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BRST formalism of Weyl Invariant Gravity and Confinement of Massive Tensor Ghost

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We present the manifestly covariant canonical operator formalism of a Weyl invariant (or equivalently, a locally scale invariant) gravity whose classical action consists of the well-known conformal gravity and Weyl invariant scalar-tensor gravity, on the basis of the Becchi-Rouet-Stora-Tyupin (BRST) formalism. Based on this formalism, we analyze the physical states by expanding the metric around a flat Minkowski background and the scalar field around a constant background. It is shown that under the assumption of no bound states the physical modes are composed of both a massive tensor ghost of five components with indefinite norm and a massless graviton of two components with positive semi-definite norm. The unitarity of the physical S-matrix is violated by the presence of the massive ghost. On the other hand, if the Weyl BRST transformation of the massive ghost has a bound state, the massive ghost is confined in the zero-norm states by the BRST-quartet mechanism so the physical S-matrix becomes unitary. This mechanism of the ghost confinement might pave the way for a long-standing problem of the unitarity violation in quadratic gravity or higher-derivative gravity.

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PoS(CORFU2023)158

1. Introduction

The Becchi-Rouet-Stora-Tyupin formalism [1–3], the *BRST formalism* in short-hand notation, has played an important role in the Yang-Mills theory and superstring theory. This is because the modern quantum field theories including superstring theory have been developed in the framework of gauge theories. The fundamental principle of gauge theories lies in an invariance of the theories under local gauge transformations. Let us recall that gauge symmetries are literally not symmetries in the conventional sense of symmetries, which act on the configuration space and as a result lead to indentical physics. Rather, gauge symmetries are a kind of redundancies in our description of the physics when we work with the configuration space rather its quotient by gauge transformations.

In order to make a quantum field theory from a certain classical field theory with gauge symmetries, somehow we must carry out a quatization by several different procedures. The most modern procedure is to make use of the BRST formalism where to remove redundancies in our description, the gauge symmetries of a classical action are fixed by suitable gauge-fixing conditions and the corresponding Faddeev-Popov (FP) ghosts are added. Consequently, instead of the local gauge symmetries, a new global and nilpotent symmetry, called the "BRST symmetry", emerges, and its BRST charge not only defines physical states as well as physical observables but also plays a role in deriving the Ward-Takahashi identities among Green functions, which are needed for the proof of the unitarity and renormalizability.

The BRST formalism of general relativity (GR) has been also constructed by Nakanishi by a series of papers [4, 5] where the existence of a huge global symmetry, which is a Poincaré-like IOSp(8|8) global symmetry, has been clarified, and the unitarity of the physical S-matrix has been proved although its renormalizability is completely obscure. On the other hand, the BRST formalism of quadratic gravity or higher-derivative gravity, of which the Einstein-Hilbert term is supplemented with R^2 and $R_{\mu\nu}R^{\mu\nu}$, has been also built in Refs. [6–8] where the renormalizability is manifest [9] while the unitarity is violated by the presence of a massive tensor ghost.

In recent years we have constructed the BRST formalism of a gobally scale invariant gravity [10], a locally scale invariant (or equivalently Weyl invariant) gravity [11], Weyl conformal gravity in Weyl geometry [12], and conformal gravity [13]. Moreover, we have clarified the existence of conformal symmetry and established the Zumino theorem [14] at the quantum level in general Weyl invariant gravitational theories [15]. One of the main motivations behind these articles is to resolve the issue of the perturbative non-renormalizability of general relativity. In the interest of renormalizability, we are accustomed to altering the Einstein-Hilbert Lagrangian in general relativity by adding to it the most general quadratic Lagrangian $\mathcal{L}$ of dimension four at most:

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda) + \alpha_r R^2 - \alpha_c C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \tag{1}$$

which is sometimes called quadratic gravity or higher-derivative gravity.[1] However, a notorious problem happens and it is associated with the last term involving conformal tensor $C_{\mu\nu\rho\sigma}$: As long as this term exists in the Lagrangian, we have a spin-2 massive ghost which makes not only the classical theory be unstable because of unbounded energy from below but also the quantum theory be non-unitary owing to the ghost with negative norm.

[1]The Gauss-Bonnet theorem enables us to use the term $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ instead of $R_{\mu\nu}R^{\mu\nu}$.

PoS(CORFU2023)158

At extremely high energies, it is expected that the kinetic term dominates the mass term and as a result all particles can be effectively regarded as massless particles. In such a situation, a global or local scale symmetry naturally appears in addition to general coordinate invariance. Since the global scale symmetry could be broken by the no-hair theorem of black holes in a curved space-time [16], it is plausible to suppose that the local scale symmetry, which we call *Weyl symmetry*, plays a role at high energies.

From this perspective, when we impose the Weyl symmetry on the Lagrangian (1) and require that Einstein's general relativity should be restored at low energies, we are forced to take account of the following Lagrangian:

$$\frac{1}{\sqrt{-g}}\mathcal{L} = \frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \alpha_c C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{2}$$

Note that in the *unitary gauge* $\phi = \sqrt{\frac{3}{4\pi G}}$ (G is the Newton constant) the terms except for conformal gravity on the right-hand side (RHS) produce the Einstein-Hilbert term.

In this article, we briefly review the manifestly covariant canonical operator formalism of the Weyl invariant gravity as defined in the Lagrangian (2) on the basis of the Becchi-Rouet-Stora-Tyupin (BRST) formalism in Sections 2-6. The detail can be found in the original paper [13]. Furthermore, we propose a new idea on ghost confinement in terms of the BRST formalism in Section 7. If the Weyl BRST transformation of the massive ghost has a bound state, then the massive ghost is confined in the zero-norm states by the BRST-quartet mechanism, thereby the physical S-matrix becoming unitary. This mechanism of the ghost confinement might pave the way for a long-standing problem of the unitarity violation in quadratic gravity or higher-derivative gravity. The Appendix gives various equal-time (anti)commutation relations in the linearized level, which are needed in computing the four-dimensional (anti)commutation relations in Section 6.

2. Classical theory

Let us consider a classical gravitational theory which is invariant under both general coordinate transformation (GCT) and Weyl transformation. Our classical Lagrangian consists of Weyl invariant scalar-tensor gravity [17] and conformal gravity[2]

$$\mathcal{L}_0 = \mathcal{L}_{WIST} + \mathcal{L}_{CG}, \tag{3}$$

where

$$\begin{aligned} \mathcal{L}_{WIST} &= \sqrt{-g}\left(\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \\ \mathcal{L}_{CG} &= -\sqrt{-g}\alpha_c C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \end{aligned} \tag{4}$$

Here ϕ is a real scalar field with a ghost-like kinetic term, R the scalar curvature, α_c a dimensionless positive coupling constant ($\alpha_c > 0$) and $C_{\mu\nu\rho\sigma}$ is the well-known conformal tensor.

[2]We follow the notation and conventions of Misner-Thorne-Wheeler (MTW) textbook [16]. Lowercase Greek letters $\mu, \nu, \ldots$ and Latin ones $i, j, \ldots$ are used for spacetime and spatial indices, respectively; for instance, $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$.

In order to perform the canonical quantization, we have to rewrite $\mathcal{L}_{CG}$ into the first-order form since it involves fourth-order derivatives. To do that, we introduce an auxiliary symmetric tensor $K_{\mu\nu} = K_{\nu\mu}$ and a Stückelberg-like vector field A_μ, which is needed to avoid the second-class constraint coming from the Bianchi identity, and rewrite $\mathcal{L}_{CG}$ as [6–8]

$$\mathcal{L}_{CG}^{(K)} \equiv \sqrt{-g}\Big\{\gamma G_{\mu\nu}K^{\mu\nu} + \alpha[(K_{\mu\nu} - \nabla_\mu A_\nu - \nabla_\nu A_\mu)^2 - (K - 2\nabla_\rho A^\rho)^2]\Big\}, \tag{5}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ denotes the Einstein tensor, and γ and α are dimensionless coupling constants which obey a relation

$$\alpha_c = \frac{\gamma^2}{8\alpha}, \tag{6}$$

where $\alpha > 0$. It is easy to verify that carrying out the path integral over $K_{\mu\nu}$ in $\mathcal{L}_{CG}^{(K)}$ produces the Lagrangian of conformal gravity $\mathcal{L}_{CG}$. Henceforth, as a classical Lagrangian $\mathcal{L}_c$ we take a linear combination of $\mathcal{L}_{WIST}$ and $\mathcal{L}_{CG}^{(K)}$

$$\begin{aligned}
\mathcal{L}_c &\equiv \mathcal{L}_{WIST} + \mathcal{L}_{CG}^{(K)} \\
&= \sqrt{-g}\Big\{\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \gamma G_{\mu\nu}K^{\mu\nu} \\
&+ \alpha[(K_{\mu\nu} - \nabla_\mu A_\nu - \nabla_\nu A_\mu)^2 - (K - 2\nabla_\rho A^\rho)^2]\Big\}.
\end{aligned} \tag{7}$$

