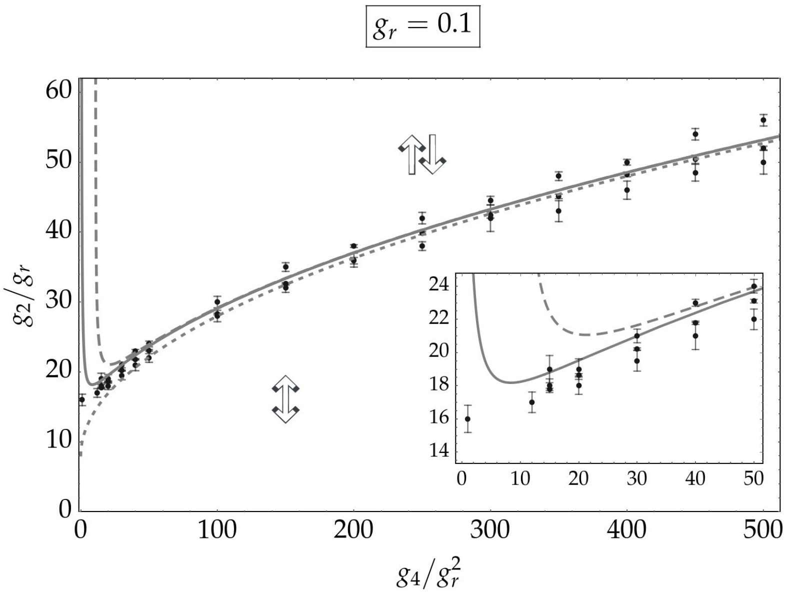

Figure 3: Comparison between the analytical phase transition lines for the matrix GW model without the kinetic term and the results of $N = 24$ numerical simulation. The dotted line is an exact solution for the $O(g_r)$ effective action, the dashed line is an exact solution for the $O(g_r^2)$ effective action and the solid line is the $O(g_r^4)$ approximation for the $O(g_r^4)$ effective action. We see from the zoomed in portion of the plot that as we add higher order terms to the effective action, the turning point of the transition line moves towards $(g_4, g_2) = (0, 16g_r)$. We also see that already the $O(g_r)$ line is a good approximation for the large g_4 regime.

Together, these two yield

$$g_2^{\text{tp}} = 16 g_r \,, \tag{4.21}$$

and simulations imply that these bounds are indeed saturated.

We can now drop the kinetic term and turn to finding the effective action by performing integration in the partition function Z:

$$Z = \int [d\Phi]\, e^{-S} = \int [d\Lambda]\, \Delta^2(\Lambda)\, e^{-N\,\mathrm{tr}\left(-g_2\Lambda^2 + g_4\Lambda^4\right)} \int [dU]\, e^{\,g_r N\,\mathrm{tr}\left(URU^\dagger \Lambda^2\right)} \,. \tag{4.22}$$

Here $\Delta(\Lambda)$ represents the Vandermonde determinant of the eigenvalue matrix Λ:

$$\Delta(\Lambda) = \prod_{1\le i<j\le N} (\lambda_j - \lambda_i)\,, \qquad \Lambda = \mathrm{diag}\,\lambda_i \,. \tag{4.23}$$

To find the rightmost integral in Z, we will consider a general integral (with normalized measure)

$$I = \int_{\mathrm{U}(N)} [dU]\, e^{t\,\mathrm{tr}\left(AUBU^\dagger\right)} \tag{4.24}$$

for arbitrary Hermitian matrices A and B.

On the one hand, I defines the correction δS to the effective action

$$S_{\text{eff}} = -g_2 N\,\mathrm{tr}\Lambda^2 + g_4 N\,\mathrm{tr}\Lambda^4 - \log\Delta^2(\Lambda) + \delta S \,, \tag{4.25}$$

	I_1	I_2	I_3	I_4	I_5	I_6
number of terms = $(n!)^2$	1	4	36	576	14,400	518,400
execution time	0.2 s	0.3 s	1 s	19 s	10 min	21 h

Table 1: RTNI package performance for Ryzen 7, 16 GB RAM + 20 GB swap, Ubuntu 20, Python. A large number of repeated terms were identified and summed with our additional code in Python, reducing the number of terms in I_n from $(n!)^2$ to p_n^2. The cause of the repetitions seems to be that RTNI sees each instance of the matrices A and B as a different matrix. It is unclear to us whether this is a general feature of the package or simply due to our unfamiliarity with the appropriate program settings. We would like to point out that it was very helpful that the package calculates integrals for unspecified matrix sizes.

as

$$I = \exp\left(-\delta S\right) = \exp\left(-\sum_{n=1}^{\infty}\frac{t^n}{n!}S_n\right). \tag{4.26}$$

On the other, we can expand it as:

$$I = 1 + \sum_{n=1}^{\infty}\frac{t^n}{n!}I_n\,, \qquad I_n = \int [dU]\,\mathrm{tr}^n\left(UAU^\dagger B\right). \tag{4.27}$$

When we compare the terms of these two series, we can extract recursive expressions for effective action terms, which are given here up to $O(t^6)$:

$$S_1 = -I_1\,, \tag{4.28}$$
$$S_2 = S_1^2 - I_2\,, \tag{4.29}$$
$$S_3 = -S_1^3 + 3S_1S_2 - I_3\,, \tag{4.30}$$
$$S_4 = S_1^4 - 6S_1^2S_2 + 3S_2^2 + 4S_1S_3 - I_4\,, \tag{4.31}$$
$$S_5 = -S_1^5 + 10S_1^3S_2 - 10S_1^2S_3 - 15S_1S_2^2 + 5S_1S_4 + 10S_2S_3 - I_5\,, \tag{4.32}$$

and

$$S_6 = S_1^6 - 15S_1^4S_2 + 45S_1^2S_2^2 - 15S_2^3 + 20S_1^3S_3 - 60S_1S_2S_3 + \\ + 10S_3^2 - 15S_1^2S_4 + 15S_2S_4 + 6S_1S_5 - I_6\,. \tag{4.33}$$