The classical Lagrangian $\mathcal{L}_c$ possesses three local transformations, those are, infinitesimal general coordinate transformation (GCT) $\delta^{(1)}$, Weyl transformation $\delta^{(2)}$ and Stückelberg transformation $\delta^{(3)}$. Concretely, the GCT takes the form

$$\begin{aligned}
&\delta^{(1)}g_{\mu\nu} = -(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) = -(\xi^\alpha\partial_\alpha g_{\mu\nu} + \partial_\mu\xi^\alpha g_{\alpha\nu} + \partial_\nu\xi^\alpha g_{\alpha\mu}), \\
&\delta^{(1)}\phi = -\xi^\alpha\partial_\alpha\phi, \quad \delta^{(1)}K_{\mu\nu} = -\xi^\alpha\nabla_\alpha K_{\mu\nu} - \nabla_\mu\xi^\alpha K_{\alpha\nu} - \nabla_\nu\xi^\alpha K_{\mu\alpha}, \\
&\delta^{(1)}A_\mu = -\xi^\alpha\nabla_\alpha A_\mu - \nabla_\mu\xi^\alpha A_\alpha.
\end{aligned} \tag{8}$$

As for the Weyl transformation, we have

$$\begin{aligned}
&\delta^{(2)}g_{\mu\nu} = 2\Lambda g_{\mu\nu}, \quad \delta^{(2)}\phi = -\Lambda\phi, \\
&\delta^{(2)}K_{\mu\nu} = \frac{\gamma}{\alpha}\nabla_\mu\nabla_\nu\Lambda - 2(A_\mu\partial_\nu\Lambda + A_\nu\partial_\mu\Lambda - g_{\mu\nu}A_\alpha\partial^\alpha\Lambda), \\
&\delta^{(2)}A_\mu = 0.
\end{aligned} \tag{9}$$

Finally, the Stückelberg transformation is given by

$$\begin{aligned}
&\delta^{(3)}g_{\mu\nu} = \delta^{(3)}\phi = 0, \quad \delta^{(3)}K_{\mu\nu} = \nabla_\mu\varepsilon_\nu + \nabla_\nu\varepsilon_\mu, \\
&\delta^{(3)}A_\mu = \varepsilon_\mu.
\end{aligned} \tag{10}$$

In the above, ξ_μ, Λ and ε_μ are infinitesimal transformation parameters.

Before closing this section, let us count the number of phyical degrees of freedom since it is known that this counting is more subtle in higher derivative theories than in conventional second-order derivative theories [18, 19]. In the formalism at hand, however, the introduction of the auxiliary field $K_{\mu\nu}$ makes it possible to rewrite conformal gravity with fourth-order derivatives to a second-order derivative theory, so we can apply the usual counting method. The fields $g_{\mu\nu}, \phi, K_{\mu\nu}$ and A_μ have 10, 1, 10 and 4 degrees of freedom, respectively. We have three kinds of local symmetries, those are, the GCT, Weyl and Stückelberg symmetries with 4, 1 and 4 degrees of freedom, respectively. Thus, we have totally $(10 + 1 + 10 + 4) - (4 + 1 + 4) \times 2 = 7$ physical degrees of freedom, which will turn out to be the massless graviton of 2 physical degrees with positive-definite norm and the spin-2 massive ghost of 5 degrees with indefinite norm.

3. Quantum theory

To fix three local symmetries and obtain a BRST invariant quantum Lagrangian, we have to introduce three kinds of gauge fixing conditions and the corresponding Faddeev-Popov (FP) ghost terms in the classical Lagrangian (7). In our previous papers [10–12], we have constructed two independent BRST transformations corresponding to general coordinate transformation (GCT) and Weyl transformation in the sense that the two nilpotent BRST charges anticommute with each other. To do so, it has been emphasized that a gauge condition for one local symmetry must respect the other symmetry [11]. However, it will turn out that we cannot find such suitable gauge fixing conditions in the present formalism since the gauge fixing condition for the Stückelberg gauge transformation necessarily breaks the Weyl symmetry [13]. Thus, in this article, instead of making three independent BRST charges we will construct only two independent BRST charges.

The suitable gauge condition for the GCT, which preserves the maximal global symmetry, is given by "the extended de Donder gauge condition" [11]

$$\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0, \tag{11}$$

where we have defined $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$. This gauge condition breaks the GCT (8) but is invariant under both the Weyl transformation (9) and the Stückelberg transformation (10).

As the gauge fixing condition for the Weyl transformation, we shall choose, what we call, "the traceless gauge condition":[3]

$$K - 2\nabla_\mu A^\mu = 0. \tag{12}$$

Let us note that the traceless gauge condition is invariant under the GCT (8) and the Stückelberg transformation (10).

Finally, let us consider the gauge fixing condition for the Stückelberg transformation. It is here that we cannot find the gauge fixing condition which breaks the Stückelberg transformation but is invariant under both the GCT and the Weyl transformation. We shall take a gauge fixing condition

$$\nabla_\mu K^{\mu\nu} = 0. \tag{13}$$

Since this gauge condition is manifestly invariant under the GCT but is not so under the Weyl transformation, we cannot define three independent BRST charges, but only two independent BRST charges. We will call this gauge condition (13) "the K-gauge".

[3] As seen in Eq. (**??**), this gauge condition is equivalent to the condition of the vanishing scalar curvature, $R = 0$ at the classical level [20–22].

The BRST transformation corresponding to the GCT, which is called GCT BRST transformation $\delta_B^{(1)}$, can be obtained from (8) by replacing the transformation parameter ξ^μ with the Faddeev-Popov (FP) ghost c^μ

$$\begin{aligned}
&\delta_B^{(1)} g_{\mu\nu} = -(\nabla_\mu c_\nu + \nabla_\nu c_\mu) = -(c^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu c^\alpha g_{\alpha\nu} + \partial_\nu c^\alpha g_{\alpha\mu}),\\
&\delta_B^{(1)} \phi = -c^\alpha \partial_\alpha \phi, \quad \delta_B^{(1)} K_{\mu\nu} = -c^\alpha \nabla_\alpha K_{\mu\nu} - \nabla_\mu c^\alpha K_{\alpha\nu} - \nabla_\nu c^\alpha K_{\mu\alpha},\\
&\delta_B^{(1)} A_\mu = -c^\alpha \nabla_\alpha A_\mu - \nabla_\mu c^\alpha A_\alpha, \qquad \delta_B^{(1)} c^\mu = -c^\alpha \partial_\alpha c^\mu,\\
&\delta_B^{(1)} \bar{c}_\mu = i B_\mu, \quad \delta_B^{(1)} B_\mu = 0, \quad \delta_B^{(1)} b_\mu = -c^\alpha \partial_\alpha b_\mu,
\end{aligned} \tag{14}$$

where $\bar{c}_\mu$ and B_μ are respectively an antighost and a Nakanishi-Lautrup (NL) field, and a new NL field b_μ is defined as

$$b_\mu = B_\mu - i c^\alpha \partial_\alpha \bar{c}_\mu, \tag{15}$$

which will be used in place of B_μ in what follows.

On the other hand, because of the K-gauge condition (13), in order to construct another BRST transformation which is independent of the GCT BRST transformation (14), we make a BRST transformation in a such way that it involves both the Weyl and the Stückelberg transformations simultaneously. This new BRST transformation $\delta_B^{(2)}$, which we call “WS BRST transformation”, can be made by replacing Λ and ε_μ with the FP ghosts c and ζ_μ, respectively, as follows

$$\begin{aligned}
&\delta_B^{(2)} g_{\mu\nu} = 2c g_{\mu\nu}, \quad \delta_B^{(2)} \phi = -c\phi,\\
&\delta_B^{(2)} K_{\mu\nu} = \frac{\gamma}{\alpha} \nabla_\mu \nabla_\nu c - 2(A_\mu \partial_\nu c + A_\nu \partial_\mu c - g_{\mu\nu} A_\alpha \partial^\alpha c) + \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu,\\
&\delta_B^{(2)} A_\mu = \zeta_\mu, \quad \delta_B^{(2)} \bar{c} = iB, \quad \delta_B^{(2)} c = \delta_B^{(2)} B = 0,\\
&\delta_B^{(2)} \bar{\zeta}_\mu = i\beta_\mu, \quad \delta_B^{(2)} \zeta_\mu = \delta_B^{(2)} \beta_\mu = 0,
\end{aligned} \tag{16}$$

where $\bar{c}$ and $\bar{\zeta}_\mu$ are antighosts, and B and β_μ are NL fields. In place of ζ_μ, it is more convenient to introduce a new FP ghost $\tilde{\zeta}_\mu$, which is defined as

$$\tilde{\zeta}_\mu = \zeta_\mu + \frac{\gamma}{2\alpha} \partial_\mu c. \tag{17}$$