In order to find S_n, we must first calculate I_n. We have done this with the help of the RTNI** computing package [24] (for its performance see Table 1). The first two integrals are

$$I_1 = \frac{\mathrm{tr}A\,\mathrm{tr}B}{N}\,, \qquad I_2 = \frac{\mathrm{tr}^2A\,\mathrm{tr}^2B + \mathrm{tr}A^2\,\mathrm{tr}B^2}{N^2-1^2} - \frac{\mathrm{tr}^2A\,\mathrm{tr}B^2 + \mathrm{tr}^2B\,\mathrm{tr}A^2}{N(N^2-1^2)}\,. \tag{4.34}$$

In general, I_n consists of p_n^2 terms, where p_n represents the number of possible partitions of the integer n. Interestingly, the first six p_n coincide with prime numbers. Due to their length, the expressions for I_3 to I_6 are left for the Appendices.

**In the meantime, an updated version of RTNI has been released [23].

It turns out that S_n can be compactly written with the help of the normalized symmetrized moments:

$$\mathcal{A}_n = \frac{\mathrm{tr}(A - (\mathrm{tr}A/N)\,\mathbb{1})^n}{N}\,, \qquad \mathcal{B}_n = \frac{\mathrm{tr}(B - (\mathrm{tr}B/N)\,\mathbb{1})^n}{N}\,. \tag{4.35}$$

The first three of them read:

$$S_1 = -\frac{\mathrm{tr}A\,\mathrm{tr}B}{N}\,, \qquad S_2 = -\frac{N^2\,\mathcal{A}_2\,\mathcal{B}_2}{N^2-1^2}\,, \qquad S_3 = -\frac{2N^3\,\mathcal{A}_3\,\mathcal{B}_3}{(N^2-1^2)(N^2-2^2)}\,. \tag{4.36}$$

To find higher order terms in the effective action, we will assume that they can similarly be written as sums of products of $\mathcal{A}$s and $\mathcal{B}$s, such that in each term the powers of both A and B separately add up to the order of the effective action term. Since in $\mathcal{A}$s, $\mathrm{tr}A^n$ is always coupled with $\mathrm{tr}^m A$, and likewise for B, it follows that the term $\mathrm{tr}A^m\,\mathrm{tr}A^n\,\mathrm{tr}B^m\,\mathrm{tr}B^n$ comes solely from $\mathcal{A}_n\mathcal{A}_m\mathcal{B}_n\mathcal{B}_m$, therefore their coefficients are the same, and can be easily extracted from the expression for S_{n+m}. The 4th order term is, for example,

$$\begin{aligned} S_4 = &-\frac{6N^2(N^2+1)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)}\cdot\mathcal{A}_4\,\mathcal{B}_4 \\ &-\frac{18N^2(N^2+1)(N^2-3)}{(N^2-1^2)^2(N^2-2^2)(N^2-3^2)}\cdot\mathcal{A}_2^2\,\mathcal{B}_2^2 \\ &+\frac{6N(2N^2-3)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)}\cdot\left(\mathcal{A}_4\,\mathcal{B}_2^2+\mathcal{B}_4\,\mathcal{A}_2^2\right), \end{aligned} \tag{4.37}$$

while the higher order terms are again given in the Appendices. These formulas correctly reproduce our earlier results for S_1 to S_4 [9] which were derived by expanding the HCIZ formula [25, 26]. In our special case where $A = R$ and $B = \Lambda^2$, we find that

$$S_3 = 0\,, \qquad S_5 = 0\,. \tag{4.38}$$

We will now prove that, in fact, all odd-order terms in the effective action vanish, except the first one.

First note that, due to its diagonality and equidistant eigenvalues, the curvature R obeys the following relation

$$\left(R - \frac{\mathrm{tr}R}{N}\,\mathbb{1}\right)^{T'} = -\left(R - \frac{\mathrm{tr}R}{N}\,\mathbb{1}\right), \tag{4.39}$$

where T' denotes the transpose w.r.t. the antidiagonal. For a diagonal matrix D, this antitranspose can be written as

$$D^{T'} = JDJ, \qquad D = JD^{T'}J, \tag{4.40}$$

where $J = J^{-1}$ is the unitary exchange matrix

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}. \tag{4.41}$$

We can use this fact about R to absorb $g_r(\mathrm{tr}R/N)\Lambda^2 = 8g_r N\Lambda^2$ into the mass term before the expansion[††], so that we are left with

$$A = R - \frac{\mathrm{tr}R}{N}\,\mathbb{1}\,, \qquad B = \Lambda^2, \tag{4.42}$$

such that $\mathrm{tr}A = 0$ and $A^{T'} = -A$. If we consider the general Hermitian matrix A with these properties, we can then write[‡‡]

$$\begin{aligned}\int [dU]\,\mathrm{tr}^n\!\left(UAU^\dagger B\right) &= \int [d(UJ)]\,\mathrm{tr}^n\left[(UJ)A(UJ)^\dagger B\right]\\ &= \int [d(UJ)]\,\mathrm{tr}^n\left[U(JAJ)U^\dagger B\right]\\ &= \int [dU]\,\mathrm{tr}^n\!\left(UA^{T'}U^\dagger B\right) = (-1)^n\int [dU]\,\mathrm{tr}^n\!\left(UAU^\dagger B\right).\end{aligned} \tag{4.43}$$