In addition to it, we introduce a new NL field b which is defined as

$$b = B + 2i\bar{c}c. \tag{18}$$

Using the new FP ghost $\tilde{\zeta}_\mu$ and the new b field, the WS BRST transformation for $K_{\mu\nu}$, A_μ, $\tilde{\zeta}_\mu$ and b can be written as

$$\begin{aligned}
&\delta_B^{(2)} K_{\mu\nu} = \nabla_\mu \tilde{\zeta}_\nu + \nabla_\nu \tilde{\zeta}_\mu - 2(A_\mu \partial_\nu c + A_\nu \partial_\mu c - g_{\mu\nu} A_\alpha \partial^\alpha c),\\
&\delta_B^{(2)} A_\mu = \tilde{\zeta}_\mu - \frac{\gamma}{2\alpha} \partial_\mu c, \quad \delta_B^{(2)} \tilde{\zeta}_\mu = 0, \quad \delta_B^{(2)} b = -2bc.
\end{aligned} \tag{19}$$

In order to make the two nilpotent BRST transformations be anticommutative, i.e., $\{\delta_B^{(1)}, \delta_B^{(2)}\} = 0$, we must determine the remaining BRST transformations: As for the GCT BRST transformation, the BRST transformations on fields, which do not appear in (14) but appear in (16) are determined in such a way that they coincide with their tensor structure, for instance,

$$\delta_B^{(1)} c = -c^\alpha \partial_\alpha c, \qquad \delta_B^{(1)} \tilde{\zeta}_\mu = -c^\alpha \nabla_\alpha \tilde{\zeta}_\mu - \nabla_\mu c^\alpha \tilde{\zeta}_\alpha. \tag{20}$$

On the other hand, in cases of the WS BRST transformations, one simply defines the vanishing BRST transformations, e.g.,

$$\delta_B^{(2)} b_\mu = \delta_B^{(2)} c^\mu = \delta_B^{(2)} \bar{c}_\mu = 0. \tag{21}$$

Now that we have chosen gauge fixing conditions and established BRST transformations, we can construct a gauge fixed and BRST invariant quantum Lagrangian by following the standard recipe

$$\begin{aligned}
\mathcal{L}_q &= \mathcal{L}_c + i\delta_B^{(1)}(\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\nu) + i\delta_B^{(2)}\{\sqrt{-g}[\bar{c}(K - 2\nabla_\mu A^\mu) + \bar{\zeta}_\nu \nabla_\mu K^{\mu\nu}]\} \\
&= \sqrt{-g}\Bigg\{\frac{1}{12}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \gamma G_{\mu\nu}K^{\mu\nu} + \alpha[(K_{\mu\nu} - \nabla_\mu A_\nu \\
&- \nabla_\nu A_\mu)^2 - (K - 2\nabla_\rho A^\rho)^2]\Bigg\} - \tilde{g}^{\mu\nu}\phi^2(\partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda) \\
&- \sqrt{-g}\, b\,(K - 2\nabla_\mu A^\mu) + i\frac{\gamma}{\alpha}\tilde{g}^{\mu\nu}\partial_\mu\bar{c}\partial_\nu c - \sqrt{-g}\nabla_\mu K^{\mu\nu}\beta_\nu \\
&+ i\sqrt{-g}\nabla^\mu\bar{\zeta}^\nu[\nabla_\mu\tilde{\zeta}_\nu + \nabla_\nu\tilde{\zeta}_\mu - 2(A_\mu\partial_\nu c + A_\nu\partial_\mu c - g_{\mu\nu}A_\alpha\partial^\alpha c)] \\
&- i\sqrt{-g}\bar{\zeta}^\mu(2K_{\mu\nu}\partial^\nu c - K\partial_\mu c),
\end{aligned} \tag{22}$$

where surface terms are dropped.

4. Canonical commutation relations

In this section, we derive the concrete expressions of canonical conjugate momenta and set up the canonical (anti)commutation relations (CCRs), which will be used in evaluating various equal-time (anti)commutation relations (ETCRs) among fundamental variables. To simplify various expressions, we obey the following abbreviations adopted in the textbook of Nakanishi and Ojima [5]:

$$\begin{aligned}
[A, B'] &= [A(x), B(x')]|_{x^0 = x'^0}, \qquad \delta^3 = \delta(\vec{x} - \vec{x}'), \\
\tilde{f} &= \frac{1}{\tilde{g}^{00}} = \frac{1}{\sqrt{-g}g^{00}},
\end{aligned} \tag{23}$$

where we assume that $\tilde{g}^{00}$ is invertible.

To remove second order derivatives of the metric involved in R and $G_{\mu\nu}$, and regard b_μ as a non-canonical variable, we perform the integration by parts and rewrite the Lagrangian (22) as

$$
\begin{aligned}
\mathcal{L}_q &= -\frac{1}{12}\tilde{g}^{\mu\nu}\phi^2(\Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\alpha} - \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\sigma\nu}) - \frac{1}{6}\phi\partial_\mu\phi(\tilde{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \tilde{g}^{\mu\nu}\Gamma^\alpha_{\nu\alpha}) \\
&+ \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \gamma\sqrt{-g}(\Gamma^\alpha_{\mu\nu}\partial_\alpha - \Gamma^\alpha_{\mu\alpha}\partial_\nu + \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu} - \Gamma^\alpha_{\mu\alpha}\Gamma^\beta_{\nu\beta})\bar{K}^{\mu\nu} \\
&+ \alpha\sqrt{-g}[(K_{\mu\nu} - \nabla_\mu A_\nu - \nabla_\nu A_\mu)^2 - (K - 2\nabla_\mu A^\mu)^2] \\
&+ \partial_\mu(\tilde{g}^{\mu\nu}\phi^2)b_\nu - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\rho\partial_\nu c^\rho - \sqrt{-g}\,b\,(K - 2\nabla_\mu A^\mu) + i\frac{\gamma}{\alpha}\tilde{g}^{\mu\nu}\partial_\mu\bar{c}\partial_\nu c \\
&- \sqrt{-g}\nabla_\mu K^{\mu\nu}\cdot\beta_\nu + i\sqrt{-g}\nabla^\mu\bar{\zeta}^\nu[\nabla_\mu\tilde{\zeta}_\nu + \nabla_\nu\tilde{\zeta}_\mu - 2(A_\mu\partial_\nu c \\
&+ A_\nu\partial_\mu c - g_{\mu\nu}A_\alpha\partial^\alpha c)] - i\sqrt{-g}\bar{\zeta}^\mu(2K_{\mu\nu}\partial^\nu c - K\partial_\mu c) + \partial_\mu\mathcal{V}^\mu,
\end{aligned} \tag{24}
$$

where $\bar{K}_{\mu\nu}$ is defined as

$$
\bar{K}_{\mu\nu} \equiv K_{\mu\nu} - \frac{1}{2}g_{\mu\nu}K, \qquad \bar{K} \equiv g^{\mu\nu}\bar{K}_{\mu\nu}, \tag{25}
$$

and a surface term $\mathcal{V}^\mu$ is given by

$$
\begin{aligned}
\mathcal{V}^\mu &= \frac{1}{12}\phi^2(\tilde{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - \tilde{g}^{\mu\nu}\Gamma^\alpha_{\nu\alpha}) + \gamma\sqrt{-g}(\Gamma^\mu_{\alpha\beta}\bar{K}^{\alpha\beta} - \Gamma^\alpha_{\alpha\nu}\bar{K}^{\mu\nu}) \\
&- \tilde{g}^{\mu\nu}\phi^2 b_\nu.
\end{aligned} \tag{26}
$$

Since the NL fields b_μ, b and β_μ have no derivatives in $\mathcal{L}_q$, we can regard them as non-canonical variables.

Using this Lagrangian (24), it is straightforward to derive the concrete expressions of canonical