In the last line we have taken advantage the fact that in the product UJ the exchange matrix merely permutes the entries of the matrix U, so that the Jacobian of this transformation is equal to 1. Applying the derived formula for odd $n = 2m-1$ results in

$$\int [dU]\,\mathrm{tr}^{2m-1}\!\left(UAU^\dagger B\right) = 0\,. \tag{4.44}$$

Since S_n in this case contains either powers of odd terms or products of odd and even terms, this means it must hold

$$S_{2m-1} = 0\,. \tag{4.45}$$

Using the expression from the Appendices and keeping only the leading terms in N, we finally arrive at the $O(g_r^6)$ effective action

$$\begin{aligned}S_{\text{eff}} =& -(g_2 - 8g_r)N\,\mathrm{tr}\Lambda^2 + \left(g_4 - \frac{32}{3}g_r^2\right)N\,\mathrm{tr}\Lambda^4 + \frac{32}{3}g_r^2\,\mathrm{tr}^2\Lambda^2\\ &+ \frac{1024}{45}g_r^4 N\,\mathrm{tr}\Lambda^8 + \frac{1024}{15}g_r^4\,\mathrm{tr}^2\Lambda^4 - \frac{4096}{45}g_r^4\,\mathrm{tr}\Lambda^6\,\mathrm{tr}\Lambda^2\\ &- \frac{262144}{2835}g_r^6 N\,\mathrm{tr}\Lambda^{12} + \frac{524288}{945}g_r^6\,\mathrm{tr}\Lambda^2\,\mathrm{tr}\Lambda^{10} - \frac{262144}{189}g_r^6\,\mathrm{tr}\Lambda^4\,\mathrm{tr}\Lambda^8 + \frac{524288}{567}g_r^6\,\mathrm{tr}^2\Lambda^6\\ &- \log\Delta^2(\Lambda)\,,\end{aligned} \tag{4.46}$$

or more transparently

$$\begin{aligned}S_{\text{eff}} =& -(g_2 - 8g_r)N\,\mathrm{tr}\Lambda^2 + \left(g_4 - \frac{1}{6}(8g_r)^2\right)N\,\mathrm{tr}\Lambda^4 + \frac{1}{6}(8g_r)^2\,\mathrm{tr}^2\Lambda^2\\ &+ \frac{1}{180}(8g_r)^4 N\,\mathrm{tr}\Lambda^8 + \frac{1}{60}(8g_r)^4\,\mathrm{tr}^2\Lambda^4 - \frac{1}{45}(8g_r)^4\,\mathrm{tr}\Lambda^6\,\mathrm{tr}\Lambda^2\\ &- \frac{1}{2835}(8g_r)^6 N\,\mathrm{tr}\Lambda^{12} + \frac{2}{945}(8g_r)^6\,\mathrm{tr}\Lambda^2\,\mathrm{tr}\Lambda^{10} - \frac{1}{189}(8g_r)^6\,\mathrm{tr}\Lambda^4\,\mathrm{tr}\Lambda^8 + \frac{2}{567}(8g_r)^6\,\mathrm{tr}^2\Lambda^6\\ &- \log\Delta^2(\Lambda)\,.\end{aligned} \tag{4.47}$$

[††] This will only shift S_1 to 0, while the higher order S_ns will remain unaffected.

[‡‡] using the normalized Haar measure

Here, the expansion/shift parameter $8g_r$ comes from a large N limit of $g_r \mathrm{tr} R/N$. The effective action is now ready to be varied and used in the eigenvalue distribution equation[§§] for the calculation of the correction to the phase transition lines and better analytical estimation of the shift of the triple point. This is part of an ongoing research.

Using the same expansion method, work has also begun on separate effects of the here disregarded kinetic term of the GW model [28], the first non-trivial terms of its effective action being:

$$S_{\mathrm{eff}}^{\mathrm{kin}} = N\,\mathrm{tr}\Lambda^2 - \mathrm{tr}^2\Lambda + \frac{97}{120}N\,\mathrm{tr}\Lambda^4 - \frac{565}{120N^2}\,\mathrm{tr}^4\Lambda + \\ + \frac{113}{12N}\,\mathrm{tr}^2\Lambda\,\mathrm{tr}\Lambda^2 - \frac{137}{60}\,\mathrm{tr}^2\Lambda^2 - \frac{97}{30}\,\mathrm{tr}\Lambda\,\mathrm{tr}\Lambda^3. \quad (4.48)$$

5. Conclusions & outlook

In this contribution, we have compiled the results of our recent studies on the phase structure of the GW model and its impact on the renormalizability of the model. First, the numerical simulations yielded the phase diagram consisting of the ordered, disordered and NC stripe phases. The obtained diagram revealed that the curvature term, which is responsible for the renormalizability of the model, causes the overall shift of the transition lines towards the larger values of the mass parameter. This was partially confirmed analytically by the 4th order perturbative derivation of the $\updownarrow \to \upharpoonleft\downharpoonright$ transition line in the strong interaction regime for the model without the kinetic term. Here we went one step further and also derived the 6th order effective action with the help of the RTNI computing package. The next step would be to calculate the 6th order correction of the $\updownarrow \to \upharpoonleft\downharpoonright$ transition line as well as the $\upharpoonleft\downharpoonright \to \upharpoonleft\upharpoonright$ transition line and then try to extrapolate the positions of their turning points in different orders of approximation in the hope of confirming the numerically obtained position of the triple point. It is also important to continue working on the kinetic term of the model and to compare and combine its effects with the effects of the curvature term.

When viewed in terms of the unscaled model parameters, the shift of the triple point leads to the removal of the stripe phase in the renormalizable GW model and also in the renormalizable $\lambda\phi^4_{\mathrm{GW}}$ model obtained by switching off the curvature coupling with increasing matrix size, but not in the original nonrenormalizable $\lambda\phi^4_\star$ model without the curvature. We believe that this correspondence between the (non)renormalizability and the presence of the stripe phase has a broader validity and also applies to other NC models. To check this, a natural step would be to simulate the related nonrenormalizable $U(1)$ gauge model [29]. This model has two possible classical vacua—the zero-vacuum and the stripe vacuum proportional to the NC coordinates—and numerical simulations would show which one is actually energetically preferred. We expect that its nonrenormalizability is also caused by the stripe phase. This would hopefully bring us a bit closer to the successful formulation of a renormalizable NC gauge model.

[§§]A nice overview of the derivation of the eigenvalue distribution and possible classes of solutions can be found in [27].