PoS(CORFU2023)158

conjugate momenta. The result is given by

$$
\begin{aligned}
\pi_g^{\mu\nu} &= \frac{\partial \mathcal{L}_q}{\partial \dot{g}_{\mu\nu}} \\
&= -\frac{1}{24}\sqrt{-g}\phi^2 \Big[-g^{0\lambda} g^{\mu\nu} g^{\sigma\tau} - g^{0\tau} g^{\mu\lambda} g^{\nu\sigma} - g^{0\sigma} g^{\mu\tau} g^{\nu\lambda} + g^{0\lambda} g^{\mu\tau} g^{\nu\sigma} \\
&+ g^{0\tau} g^{\mu\nu} g^{\lambda\sigma} + g^{0(\mu} g^{\nu)\lambda} g^{\sigma\tau} \Big] \partial_\lambda g_{\sigma\tau} - \frac{1}{6}\sqrt{-g}\Big[g^{0(\mu} g^{\nu)\rho} - g^{\mu\nu} g^{0\rho} \Big] \phi \partial_\rho \phi \\
&- \frac{1}{2}\sqrt{-g}(2 g^{0(\mu} g^{\nu)\rho} - g^{\mu\nu} g^{0\rho})(\phi^2 b_\rho + 2 b A_\rho) \\
&- \gamma\sqrt{-g}\Bigg[\nabla^{(\mu} \bar{K}^{\nu)0} - \frac{1}{2}\nabla^0 \bar{K}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \partial_\alpha \bar{K}^{0\alpha} - g^{\mu\nu} \Gamma^\beta_{\alpha\beta} \bar{K}^{0\alpha} \\
&- 2\Gamma^0_{\rho\sigma} g^{\rho(\mu} \bar{K}^{\nu)\sigma} + \Gamma^\alpha_{\rho\alpha} (g^{\rho(\mu} \bar{K}^{\nu)0} + g^{0(\mu} \bar{K}^{\nu)\rho}) \Bigg] \\
&+ 2\alpha\sqrt{-g}\Big[2\hat{K}^{0(\mu} A^{\nu)} - \hat{K}^{\mu\nu} A^0 - \hat{K}(2 g^{0(\mu} A^{\nu)} - g^{\mu\nu} A^0) \Big] \\
&+ \frac{1}{2}\sqrt{-g}(2 g^{0(\mu} K^{\nu)\rho} + g^{0\rho} K^{\mu\nu} - g^{\mu\nu} K^{0\rho})\beta_\rho, \\
&- i\sqrt{-g}(2 g^{0\alpha} g^{\beta(\mu} \bar{\zeta}^{\nu)} - \bar{\zeta}^0 g^{\alpha\mu} g^{\beta\nu})(\nabla_{(\alpha} \tilde{\zeta}_{\beta)} - 2 A_{(\alpha} \partial_{\beta)} c + g_{\alpha\beta} A_\gamma \partial^\gamma c) \\
&- i\sqrt{-g}[(\nabla^0 \bar{\zeta}^{(\mu} + \nabla^{(\mu} \bar{\zeta}^{|0|}) \tilde{\zeta}^{\nu)} - \nabla^{(\mu} \bar{\zeta}^{\nu)} \tilde{\zeta}^0], \\
\pi_\phi &= \frac{\partial \mathcal{L}_q}{\partial \dot{\phi}} = \tilde{g}^{0\mu} \partial_\mu \phi + 2\tilde{g}^{0\mu} \phi b_\mu + \frac{1}{6}\phi(-\tilde{g}^{\alpha\beta} \Gamma^0_{\alpha\beta} + \tilde{g}^{0\alpha} \Gamma^\beta_{\alpha\beta}), \\
\pi_K^{\mu\nu} &= \frac{\partial \mathcal{L}_q}{\partial \dot{K}_{\mu\nu}} = -\gamma\sqrt{-g}\Big[(g^{\mu\rho} g^{\nu\sigma} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma}) \Gamma^0_{\rho\sigma} - \frac{1}{2}(g^{0\mu} g^{\nu\rho} + g^{0\nu} g^{\mu\rho} - g^{\mu\nu} g^{0\rho}) \Gamma^\sigma_{\rho\sigma} \Big] \\
&- \frac{1}{2}\sqrt{-g}(g^{0\mu} \beta^\nu + g^{0\nu} \beta^\mu), \\
\pi_A^\mu &= \frac{\partial \mathcal{L}_q}{\partial \dot{A}_\mu} = -4\alpha\sqrt{-g}(\hat{K}^{0\mu} - g^{0\mu} \hat{K}) + 2\tilde{g}^{0\mu} b, \\
\pi_{c\mu} &= \frac{\partial \mathcal{L}_q}{\partial \dot{c}^\mu} = -i\tilde{g}^{0\nu} \phi^2 \partial_\nu \bar{c}_\mu, \quad \pi_{\bar{c}}^\mu = \frac{\partial \mathcal{L}_q}{\partial \dot{\bar{c}}_\mu} = i\tilde{g}^{0\nu} \phi^2 \partial_\nu c^\mu, \\
\pi_c &= \frac{\partial \mathcal{L}_q}{\partial \dot{c}} = i\frac{\gamma}{\alpha} \tilde{g}^{0\mu} \partial_\mu \bar{c} - 2i\sqrt{-g}[(\nabla^0 \bar{\zeta}^\mu + \nabla^\mu \bar{\zeta}^0) A_\mu - \nabla_\rho \bar{\zeta}^\rho A^0] - i\sqrt{-g}(2\bar{\zeta}^\mu K_\mu{}^0 - \bar{\zeta}^0 K), \\
\pi_{\bar{c}} &= \frac{\partial \mathcal{L}_q}{\partial \dot{\bar{c}}} = -i\frac{\gamma}{\alpha} \tilde{g}^{0\mu} \partial_\mu c, \\
\pi_{\tilde{\zeta}}^\mu &= \frac{\partial \mathcal{L}_q}{\partial \dot{\tilde{\zeta}}_\mu} = i\sqrt{-g}(\nabla^\mu \bar{\zeta}^0 + \nabla^0 \bar{\zeta}^\mu), \\
\pi_{\bar{\zeta}}^\mu &= \frac{\partial \mathcal{L}_q}{\partial \dot{\bar{\zeta}}_\mu} = -i\sqrt{-g}[\nabla^\mu \tilde{\zeta}^0 + \nabla^0 \tilde{\zeta}^\mu - 2(A^\mu \partial^0 c + A^0 \partial^\mu c - g^{0\mu} A_\rho \partial^\rho c)],
\end{aligned}
\tag{27}
$$

where we have defined the time derivative such as $\dot{g}_{\mu\nu} \equiv \frac{\partial g_{\mu\nu}}{\partial t} \equiv \frac{\partial g_{\mu\nu}}{\partial x^0} \equiv \partial_0 g_{\mu\nu}$, and differentiation of ghosts is taken from the right.

Next let us set up the canonical (anti)commutation relations:

$$
\begin{aligned}
&[g_{\mu\nu}, \pi_g^{\rho\lambda\prime}] = [K_{\mu\nu}, \pi_K^{\rho\lambda\prime}] = i\frac{1}{2}(\delta_\mu^\rho \delta_\nu^\lambda + \delta_\mu^\lambda \delta_\nu^\rho)\delta^3, \\
&[\phi, \pi_\phi'] = i\delta^3, \quad [A_\mu, \pi_A^{\nu\prime}] = i\delta_\mu^\nu \delta^3, \\
&\{c^\mu, \pi_{c\nu}'\} = \{\bar{c}_\nu, \pi_{\bar{c}}^{\mu\prime}\} = i\delta_\nu^\mu \delta^3, \quad \{c, \pi_c'\} = \{\bar{c}, \pi_{\bar{c}}'\} = i\delta^3, \\
&\{\bar{\zeta}_\mu, \pi_{\bar{\zeta}}^{\nu\prime}\} = \{\tilde{\zeta}_\mu, \pi_{\tilde{\zeta}}^{\nu\prime}\} = i\delta_\mu^\nu \delta^3,
\end{aligned} \tag{28}
$$

where the other (anti)commutation relations vanish. In setting up these CCRs, it is valuable to distinguish non-canonical variables from canonical ones. Recall again that in our formalism, the NL fields b_μ, b and β_μ are not canonical variables. On the basis of these CCRs, field equations and the BRST transformations, it is lengthy but straightforward to calculate all the equal-time (anti)commutation relations (ETCRs).

5. Linearized field equations

In this section, we analyze asymptotic fields under the assumption that all fields have their own asymptotic fields and there is no bound state. We also assume that all asymptotic fields are governed by the quadratic part of the quantum Lagrangian apart from possible renormalization.

First, let us define the gravitational field $\varphi_{\mu\nu}$ on a flat Minkowski metric $\eta_{\mu\nu}$ and the scalar fluctuation $\tilde{\phi}$ on a nonzero fixed scalar field ϕ_0:

$$
g_{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu}, \qquad \phi = \phi_0 + \tilde{\phi}. \tag{29}
$$

For sake of simplicity, we use the same notation for the other asymptotic fields as that for the interacting fields. Then, up to surface terms the quadratic part of the quantum Lagrangian (22) reads

$$
\begin{aligned}
\mathcal{L}_q = &\frac{1}{12}\phi_0^2\Big(\frac{1}{4}\varphi_{\mu\nu}\Box\varphi^{\mu\nu} - \frac{1}{4}\varphi\Box\varphi - \frac{1}{2}\varphi^{\mu\nu}\partial_\mu\partial_\rho\varphi_\nu{}^\rho + \frac{1}{2}\varphi^{\mu\nu}\partial_\mu\partial_\nu\varphi\Big) \\
&+\frac{1}{6}\phi_0\tilde{\phi}\left(-\Box\varphi + \partial_\mu\partial_\nu\varphi^{\mu\nu}\right) + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}\gamma(2\partial_\mu\partial_\rho\varphi_\nu{}^\rho - \Box\varphi_{\mu\nu} \\
&-\partial_\mu\partial_\nu\varphi)\bar{K}^{\mu\nu} + \alpha[(K_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu)^2 - (K - 2\partial_\mu A^\mu)^2] \\
&-\Big(2\phi_0\eta^{\mu\nu}\tilde{\phi} - \phi_0^2\varphi^{\mu\nu} + \frac{1}{2}\phi_0^2\eta^{\mu\nu}\varphi\Big)\partial_\mu b_\nu - i\phi_0^2\partial_\mu\bar{c}_\rho\partial^\mu c^\rho \\
&-b(K - 2\partial_\mu A^\mu) + i\frac{\gamma}{\alpha}\partial_\mu\bar{c}\partial^\mu c - \partial_\mu K^{\mu\nu}\beta_\nu + i\partial^\mu\bar{\zeta}^\nu(\partial_\mu\tilde{\zeta}_\nu + \partial_\nu\tilde{\zeta}_\mu).
\end{aligned} \tag{30}
$$

In this and next sections, the spacetime indices $\mu, \nu, \ldots$ are raised or lowered by the Minkowski metric $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and we define $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, $\varphi \equiv \eta^{\mu\nu}\varphi_{\mu\nu}$ and $\bar{K}_{\mu\nu} \equiv K_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}K$.