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A. 3rd order integral

$$
\begin{aligned}
I_3 = \frac{1}{N(N^2-1^2)(N^2-2^2)} \times \Big(& (N^2-2)\cdot \mathrm{tr}^3 A\, \mathrm{tr}^3 B \\
& +2N^2 \cdot \mathrm{tr} A^3\, \mathrm{tr} B^3 \\
& +4\cdot (\mathrm{tr}^3 A\, \mathrm{tr} B^3 + \mathrm{tr}^3 B\, \mathrm{tr} A^3) \\
& -3N\cdot (\mathrm{tr}^3 A\, \mathrm{tr} B^2\, \mathrm{tr} B + \mathrm{tr}^3 B\, \mathrm{tr} A^2\, \mathrm{tr} A) \\
& -6N\cdot (\mathrm{tr} A^3\, \mathrm{tr} B^2\, \mathrm{tr} B + \mathrm{tr} B^3\, \mathrm{tr} A^2\, \mathrm{tr} A) \\
& +3(N^2+2)\cdot \mathrm{tr} A^2\, \mathrm{tr} A\, \mathrm{tr} B^2\, \mathrm{tr} B \Big)
\end{aligned} \tag{A.49}
$$

B. 4th order integral

$$
I_4 = \frac{1}{N^2(N^2-1^2)(N^2-2^2)(N^2-3^2)} \times
$$

$$
\begin{aligned}
\times\Big(& (N^4-8N^2+6)\cdot \mathrm{tr}^4 A\, \mathrm{tr}^4 B \\
& +6N^2(N^2+1)\cdot \mathrm{tr} A^4\, \mathrm{tr} B^4 \\
& +3(N^4-6N^2+18)\cdot \mathrm{tr}^2 A^2\, \mathrm{tr}^2 B^2 \\
& +8(N^4+3N^2+12)\cdot \mathrm{tr} A^3\, \mathrm{tr} A\, \mathrm{tr} B^3\, \mathrm{tr} B \\
& +6N^2(N^2+1)\cdot \mathrm{tr}^2 A\, \mathrm{tr} A^2\, \mathrm{tr}^2 B\, \mathrm{tr} B^2 \\
& -30N\cdot (\mathrm{tr}^4 A\, \mathrm{tr} B^4 + \mathrm{tr}^4 B\, \mathrm{tr} A^4) \\
& +3(N^2+6)\cdot (\mathrm{tr}^4 A\, \mathrm{tr}^2 B^2 + \mathrm{tr}^4 B\, \mathrm{tr}^2 A^2) \\
& -6N(2N^2-3)\cdot (\mathrm{tr} A^4\, \mathrm{tr}^2 B^2 + \mathrm{tr} B^4\, \mathrm{tr}^2 A^2) \\
& +8(2N^2-3)\cdot (\mathrm{tr}^4 A\, \mathrm{tr} B^3\, \mathrm{tr} B + \mathrm{tr}^4 B\, \mathrm{tr} A^3\, \mathrm{tr} A) \\
& -24N(N^2+1)\cdot (\mathrm{tr} A^4\, \mathrm{tr} B^3\, \mathrm{tr} B + \mathrm{tr} B^4\, \mathrm{tr} A^3\, \mathrm{tr} A) \\
& -6N(N^2-2^2)\cdot (\mathrm{tr}^4 A\, \mathrm{tr}^2 B\, \mathrm{tr} B^2 + \mathrm{tr}^4 B\, \mathrm{tr}^2 A\, \mathrm{tr} A^2) \\
& +60N^2\cdot (\mathrm{tr} A^4\, \mathrm{tr}^2 B\, \mathrm{tr} B^2 + \mathrm{tr} B^4\, \mathrm{tr}^2 A\, \mathrm{tr} A^2) \\
& -6N(N^2+6)\cdot (\mathrm{tr}^2 A^2\, \mathrm{tr}^2 B\, \mathrm{tr} B^2 + \mathrm{tr}^2 B^2\, \mathrm{tr}^2 A\, \mathrm{tr} A^2) \\
& +24(2N^2-3)\cdot (\mathrm{tr} A^3\, \mathrm{tr} A\, \mathrm{tr}^2 B^2 + \mathrm{tr} B^3\, \mathrm{tr} B\, \mathrm{tr}^2 A^2) \\
& -24N(N^2+1)\cdot (\mathrm{tr} A^3\, \mathrm{tr} A\, \mathrm{tr}^2 B\, \mathrm{tr} B^2 + \mathrm{tr} B^3\, \mathrm{tr} B\, \mathrm{tr}^2 A\, \mathrm{tr} A^2)\Big)
\end{aligned} \tag{B.50}
$$

C. 5th order integral

$$
\begin{aligned}
I_5 = \frac{1}{N^2(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)}\times& \\
\times\Big(N(N^4-20N^2+78)&\cdot \mathrm{tr}^5A\,\mathrm{tr}^5B \\
+24N^2(N^2+5)N&\cdot \mathrm{tr}A^5\,\mathrm{tr}B^5 \\
+336N&\cdot(\mathrm{tr}^5A\,\mathrm{tr}B^5+\mathrm{tr}^5B\,\mathrm{tr}A^5) \\
+30N(N^4+5N^2+84)&\cdot \mathrm{tr}A^4\,\mathrm{tr}A\,\mathrm{tr}B^4\,\mathrm{tr}B \\
+10N(N^2-3^2)(N^2+4)&\cdot \mathrm{tr}^3A\,\mathrm{tr}A^2\,\mathrm{tr}^3B\,\mathrm{tr}B^2 \\
+20N(N^4+24)&\cdot(\mathrm{tr}A^3\,\mathrm{tr}^2A\,\mathrm{tr}B^3\,\mathrm{tr}^2B+\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}B^3\,\mathrm{tr}B^2) \\
+15N(N^4-10N^2+114)&\cdot \mathrm{tr}^2A^2\,\mathrm{tr}A\,\mathrm{tr}^2B^2\,\mathrm{tr}B \\
-30(5N^2-24)&\cdot(\mathrm{tr}^5A\,\mathrm{tr}B^4\,\mathrm{tr}B+\mathrm{tr}^5B\,\mathrm{tr}A^4\,\mathrm{tr}A) \\
-120N^2(N^2+5)&\cdot(\mathrm{tr}A^5\,\mathrm{tr}B^4\,\mathrm{tr}B+\mathrm{tr}B^5\,\mathrm{tr}A^4\,\mathrm{tr}A) \\
-10(N^2-12)(N^2-2)&\cdot(\mathrm{tr}^5A\,\mathrm{tr}^3B\,\mathrm{tr}B^2+\mathrm{tr}^5B\,\mathrm{tr}^3A\,\mathrm{tr}A^2) \\
-840N^2&\cdot(\mathrm{tr}A^5\,\mathrm{tr}^3B\,\mathrm{tr}B^2+\mathrm{tr}B^5\,\mathrm{tr}^3A\,\mathrm{tr}A^2) \\
+40N(N^2-3^2)&\cdot(\mathrm{tr}^5A\,\mathrm{tr}B^3\,\mathrm{tr}^2B+\mathrm{tr}^5B\,\mathrm{tr}A^3\,\mathrm{tr}^2A) \\
-40(N^2+12)&\cdot(\mathrm{tr}^5A\,\mathrm{tr}B^3\,\mathrm{tr}B^2+\mathrm{tr}^5B\,\mathrm{tr}A^3\,\mathrm{tr}A^2) \\
+120N(3N^2+8)&\cdot(\mathrm{tr}A^5\,\mathrm{tr}B^3\,\mathrm{tr}^2B+\mathrm{tr}B^5\,\mathrm{tr}A^3\,\mathrm{tr}^2A) \\
-120N^2(N^2-2)&\cdot(\mathrm{tr}A^5\,\mathrm{tr}B^3\,\mathrm{tr}B^2+\mathrm{tr}B^5\,\mathrm{tr}A^3\,\mathrm{tr}A^2) \\
+15N(N^2-2)&\cdot(\mathrm{tr}^5A\,\mathrm{tr}^2B^2\,\mathrm{tr}B+\mathrm{tr}^5B\,\mathrm{tr}^2A^2\,\mathrm{tr}A) \\
+360N(N^2-2)&\cdot(\mathrm{tr}A^5\,\mathrm{tr}^2B^2\,\mathrm{tr}B+\mathrm{tr}B^5\,\mathrm{tr}^2A^2\,\mathrm{tr}A) \\
+300N(N^2-2)&\cdot(\mathrm{tr}A^4\,\mathrm{tr}A\,\mathrm{tr}^3B\,\mathrm{tr}B^2+\mathrm{tr}B^4\,\mathrm{tr}B\,\mathrm{tr}^3A\,\mathrm{tr}A^2) \\
-120(N^4+24)&\cdot(\mathrm{tr}A^4\,\mathrm{tr}A\,\mathrm{tr}B^3\,\mathrm{tr}^2B+\mathrm{tr}B^4\,\mathrm{tr}B\,\mathrm{tr}A^3\,\mathrm{tr}^2A) \\
+600N(N^2-2)&\cdot(\mathrm{tr}A^4\,\mathrm{tr}A\,\mathrm{tr}B^3\,\mathrm{tr}B^2+\mathrm{tr}B^4\,\mathrm{tr}B\,\mathrm{tr}A^3\,\mathrm{tr}A^2) \\
-30(2N^4+25N^2-72)&\cdot(\mathrm{tr}A^4\,\mathrm{tr}A\,\mathrm{tr}^2B^2\,\mathrm{tr}B+\mathrm{tr}B^4\,\mathrm{tr}B\,\mathrm{tr}^2A^2\,\mathrm{tr}A) \\
-20(N^2-6)(3N^2+8)&\cdot(\mathrm{tr}^3A\,\mathrm{tr}A^2\,\mathrm{tr}B^3\,\mathrm{tr}^2B+\mathrm{tr}^3B\,\mathrm{tr}B^2\,\mathrm{tr}A^3\,\mathrm{tr}^2A) \\
+100N(N^2+12)&\cdot(\mathrm{tr}^3A\,\mathrm{tr}A^2\,\mathrm{tr}B^3\,\mathrm{tr}B^2+\mathrm{tr}^3B\,\mathrm{tr}B^2\,\mathrm{tr}A^3\,\mathrm{tr}A^2) \\
-30(N^4+24)&\cdot(\mathrm{tr}^3A\,\mathrm{tr}A^2\,\mathrm{tr}^2B^2\,\mathrm{tr}B+\mathrm{tr}^3B\,\mathrm{tr}B^2\,\mathrm{tr}^2A^2\,\mathrm{tr}A) \\
-20(N^4+60N^2-96)&\cdot(\mathrm{tr}A^3\,\mathrm{tr}^2A\,\mathrm{tr}B^3\,\mathrm{tr}B^2+\mathrm{tr}B^3\,\mathrm{tr}^2B\,\mathrm{tr}A^3\,\mathrm{tr}A^2) \\
+300N(N^2-2)&\cdot(\mathrm{tr}A^3\,\mathrm{tr}^2A\,\mathrm{tr}^2B^2\,\mathrm{tr}B+\mathrm{tr}B^3\,\mathrm{tr}^2B\,\mathrm{tr}^2A^2\,\mathrm{tr}A) \\
-60(N^4+24)&\cdot(\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}^2B^2\,\mathrm{tr}B+\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}^2A^2\,\mathrm{tr}A)\Big)
\end{aligned}
\tag{C.51}
$$