Based on this Lagrangian, it is straightforward to derive the linearized field equations:

$$\frac{1}{12}\phi_0^2\Big(\frac{1}{2}\Box\varphi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box\varphi - \partial_\rho\partial_{(\mu}\varphi_{\nu)}{}^\rho + \frac{1}{2}\partial_\mu\partial_\nu\varphi + \frac{1}{2}\eta_{\mu\nu}\partial_\rho\partial_\sigma\varphi^{\rho\sigma}\Big)$$
$$+\frac{1}{6}\phi_0\left(-\eta_{\mu\nu}\Box + \partial_\mu\partial_\nu\right)\tilde{\phi} + \frac{\gamma}{2}\Big(2\partial_\rho\partial_{(\mu}\bar{K}_{\nu)}{}^\rho - \Box\bar{K}_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{K}^{\rho\sigma}\Big)$$
$$+\phi_0^2\Big(\partial_{(\mu}b_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\partial_\rho b^\rho\Big) = 0. \tag{31}$$
$$\frac{1}{6}\phi_0(\Box\varphi - \partial_\mu\partial_\nu\varphi^{\mu\nu}) + \Box\tilde{\phi} + 2\phi_0\partial_\mu b^\mu = 0. \tag{32}$$
$$2\partial_\rho\partial_{(\mu}\varphi_{\nu)}{}^\rho - \Box\varphi_{\mu\nu} - \partial_\mu\partial_\nu\varphi - \eta_{\mu\nu}(\partial_\rho\partial_\sigma\varphi^{\rho\sigma} - \Box\varphi) + \frac{4\alpha}{\gamma}\Big[K_{\mu\nu}$$
$$-\partial_\mu A_\nu - \partial_\nu A_\mu - \eta_{\mu\nu}(K - 2\partial_\rho A^\rho)\Big] + \frac{2}{\gamma}(-\eta_{\mu\nu}b + \partial_{(\mu}\beta_{\nu)}) = 0. \tag{33}$$
$$\partial^\nu\Big[K_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu - \eta_{\mu\nu}(K - 2\partial_\rho A^\rho)\Big] - \frac{1}{2\alpha}\partial_\mu b = 0. \tag{34}$$
$$\partial_\mu\tilde{\phi} - \frac{1}{2}\phi_0\Big(\partial^\nu\varphi_{\mu\nu} - \frac{1}{2}\partial_\mu\varphi\Big) = 0. \tag{35}$$
$$K - 2\partial_\mu A^\mu = 0. \tag{36}$$
$$\partial_\mu K^{\mu\nu} = 0. \tag{37}$$
$$\Box c^\mu = \Box\bar{c}_\mu = \Box c = \Box\bar{c} = 0. \tag{38}$$
$$\Box\tilde{\zeta}_\mu + \partial_\mu\partial_\nu\tilde{\zeta}^\nu = \Box\bar{\zeta}_\mu + \partial_\mu\partial_\nu\bar{\zeta}^\nu = 0. \tag{39}$$

Now we are ready to simplify the field equations obtained above. Before doing so, it is more convenient to make use of the linearized BRST transformations in order to seek for the linearized field equations for the NL fields b_μ, b and β_μ. Taking the linearized GCT BRST transformation $\delta_B^{(1L)}\bar{c}_\mu = ib_\mu$ of $\Box\bar{c}_\mu = 0$ in Eq. (38) gives us

$$\Box b_\mu = 0. \tag{40}$$

Similarly, the linearized WS BRST transformation $\delta_B^{(2L)}\bar{c} = ib$ of $\Box\bar{c} = 0$ in Eq. (38) produces

$$\Box b = 0. \tag{41}$$

Finally, the linearized WS BRST transformation $\delta_B^{(2L)}\bar{\zeta}_\mu = i\beta_\mu$ of $\Box\bar{\zeta}_\mu + \partial_\mu\partial_\nu\zeta^\nu = 0$ in Eq. (39) yields

$$\Box\beta_\mu + \partial_\mu\partial_\nu\beta^\nu = 0. \tag{42}$$

Of course, Eqs. (40), (41) and (42) can be also derived by solving the linearized field equations directly.

Next, operating ∂^μ on Eq. (42) leads to

$$\Box\partial_\mu\beta^\mu = 0. \tag{43}$$

Moreover, acting $\Box$ on Eq. (42) and using Eq. (43), we have

$$\Box^2\beta_\mu = 0, \tag{44}$$

which implies that β_μ is a dipole field. In a perfectly similar manner, Eq. (39) gives us

$$\Box\partial_\mu\tilde{\zeta}^\mu = \Box^2\tilde{\zeta}_\mu = 0, \quad \Box\partial_\mu\bar{\zeta}^\mu = \Box^2\bar{\zeta}_\mu = 0. \tag{45}$$

Now it is easy to see that with the help of Eqs. (36) and (37), Eq. (34) provides[4]

$$\Box A_\mu + \partial_\mu\partial_\nu A^\nu + \frac{1}{2\alpha}\partial_\mu b = 0. \tag{46}$$

Given Eq. (41), this equation shows that the gauge field A_μ is a dipole field obeying

$$\Box\partial_\mu A^\mu = \Box^2 A_\mu = 0. \tag{47}$$

By use of Eq. (36), this equation means that K is a simple field:

$$\Box K = 0. \tag{48}$$

Next, to exhibit that the scalar field $\tilde{\phi}$ is also a dipole field, let us take the trace of Eq. (33) whose result can be written as

$$\Box\varphi - \partial_\mu\partial_\nu\varphi^{\mu\nu} = \frac{1}{\gamma}(4b - \partial_\mu\beta^\mu), \tag{49}$$

where Eq. (36) was utilized. Substituting this equation into Eq. (32) yields

$$\Box\tilde{\phi} = -\frac{\phi_0}{6\gamma}(4b - \partial_\mu\beta^\mu + 12\gamma\partial_\mu b^\mu). \tag{50}$$

Operating $\Box$ on this equation produces the desired result that $\tilde{\phi}$ is a dipole field:

$$\Box^2\tilde{\phi} = 0, \tag{51}$$

where we used Eqs. (40), (41) and (43). The divergence of Eq. (35) takes the form

$$\Box\tilde{\phi} = \frac{1}{2}\phi_0(\partial_\mu\partial_\nu\varphi^{\mu\nu} - \frac{1}{2}\Box\varphi). \tag{52}$$

Using three equations (49), (50) and (52), we can describe $\Box\varphi$ and $\partial_\mu\partial_\nu\varphi^{\mu\nu}$ as

$$\begin{aligned} \Box\varphi &= \frac{4}{3\gamma}(4b - \partial_\mu\beta^\mu - 6\gamma\partial_\mu b^\mu), \\ \partial_\mu\partial_\nu\varphi^{\mu\nu} &= \frac{1}{3\gamma}(4b - \partial_\mu\beta^\mu - 24\gamma\partial_\mu b^\mu), \end{aligned} \tag{53}$$

which imply two equations:

$$\Box^2\varphi = \Box\partial_\mu\partial_\nu\varphi^{\mu\nu} = 0. \tag{54}$$

[4]Note that as a consistency check, the WS BRST transformation of this equation gives rise to the linearized field equation for $\tilde{\zeta}_\mu$ in Eq. (39) when we use the field equation $\Box c = 0$ and ignore the quadratic term bc.