D. 6th order integral

$$
\begin{aligned}
I_6 = \frac{1}{N^2(N^2-1^2)^2(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\times \\
\times\Big(&(N^8-41N^6+458N^4-1258N^2+240)\cdot \mathrm{tr}^6A\,\mathrm{tr}^6B \\
&+120N^2(N^2-1)(N^4+15N^2+8)\cdot \mathrm{tr}A^6\,\mathrm{tr}B^6 \\
&-5040N(N^2-1)\cdot(\mathrm{tr}^6A\,\mathrm{tr}B^6+\mathrm{tr}^6B\,\mathrm{tr}A^6) \\
&+40(N^8-28N^6+477N^4-1890N^2-2400)\cdot \mathrm{tr}^2A^3\,\mathrm{tr}^2B^3 \\
&+15(N^8-35N^6+376N^4+18N^2-3600)\cdot \mathrm{tr}^3A^2\,\mathrm{tr}^3B^2 \\
&+160(N^4+29N^2-90)\cdot(\mathrm{tr}^6A\,\mathrm{tr}^2B^3+\mathrm{tr}^6B\,\mathrm{tr}^2A^3) \\
&-15N(N^4+N^2+358)\cdot(\mathrm{tr}^6A\,\mathrm{tr}^3B^2+\mathrm{tr}^6B\,\mathrm{tr}^3A^2) \\
&-120N(N^2-1)(3N^4-11N^2+80)\cdot(\mathrm{tr}A^6\,\mathrm{tr}^2B^3+\mathrm{tr}B^6\,\mathrm{tr}^2A^3) \\
&+840N^2(N^2-1)(N^2-7)\cdot(\mathrm{tr}A^6\,\mathrm{tr}^3B^2+\mathrm{tr}B^6\,\mathrm{tr}^3A^2) \\
&-480N(4N^4-59N^2+235)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}^3B^2+\mathrm{tr}^2B^3\,\mathrm{tr}^3A^2) \\
&+2016(N^2-1^2)(N^2-10)\cdot(\mathrm{tr}^6A\,\mathrm{tr}B^5\,\mathrm{tr}B+\mathrm{tr}^6B\,\mathrm{tr}A^5\,\mathrm{tr}A) \\
&-720N(N^2-1)(N^4+15N^2+8)\cdot(\mathrm{tr}A^6\,\mathrm{tr}B^5\,\mathrm{tr}B+\mathrm{tr}B^6\,\mathrm{tr}A^5\,\mathrm{tr}A) \\
&-15N(N^2-3^2)(N^4-24N^2+38)\cdot(\mathrm{tr}^6A\,\mathrm{tr}^4B\,\mathrm{tr}B^2+\mathrm{tr}^6B\,\mathrm{tr}^4A\,\mathrm{tr}A^2) \\
&-90N(N^2-4^2)(5N^2-13)\cdot(\mathrm{tr}^6A\,\mathrm{tr}B^4\,\mathrm{tr}^2B+\mathrm{tr}^6B\,\mathrm{tr}A^4\,\mathrm{tr}^2A) \\
&+450(N^4+15N^2+8)\cdot(\mathrm{tr}^6A\,\mathrm{tr}B^4\,\mathrm{tr}B^2+\mathrm{tr}^6B\,\mathrm{tr}A^4\,\mathrm{tr}A^2) \\
&+15120N^2(N^2-1)\cdot(\mathrm{tr}A^6\,\mathrm{tr}^4B\,\mathrm{tr}B^2+\mathrm{tr}B^6\,\mathrm{tr}^4A\,\mathrm{tr}A^2) \\
&+2520N^2(N^2-1)(N^2+11)\cdot(\mathrm{tr}A^6\,\mathrm{tr}B^4\,\mathrm{tr}^2B+\mathrm{tr}B^6\,\mathrm{tr}A^4\,\mathrm{tr}^2A) \\
&-720N(N^2-1^2)(N^2-2^2)(N^2+5)\cdot(\mathrm{tr}A^6\,\mathrm{tr}B^4\,\mathrm{tr}B^2+\mathrm{tr}B^6\,\mathrm{tr}A^4\,\mathrm{tr}A^2) \\
&+45(N^2-2^2)(N^4-3N^2+10)\cdot(\mathrm{tr}^6A\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}^6B\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A) \\
&-5040N(N^2-1)(2N^2-5)\cdot(\mathrm{tr}A^6\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}B^6\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A) \\
&+40(2N^6-51N^4+229N^2-60)\cdot(\mathrm{tr}^6A\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}^6B\,\mathrm{tr}^3A\,\mathrm{tr}A^3) \\
&-6720N(N^2-1)(N^2+5)\cdot(\mathrm{tr}A^6\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}B^6\,\mathrm{tr}^3A\,\mathrm{tr}A^3) \\
&-120N(N^2-3^2)(2N^2+13)\cdot(\mathrm{tr}^6A\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}^6B\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A) \\
&+5040N^2(N^2-1^2)^2\cdot(\mathrm{tr}A^6\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}B^6\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A) \\
&+720(N^2-1^2)(3N^4-11N^2+80)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}B^5\,\mathrm{tr}B+\mathrm{tr}^2B^3\,\mathrm{tr}A^5\,\mathrm{tr}A) \\
&-480N(N^4+29N^2-90)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}^4B\,\mathrm{tr}B^2+\mathrm{tr}^2B^3\,\mathrm{tr}^4A\,\mathrm{tr}A^2) \\
&-7200N(N^4-6N^2+29)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}B^4\,\mathrm{tr}^2B+\mathrm{tr}^2B^3\,\mathrm{tr}A^4\,\mathrm{tr}^2A) \\
&+1800(N^6-10N^4+25N^2+80)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}B^4\,\mathrm{tr}B^2+\mathrm{tr}^2B^3\,\mathrm{tr}A^4\,\mathrm{tr}A^2) \\
&+360(N^6+18N^4-59N^2-200)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}^2B^3\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A) \\
&+160(N^6+62N^4-183N^2+600)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}^2B^3\,\mathrm{tr}^3A\,\mathrm{tr}A^3) \\
&-240N(N^6+2N^4+177N^2-1140)\cdot(\mathrm{tr}^2A^3\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}^2B^3\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A) \\
&-5040N(N^2-1)(N^2-7)\cdot(\mathrm{tr}^3A^2\,\mathrm{tr}B^5\,\mathrm{tr}B+\mathrm{tr}^3B^2\,\mathrm{tr}A^5\,\mathrm{tr}A)
\end{aligned}
$$