Here it is useful to express $K_{\mu\nu}$ in terms of the other fields by starting with Eq. (33) and utilizing some equations obtained thus far, whose result is described as

$$\begin{aligned} K_{\mu\nu} &= \partial_\mu A_\nu + \partial_\nu A_\mu + \frac{\gamma}{4\alpha}\Box\varphi_{\mu\nu} - \frac{\gamma}{\alpha\phi_0}\partial_\mu\partial_\nu\tilde{\phi} \\ &- \frac{1}{2\alpha}\left(\eta_{\mu\nu}b + \partial_{(\mu}\beta_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\beta^\rho\right). \end{aligned} \tag{55}$$

Finally, let us focus on the linearized Einstein equation (31). After some calculations using several equations, it turns out Eq. (31) can be rewritten into a more compact form

$$\begin{aligned} &\Box(\Box - m^2)\varphi_{\mu\nu} + \frac{1}{3\gamma}m^2\eta_{\mu\nu}(4b - \partial_\rho\beta^\rho) - \frac{4}{3\gamma}\partial_\mu\partial_\nu b + \frac{4}{3\gamma}\partial_\mu\partial_\nu\partial_\rho\beta^\rho \\ &+8\partial_\mu\partial_\nu\partial_\rho b^\rho - 24m^2\left(\partial_{(\mu}b_{\nu)} - \frac{1}{6}\eta_{\mu\nu}\partial_\rho b^\rho\right) = 0, \end{aligned} \tag{56}$$

where we have defined mass squared, $m^2 \equiv \frac{\phi_0^2}{24\alpha_c} = \frac{\alpha\phi_0^2}{3\gamma^2}$, which demands us to take the positive α as assumed before. Furthermore, operating $\Box$ on (56), we can obtain the gravitational equation for $\varphi_{\mu\nu}$:

$$\Box^2\left(\Box - m^2\right)\varphi_{\mu\nu} = 0. \tag{57}$$

Eq. (57) implies that there are both massless and massive modes in $\varphi_{\mu\nu}$. In order to disentangle these two modes, let us act $\Box$ on Eq. (55):

$$\Box\left(K_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{\gamma}{\alpha\phi_0}\partial_\mu\partial_\nu\tilde{\phi} + \frac{1}{2\alpha}\partial_{(\mu}\beta_{\nu)}\right) = \frac{\gamma}{4\alpha}\Box^2\varphi_{\mu\nu}. \tag{58}$$

This RHS can be further rewritten by using Eqs. (55) and (56) as

$$\begin{aligned} &\Box\left(K_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{\gamma}{\alpha\phi_0}\partial_\mu\partial_\nu\tilde{\phi} + \frac{1}{2\alpha}\partial_{(\mu}\beta_{\nu)}\right) \\ &= m^2\Big[K_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{\gamma}{\alpha\phi_0}\partial_\mu\partial_\nu\tilde{\phi} + \frac{1}{2\alpha}\partial_{(\mu}\beta_{\nu)} + \frac{1}{6\alpha}\eta_{\mu\nu}(b - \partial_\rho\beta^\rho) \\ &+\frac{1}{3\alpha m^2}\partial_\mu\partial_\nu(b - \partial_\rho\beta^\rho - 6\gamma\partial_\rho b^\rho) + \frac{6\gamma}{\alpha}\left(\partial_{(\mu}b_{\nu)} - \frac{1}{6}\eta_{\mu\nu}\partial_\rho b^\rho\right)\Big]. \end{aligned} \tag{59}$$

Provided that we take a linear combination of fields given as

$$\begin{aligned} \psi_{\mu\nu} &= K_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{\gamma}{\alpha\phi_0}\partial_\mu\partial_\nu\tilde{\phi} + \frac{1}{2\alpha}\partial_{(\mu}\beta_{\nu)} + \frac{1}{6\alpha}\eta_{\mu\nu}(b - \partial_\rho\beta^\rho) \\ &+\frac{1}{3\alpha m^2}\partial_\mu\partial_\nu(b - \partial_\rho\beta^\rho - 6\gamma\partial_\rho b^\rho) + \frac{6\gamma}{\alpha}\left(\partial_{(\mu}b_{\nu)} - \frac{1}{6}\eta_{\mu\nu}\partial_\rho b^\rho\right), \end{aligned} \tag{60}$$

we find that $\psi_{\mu\nu}$ corresponds to an infamous massive ghost of spin-2 of 5 physical degrees of freedom since it satisfies the equations of motion

$$(\Box - m^2)\psi_{\mu\nu} = \psi_\mu^\mu = \partial^\nu\psi_{\mu\nu} = 0. \tag{61}$$

On the other hand, if we choose the following linear combination

$$h_{\mu\nu} = \varphi_{\mu\nu} - \frac{12\gamma}{\phi_0^2}\psi_{\mu\nu} + \frac{2}{\phi_0}\eta_{\mu\nu}\tilde{\phi}, \tag{62}$$

we find that $h_{\mu\nu}$ obeys the field equation

$$\Box h_{\mu\nu} = -\frac{4}{3\gamma m^2}\partial_\mu\partial_\nu(b - \partial_\rho\beta^\rho - 6\gamma\partial_\rho b^\rho) - 24\partial_{(\mu}b_{\nu)},$$
$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{63}$$

Then, Eq. (63) implies that $h_{\mu\nu}$ is a dipole field satisfying

$$\Box^2 h_{\mu\nu} = 0. \tag{64}$$

In the next section we will show that two transverse components of $h_{\mu\nu}$ is nothing but a massless spin-2 graviton.

6. Analysis of physical states

Following the standard technique, we can calculate the four-dimensional (anti)commutation relations (4D CRs) between asymptotic fields. The point is that the simple pole fields, for instance, the Nakanishi-Lautrup field $b_\mu(x)$ obeying $\Box b_\mu = 0$, can be expressed in terms of the invariant delta function $D(x)$ as

$$b_\mu(x) = \int d^3z D(x-z)\overleftrightarrow{\partial}{}^z_0 b_\mu(z). \tag{65}$$

Here the invariant delta function $D(x)$ for massless simple pole fields and its properties are described as

$$D(x) = -\frac{i}{(2\pi)^3}\int d^4k\, \epsilon(k_0)\delta(k^2)e^{ikx}, \qquad \Box D(x) = 0,$$
$$D(-x) = -D(x), \qquad D(0,\vec{x}) = 0, \qquad \partial_0 D(0,\vec{x}) = \delta^3(x), \tag{66}$$

where $\epsilon(k_0) \equiv \frac{k_0}{|k_0|}$. With these properties, it is easy to see that the right-hand side (RHS) of Eq. (65) is independent of z^0, and this fact will be used in evaluating 4D CRs via the ETCRs shortly.

To illustrate the detail of the calculation, let us evaluate a 4D CR, $[h_{\mu\nu}(x), b_\rho(y)]$ explicitly. Using Eq. (65), it can be described as

$$\begin{aligned} & [h_{\mu\nu}(x), b_\rho(y)] \\ = & \int d^3z D(y-z)\overleftrightarrow{\partial}{}^z_0[h_{\mu\nu}(x), b_\rho(z)] \\ = & \int d^3z\Big(D(y-z)[h_{\mu\nu}(x), \dot{b}_\rho(z)] - \partial^z_0 D(y-z)[h_{\mu\nu}(x), b_\rho(z)]\Big). \end{aligned} \tag{67}$$

As mentioned above, since the RHS of Eq. (65) is independent of z^0, we put $z^0 = x^0$ in (67) and use relevant ETCRs to obtain

$$[h_{\mu\nu}(x), b_\rho(z)] = i\phi_0^{-2}(\delta^0_\mu\eta_{\rho\nu} + \delta^0_\nu\eta_{\rho\mu})\delta^3(x-z),$$
$$[h_{\mu\nu}(x), \dot{b}_\rho(z)] = -i\phi_0^{-2}(\delta^k_\mu\eta_{\rho\nu} + \delta^k_\nu\eta_{\rho\mu})\partial_k\delta^3(x-z). \tag{68}$$

Substituting Eq. (68) into Eq. (67), we can obtain the 4D CR

$$[h_{\mu\nu}(x), b_\rho(y)] = i\phi_0^{-2}(\eta_{\mu\rho}\partial_\nu + \eta_{\nu\rho}\partial_\mu)D(x-y). \tag{69}$$

In a similar manner, we can compute 4D CRs among $\psi_{\mu\nu}$, $h_{\mu\nu}$ and b_μ etc. To do that, let us note that since $\psi_{\mu\nu}$ obeys a massive simple pole equation (61), it can be expressed in terms of the invariant delta function $\Delta(x; m^2)$ for massive simple pole fields as

$$\psi_{\mu\nu}(x) = \int d^3z \Delta(x-z; m^2) \overleftrightarrow{\partial}{}_0^z \psi_{\mu\nu}(z), \tag{70}$$

where $\Delta(x; m^2)$ is defined as

$$\begin{aligned}
&\Delta(x; m^2) = -\frac{i}{(2\pi)^3}\int d^4k\, \epsilon(k_0)\delta(k^2+m^2)e^{ikx}, \quad (\Box - m^2)\Delta(x; m^2) = 0, \\
&\Delta(-x; m^2) = -\Delta(x; m^2), \quad \Delta(0, \vec{x}; m^2) = 0, \\
&\partial_0\Delta(0, \vec{x}; m^2) = \delta^3(x), \qquad \Delta(x; 0) = D(x).
\end{aligned} \tag{71}$$

As for $h_{\mu\nu}$, since it is a massless dipole field as can be seen in Eq. (64), it can be described as

$$h_{\mu\nu}(x) = \int d^3z \left[D(x-z)\overleftrightarrow{\partial}{}_0^z h_{\mu\nu}(z) + E(x-z)\overleftrightarrow{\partial}{}_0^z \Box h_{\mu\nu}(z)\right], \tag{72}$$

where we have introduced the invariant delta function $E(x)$ for massless dipole fields and its properties are given by

$$\begin{aligned}
&E(x) = -\frac{i}{(2\pi)^3}\int d^4k\, \epsilon(k_0)\delta'(k^2)e^{ikx}, \qquad \Box E(x) = D(x), \\
&E(-x) = -E(x), \qquad E(0, \vec{x}) = \partial_0 E(0, \vec{x}) = \partial_0^2 E(0, \vec{x}) = 0, \\
&\partial_0^3 E(0, \vec{x}) = -\delta^3(x).
\end{aligned} \tag{73}$$

As in Eq. (65), we can also show that the RHS of both (70) and (72) is independent of z^0.

By using the ETCRs summarized in Appendix A, after a lengthy but straightforward calculation, we find the following 4D CRs among $\psi_{\mu\nu}, h_{\mu\nu}, \tilde{\phi}, b_\mu, b, \beta_\mu, c^\mu, \bar{c}_\mu, c$ and $\bar{c}$:

$$\begin{aligned}
&[\psi_{\mu\nu}(x), \psi_{\sigma\tau}(y)] = -i\frac{\phi_0^2}{12\gamma^2}\Big[\frac{2}{3}\eta_{\mu\nu}\eta_{\sigma\tau} - \eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma} \\
&+\frac{1}{m^2}(\eta_{\mu\sigma}\partial_\nu\partial_\tau + \eta_{\mu\tau}\partial_\nu\partial_\sigma + \eta_{\nu\sigma}\partial_\mu\partial_\tau + \eta_{\nu\tau}\partial_\mu\partial_\sigma \\
&-\frac{2}{3}\eta_{\mu\nu}\partial_\sigma\partial_\tau - \frac{2}{3}\eta_{\sigma\tau}\partial_\mu\partial_\nu) - \frac{4}{3m^4}\partial_\mu\partial_\nu\partial_\sigma\partial_\tau\Big]\Delta(x-y; m^2). &(74)\\
&[h_{\mu\nu}(x), h_{\sigma\tau}(y)] = i\frac{12}{\phi_0^2}\Big[\eta_{\mu\nu}\eta_{\sigma\tau} - \eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma} \\
&+\frac{1}{m^2}(\eta_{\mu\sigma}\partial_\nu\partial_\tau + \eta_{\mu\tau}\partial_\nu\partial_\sigma + \eta_{\nu\sigma}\partial_\mu\partial_\tau + \eta_{\nu\tau}\partial_\mu\partial_\sigma \\
&-\frac{2}{3}\eta_{\mu\nu}\partial_\sigma\partial_\tau - \frac{2}{3}\eta_{\sigma\tau}\partial_\mu\partial_\nu) - \frac{4}{3m^4}\partial_\mu\partial_\nu\partial_\sigma\partial_\tau\Big]D(x-y) \\
&+i\frac{12}{\phi_0^2}\Big(\eta_{\mu\sigma}\partial_\nu\partial_\tau + \eta_{\mu\tau}\partial_\nu\partial_\sigma + \eta_{\nu\sigma}\partial_\mu\partial_\tau + \eta_{\nu\tau}\partial_\mu\partial_\sigma \\
&-\frac{4}{3m^2}\partial_\mu\partial_\nu\partial_\sigma\partial_\tau\Big)E(x-y). &(75)\\
&[h_{\mu\nu}(x), \psi_{\sigma\tau}(y)] = 0. &(76)\\
&[\psi_{\mu\nu}(x), b_\rho(y)] = [\psi_{\mu\nu}(x), b(y)] = [\psi_{\mu\nu}(x), \beta_\rho(y)] = 0. &(77)\\
&[h_{\mu\nu}(x), b_\rho(y)] = i\phi_0^{-2}(\eta_{\mu\rho}\partial_\nu + \eta_{\nu\rho}\partial_\mu)D(x-y). &(78)\\
&[h_{\mu\nu}(x), b(y)] = [h_{\mu\nu}(x), \beta_\rho(y)] = 0. &(79)\\
&[\tilde{\phi}(x), \tilde{\phi}(y)] = -i[D(x-y) - 2m^2E(x-y)]. &(80)\\
&[\tilde{\phi}(x), \psi_{\sigma\tau}(y)] = 0. &(81)\\
&[\tilde{\phi}(x), h_{\sigma\tau}(y)] = 2i\phi_0^{-1}[\eta_{\sigma\tau}D(x-y) + 2\partial_\sigma\partial_\tau E(x-y)]. &(82)\\
&[\tilde{\phi}(x), b(y)] = -i\frac{\alpha}{\gamma}\phi_0 D(x-y). &(83)\\
&[\tilde{\phi}(x), b_\rho(y)] = [\tilde{\phi}(x), \beta_\rho(y)] = 0. &(84)\\
&\{c^\mu(x), \bar{c}_\nu(y)\} = -\phi_0^{-2}\delta_\nu^\mu D(x-y). &(85)\\
&\{c(x), \bar{c}(y)\} = \frac{\alpha}{\gamma}D(x-y). &(86)
\end{aligned}$$

In particular, note that the negative sign in front of the RHS of Eq. (74) implies that the massive spin-2 field $\psi_{\mu\nu}$ has indefinite norm so it is sometimes called "*massive ghost*".

As usual, the physical Hilbert space $|\text{phys}\rangle$ is defined by the Kugo-Ojima subsidiary conditions [3]

$$\mathrm{Q_B^{(1)}}|\text{phys}\rangle = \mathrm{Q_B^{(2)}}|\text{phys}\rangle = 0, \tag{87}$$

where $\mathrm{Q_B^{(1)}}$ and $\mathrm{Q_B^{(2)}}$ are respectively BRST charges corresponding to the GCT and WS BRST transformations.

The GCT BRST transformation for the asymptotic fields[5] is given by

$$\begin{aligned}
&\delta_B^{(1)}\psi_{\mu\nu} = 0, \qquad \delta_B^{(1)} h_{\mu\nu} = -(\partial_\mu c_\nu + \partial_\nu c_\mu), \qquad \delta_B^{(1)}\tilde{\phi} = 0,\\
&\delta_B^{(1)} b_\mu = \delta_B^{(1)} b = \delta_B^{(1)}\beta_\mu = 0,\\
&\delta_B^{(1)}\bar{c}_\mu = i b_\mu, \qquad \delta_B^{(1)} c^\mu = \delta_B^{(1)} c = \delta_B^{(1)}\bar{c} = 0.
\end{aligned} \tag{88}$$

And the WS transformation for the asymptotic fields takes the form

$$\begin{aligned}
&\delta_B^{(2)}\psi_{\mu\nu} = \delta_B^{(2)} h_{\mu\nu} = 0, \qquad \delta_B^{(2)}\tilde{\phi} = -\phi_0 c,\\
&\delta_B^{(2)} b_\mu = \delta_B^{(2)} b = \delta_B^{(2)}\beta_\mu = 0,\\
&\delta_B^{(2)}\bar{c} = i b, \qquad \delta_B^{(2)} c^\mu = \delta_B^{(2)}\bar{c}_\mu = \delta_B^{(2)} c = 0.
\end{aligned} \tag{89}$$

Given the physical state conditions (87) and the two BRST transformations (88) and (89), it is easy to clarify the physical content of the theory under consideration: The physical modes are composed of both a spin-2 massive ghost $\psi_{\mu\nu}$ of mass m which has five physical degrees of freedom, and a spin-2 massless graviton which corresponds to two components of $h_{\mu\nu}$ (for instance, in the specific Lorentz frame $p_\mu = (p, 0, 0, p)$, the graviton corresponds to $\frac{1}{\sqrt{2}}(h_{11} - h_{22})$ and h_{12}.). On the other hand, the remaining four components of $h_{\mu\nu}$, b_μ, c^μ and $\bar{c}_\mu$ belong to a GCT-BRST quartet while $\tilde{\phi}$, b, c and $\bar{c}$ does a WS-BRST quartet. These quartets appear in the physical subspace only as zero norm states by the Kugo-Ojima subsidiary conditions (87). It is worthwhile to stress that the massive ghost with indefinite norm appears in the physical Hilbert space so the unitarity of the physical S-matrix is explicitly violated when there is no bound state.

7. A possible mechanism of ghost confinement

In the previous section, we have shown that when there is no bound state, the unitarity of the physical S-matrix is violated because of the presence of the massive ghost so the quantum conformal gravity under consideration is not regarded as a physically meaningful theory. In this section, we wish to present a possible mechanism of the confinement of the massive tensor ghost. Our idea stems from previous two ideas, one of which is a mechanism of color confinement in quantum chromodynamics (QCD) [23] and the other is the one of a massive ghost in the renormalizable quadratic quantum gravity [24].