$$
\begin{aligned}
&+45N^2(N^4+N^2+358)\cdot(\mathrm{tr}^3A^2\,\mathrm{tr}^4B\,\mathrm{tr}B^2+\mathrm{tr}^3B^2\,\mathrm{tr}^4A\,\mathrm{tr}A^2)\\
&+90(2N^6+95N^4-577N^2-600)\cdot(\mathrm{tr}^3A^2\,\mathrm{tr}B^4\,\mathrm{tr}^2B+\mathrm{tr}^3B^2\,\mathrm{tr}A^4\,\mathrm{tr}^2A)\\
&-90N(N^2-2^2)(2N^4-37N^2+395)\cdot(\mathrm{tr}^3A^2\,\mathrm{tr}B^4\,\mathrm{tr}B^2+\mathrm{tr}^3B^2\,\mathrm{tr}A^4\,\mathrm{tr}A^2)\\
&-45N(N^6-11N^4+220N^2+870)\cdot(\mathrm{tr}^3A^2\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}^3B^2\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A)\\
&-360N(2N^4+23N^2-145)\cdot(\mathrm{tr}^3A^2\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}^3B^2\,\mathrm{tr}^3A\,\mathrm{tr}A^3)\\
&+360(2N^6-13N^4+71N^2+300)\cdot(\mathrm{tr}^3A^2\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}^3B^2\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A)\\
&+144(N^2-1^2)(N^6+10N^4+325N^2+240)\cdot\mathrm{tr}A^5\,\mathrm{tr}A\,\mathrm{tr}B^5\,\mathrm{tr}B\\
&-5040N(N^2-1)(N^2-7)\cdot(\mathrm{tr}A^5\,\mathrm{tr}A\,\mathrm{tr}^4B\,\mathrm{tr}B^2+\mathrm{tr}B^5\,\mathrm{tr}B\,\mathrm{tr}^4A\,\mathrm{tr}A^2)\\
&-720N(N^2-1)(N^4+N^2+106)\cdot(\mathrm{tr}A^5\,\mathrm{tr}A\,\mathrm{tr}B^4\,\mathrm{tr}^2B+\mathrm{tr}B^5\,\mathrm{tr}B\,\mathrm{tr}A^4\,\mathrm{tr}^2A)\\
&+4320(N^2-1^2)(N^2-2^2)(N^2+5)\cdot(\mathrm{tr}A^5\,\mathrm{tr}A\,\mathrm{tr}B^4\,\mathrm{tr}B^2+\mathrm{tr}B^5\,\mathrm{tr}B\,\mathrm{tr}A^4\,\mathrm{tr}A^2)\\
&+2160(N^2-1^2)(N^2-2^2)(N^2+5)\cdot(\mathrm{tr}A^5\,\mathrm{tr}A\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}B^5\,\mathrm{tr}B\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A)\\
&+720(N^2-1^2)(3N^4-11N^2+80)\cdot(\mathrm{tr}A^5\,\mathrm{tr}A\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}B^5\,\mathrm{tr}B\,\mathrm{tr}^3A\,\mathrm{tr}A^3)\\
&-720N(N^2-1)(N^4+15N^2+8)\cdot(\mathrm{tr}A^5\,\mathrm{tr}A\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}B^5\,\mathrm{tr}B\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A)\\
&+15(N^8-19N^6-64N^4+922N^2-1200)\cdot\mathrm{tr}^4A\,\mathrm{tr}A^2\,\mathrm{tr}^4B\,\mathrm{tr}B^2\\
&+90(N^2-2^2)(N^4-N^2+120)(N^2+5)\cdot\mathrm{tr}A^4\,\mathrm{tr}^2A\,\mathrm{tr}B^4\,\mathrm{tr}^2B\\
&+90(N^2-2^2)(N^4-N^2+120)(N^2+5)\cdot\mathrm{tr}A^4\,\mathrm{tr}A^2\,\mathrm{tr}B^4\,\mathrm{tr}B^2\\
&+45(N^8-21N^6+270N^4-130N^2-1200)\cdot\mathrm{tr}^2A^2\,\mathrm{tr}^2A\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B\\
&+450(2N^6-25N^4-25N^2+120)\cdot(\mathrm{tr}^4A\,\mathrm{tr}A^2\,\mathrm{tr}B^4\,\mathrm{tr}^2B+\mathrm{tr}^4B\,\mathrm{tr}B^2\,\mathrm{tr}A^4\,\mathrm{tr}^2A)\\
&-1350N(N^4+15N^2+8)\cdot(\mathrm{tr}^4A\,\mathrm{tr}A^2\,\mathrm{tr}B^4\,\mathrm{tr}B^2+\mathrm{tr}^4B\,\mathrm{tr}B^2\,\mathrm{tr}A^4\,\mathrm{tr}A^2)\\
&-90N(N^2-2^2)(N^2+5)(N^2+119)\cdot(\mathrm{tr}A^4\,\mathrm{tr}^2A\,\mathrm{tr}B^4\,\mathrm{tr}B^2+\mathrm{tr}B^4\,\mathrm{tr}^2B\,\mathrm{tr}A^4\,\mathrm{tr}A^2)\\
&-45N(2N^6-29N^4+97N^2-430)\cdot(\mathrm{tr}^4A\,\mathrm{tr}A^2\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}^4B\,\mathrm{tr}B^2\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A)\\
&-90N(N^2-2^2)(2N^4+53N^2-415)\cdot(\mathrm{tr}A^4\,\mathrm{tr}^2A\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}B^4\,\mathrm{tr}^2B\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A)\\
&+270(4N^6+15N^4+141N^2+200)\cdot(\mathrm{tr}A^4\,\mathrm{tr}A^2\,\mathrm{tr}^2B^2\,\mathrm{tr}^2B+\mathrm{tr}B^4\,\mathrm{tr}B^2\,\mathrm{tr}^2A^2\,\mathrm{tr}^2A)\\
&-120N(N^6-16N^4-30N^2+165)\cdot(\mathrm{tr}^4A\,\mathrm{tr}A^2\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}^4B\,\mathrm{tr}B^2\,\mathrm{tr}^3A\,\mathrm{tr}A^3)\\
&-360N(N^2-3^2)(N^4-N^2+30)\cdot(\mathrm{tr}A^4\,\mathrm{tr}^2A\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}B^4\,\mathrm{tr}^2B\,\mathrm{tr}^3A\,\mathrm{tr}A^3)\\
&+360(N^6+60N^4-101N^2-200)\cdot(\mathrm{tr}A^4\,\mathrm{tr}A^2\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}B^4\,\mathrm{tr}B^2\,\mathrm{tr}^3A\,\mathrm{tr}A^3)\\
&+360(3N^6-37N^4+54N^2+100)\cdot(\mathrm{tr}^2A^2\,\mathrm{tr}^2A\,\mathrm{tr}^3B\,\mathrm{tr}B^3+\mathrm{tr}^2B^2\,\mathrm{tr}^2B\,\mathrm{tr}^3A\,\mathrm{tr}A^3)\\
&+120(5N^6-N^4-64N^2-300)\cdot(\mathrm{tr}^4A\,\mathrm{tr}A^2\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}^4B\,\mathrm{tr}B^2\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A)\\
&-360N(N^6-8N^4+202N^2-555)\cdot(\mathrm{tr}^2A^2\,\mathrm{tr}^2A\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}^2B^2\,\mathrm{tr}^2B\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A)\\
&+360(11N^6-40N^4+149N^2+600)\cdot(\mathrm{tr}A^4\,\mathrm{tr}^2A\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}B^4\,\mathrm{tr}^2B\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A)\\
&-360N(N^6+30N^4-201N^2+890)\cdot(\mathrm{tr}A^4\,\mathrm{tr}A^2\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}B^4\,\mathrm{tr}B^2\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A)\\
&+40N^2(N^6-12N^4+37N^2-986)\cdot\mathrm{tr}^3A\,\mathrm{tr}A^3\,\mathrm{tr}^3B\,\mathrm{tr}B^3\\
&-120N(N^6+54N^4-415N^2+1320)\cdot(\mathrm{tr}^3A\,\mathrm{tr}A^3\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B+\mathrm{tr}^3B\,\mathrm{tr}B^3\,\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A)\\
&+120(N^8-2N^6+433N^4-912N^2-2400)\cdot\mathrm{tr}A^3\,\mathrm{tr}A^2\,\mathrm{tr}A\,\mathrm{tr}B^3\,\mathrm{tr}B^2\,\mathrm{tr}B\Big)
\end{aligned}
\tag{D.52}
$$

E. 5th order effective action

$$\begin{aligned}
S_5 = &-\frac{24N(N^2+5)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)}\cdot\mathcal{A}_5\,\mathcal{B}_5\\
&-\frac{480(N^4-5N^2-1)}{N(N^2-1^2)^2(N^2-2^2)(N^2-3^2)(N^2-4^2)}\cdot\mathcal{A}_3\,\mathcal{A}_2\,\mathcal{B}_3\,\mathcal{B}_2\\
&+\frac{120(N^2-2)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)}\cdot(\mathcal{A}_5\,\mathcal{B}_3\,\mathcal{B}_2+\mathcal{B}_5\,\mathcal{A}_3\,\mathcal{A}_2)
\end{aligned} \tag{E.53}$$

F. 6th order effective action

$$\begin{aligned}
S_6 = &-\frac{120(N^4+15N^2+8)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\mathcal{A}_6\,\mathcal{B}_6\\
&-\frac{1800(2N^6-13N^4-13N^2-120)}{N^2(N^2-1^2)^2(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\mathcal{A}_4\,\mathcal{A}_2\,\mathcal{B}_4\,\mathcal{B}_2\\
&-\frac{240(3N^8-30N^6-33N^4+860N^2+1600)}{N^2(N^2-1^2)^2(N^2-2^2)^2(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\mathcal{A}_3^2\,\mathcal{B}_3^3\\
&-\frac{1080(3N^6-32N^4-81N^2+350)}{N^2(N^2-1^2)^3(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\mathcal{A}_2^3\,\mathcal{B}_2^3\\
&+\frac{720(N^2+5)}{(N^2-1^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot(\mathcal{A}_6\,\mathcal{B}_4\,\mathcal{B}_2+\mathcal{B}_6\,\mathcal{A}_4\,\mathcal{A}_2)\\
&+\frac{120(3N^4-11N^2+80)}{N(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\left(\mathcal{A}_6\,\mathcal{B}_3^2+\mathcal{B}_6\,\mathcal{A}_3^2\right)\\
&-\frac{840(N^2-7)}{(N^2-1^2)(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\left(\mathcal{A}_6\,\mathcal{B}_2^3+\mathcal{B}_6\,\mathcal{A}_2^3\right)\\
&-\frac{1800(N^6-10N^4+25N^2+80)}{N^2(N^2-1^2)^2(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\left(\mathcal{A}_4\,\mathcal{A}_2\,\mathcal{B}_3^2+\mathcal{B}_4\,\mathcal{B}_2\,\mathcal{A}_3^2\right)\\
&+\frac{1800(2N^4-19N^2-19)}{N(N^2-1^2)^2(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\left(\mathcal{A}_4\,\mathcal{A}_2\,\mathcal{B}_2^3+\mathcal{B}_4\,\mathcal{B}_2\,\mathcal{A}_2^3\right)\\
&+\frac{480(4N^4-59N^2+235)}{N(N^2-1^2)^2(N^2-2^2)(N^2-3^2)(N^2-4^2)(N^2-5^2)}\cdot\left(\mathcal{A}_3^2\,\mathcal{B}_2^3+\mathcal{B}_3^2\,\mathcal{A}_2^3\right)
\end{aligned} \tag{F.54}$$

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