Before delving into the detail, let us recall the main idea of this confinement mechanism. In the standard BRST formalism, using the BRST charge Q_B the physical Hilbert space $|\text{phys}\rangle$ is determined by

$$Q_B |\text{phys}\rangle = 0. \tag{90}$$

For instance, in QCD, this physical state condition is a sufficient condition to prohibit the appearance of unphysical modes such as the FP ghosts c and $\bar{c}$, the scalar component of A_μ (or the Nakanishi-Lautrup auxiliary field B) and the unphysical Higgs mode. The main idea of the confinement

[5] Recall that we use same fields for the interacting and the asymptotic fields. In this section, all the fields describe the asymptotic ones.

mechanism is to regard the very condition (90) as a sufficient condition for the confinement of the color or the massive ghost as well.

Actually, we can prove the following statement [23]: If one can prove the existence of a bound-state pole in C for any gauge-variant operator Φ transforming as

$$[iQ_B, \Phi(x)] = C(x), \tag{91}$$

then the operator Φ with vanishing FP ghost number is confined by the BRST-quartet mechanism [3]. Let us make this statement be more precise. First, we assume that the field operators $\Phi(x)$ and $C(x)$ have respectively asymptotic fields $\chi(x)$ and $\gamma(x)$. Then, Eq. (91) leads to

$$[iQ_B, \chi(x)] = \gamma(x), \tag{92}$$

which implies that the pair $\{\chi(x), \gamma(x)\}$ forms a BRST-doublet. Moreover, from [3], we can always find another Heisenberg operator $\overline{C}(x)$ with the FP ghost number -1 having the asymptotic field $\overline{\gamma}(x)$, which is FP-conjugate to $\gamma(x)$. This asymptotic field $\gamma(x)$ also forms another BRST-doublet

$$\{Q_B, \overline{\gamma}(x)\} = \beta(x), \tag{93}$$

where $\beta(x)$ is the asymptotic field of an operator $\mathcal{B}(x)$ with vanishing FP ghost number, which is defined as

$$\{Q_B, \overline{C}(x)\} = \mathcal{B}(x). \tag{94}$$

Finally, the two pairs of BRST-doublets $\{\chi(x), \gamma(x)\}$ and $\{\overline{\gamma}(x), \beta(x)\}$ constitute a BRST-quartet having the same quantum numbers such as mass and spin, and consequently these asymptotic fields emerge in the physical Hilbert space only as the zero-norm states, which means the confinement of the BRST-quartet.

In the articles [7, 24], the issue of the ghost confinement was considered in the quadratic gravity (or higher-derivative gravity), but there seems to be a problem. In the quadratic gravity, the local gauge symmetry is only the general coordinate symmetry, but this symmetry does not tell the massive ghost from the graviton but treat them as tensor fields on an equal footing. Thus, if the massive ghost were confined by the quartet mechanism, the graviton would be confined as well and vice versa, which is against experiments.

On the other hand, in the quantum conformal gravity under consideration, there are three different local gauge symmetries, from which one can construct two BRST transformations, those are, GCT BRST transformation and WS BRST transformation. In particular, since the WS BRST transformation includes Weyl BRST transformation, which has the ability to distinguish a massless particle and a massive one, there might be a possibility of confining only the massive ghost without touching the massless graviton. We will purpue this possibility in what follows.

For this purpose, let us first consider the generally covariant tensor fields corresponding to the massive ghost (60) and the massless graviton (62):

$$\begin{aligned}
\Psi_{\mu\nu} &= K_{\mu\nu} - \nabla_\mu A_\nu - \nabla_\nu A_\mu + \frac{\gamma}{\alpha\phi_0}\nabla_\mu\nabla_\nu\phi + \frac{1}{2\alpha}\nabla_{(\mu}\beta_{\nu)} + \frac{1}{6\alpha}g_{\mu\nu}(b - \nabla_\rho\beta^\rho) \\
&+ \frac{1}{3\alpha m^2}\nabla_{(\mu}\nabla_{\nu)}(b - \nabla_\rho\beta^\rho - 6\gamma\nabla_\rho b^\rho) + \frac{6\gamma}{\alpha}\Big(\nabla_{(\mu}b_{\nu)} - \frac{1}{6}g_{\mu\nu}\nabla_\rho b^\rho\Big), \\
H_{\mu\nu} &= g_{\mu\nu} - \frac{12\gamma}{\phi_0^2}\Psi_{\mu\nu} + \frac{2}{\phi_0}g_{\mu\nu}\phi.
\end{aligned} \tag{95}$$

Next, taking the WS BRST transformation of $\Psi_{\mu\nu}$ produces

$$\begin{aligned} \delta_B^{(2)}\Psi_{\mu\nu} &\equiv \{iQ_B^{(2)}, \Psi_{\mu\nu}\} \\ &= \frac{\gamma}{\alpha}(1-\frac{\phi}{\phi_0})\nabla_\mu\nabla_\nu c + \frac{\gamma}{\alpha\phi_0}(-4\nabla_{(\mu}c\nabla_{\nu)}\phi - c\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\nabla_\rho c\nabla^\rho\phi) + \dots, \end{aligned} \tag{96}$$

where ... denotes terms involving the Nakanishi-Lautrup fields, b_μ, β_μ and b.

At this point, we assume that the RHS has a bound state $C_{\mu\nu}$, i.e.,

$$\{iQ_B^{(2)}, \Psi_{\mu\nu}(x)\} = C_{\mu\nu}(x). \tag{97}$$

When we define the asymptotic fields, $\Psi_{\mu\nu} \rightarrow \chi_{\mu\nu}$ and $C_{\mu\nu} \rightarrow \gamma_{\mu\nu}$ for $|x^0| \rightarrow \infty$, Eq. (97) gives us

$$\{iQ_B^{(2)}, \chi_{\mu\nu}(x)\} = \gamma_{\mu\nu}(x), \tag{98}$$

which means that the pair $\{\chi_{\mu\nu}(x), \gamma_{\mu\nu}(x)\}$ forms a BRST-doublet.

Then, the operator $\overline{C}_{\mu\nu}$, which is FP-conjugate to $C_{\mu\nu}$, can be made by replacing the ghost c by the antighost $\bar{c}$ on the RHS of Eq. (96). Moreover, we can construct an operator $\mathcal{B}$ with vanishing FP ghost number by taking the WS BRST transformation

$$\{Q_B, \overline{C}_{\mu\nu}(x)\} = \mathcal{B}_{\mu\nu}(x). \tag{99}$$

When we define the asymptotic fields, $\overline{C}_{\mu\nu} \rightarrow \overline{\gamma}_{\mu\nu}$ and $\mathcal{B}_{\mu\nu} \rightarrow \beta_{\mu\nu}$ for $|x^0| \rightarrow \infty$, Eq. (99) leads to

$$\{Q_B, \overline{\gamma}_{\mu\nu}(x)\} = \beta_{\mu\nu}(x). \tag{100}$$

The asymptotic fields $\{\overline{\gamma}_{\mu\nu}(x), \beta_{\mu\nu}(x)\}$ therefore also form another BRST-doublet. Finally, the two pairs of BRST-doublets $\{\chi_{\mu\nu}(x), \gamma_{\mu\nu}(x)\}$ and $\{\overline{\gamma}_{\mu\nu}(x), \beta_{\mu\nu}(x)\}$ constitute a BRST-quartet and thus these asymptotic fields appear in the physical Hilbert space in the zero-norm combinations, meaning the confinement of the massive ghost.

The key observation is that even if the massive ghost is confined, there is a chance that the graviton is not confined but appear in the physical Hilbert space with positive-definite norm. To show this possibility, let us take the WS BRST transformation of $H_{\mu\nu}$ in Eq. (95) whose result takes the form

$$\begin{aligned} \delta_B^{(2)} H_{\mu\nu} &\equiv \{iQ_B^{(2)}, H_{\mu\nu}\} \\ &= 2(1+\frac{\phi}{\phi_0})c g_{\mu\nu} - \frac{12\gamma}{\phi_0^2}C_{\mu\nu}. \end{aligned} \tag{101}$$

Provided that the first term on the RHS has also a bound state in such a way that it cancles out the second term, we can have

$$\{iQ_B^{(2)}, H_{\mu\nu}\} = 0. \tag{102}$$

As a result, the massless graviton belongs to the physical Hilbert space with positive-definite norm. We shall detail the issue of the ghost confinement in a separate publication in future.

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Appendix

A. Various equal-time commutation relations in the linearized level

In this Appendix, we simply write down various equal-time (anti)commutation relations (ETCRs) which are useful in deriving the four-dimensional commutation relations (4D CRs) in Eqs. (74)-(86). These ETCRs can be derived by using the canonical (anti)commutation relations (CRs), the BRST transformations and the linearized field equations.

$$
\begin{aligned}
&[\dot{\varphi}_{\mu\nu}, \varphi'_{\sigma\tau}] = 16 i \phi_0^{-2} \delta_\mu^0 \delta_\nu^0 \delta_\sigma^0 \delta_\tau^0 \delta^3, \\
&[\varphi_{\mu\nu}, \dot{\tilde{\phi}}'] = 4 i \phi_0^{-1} \delta_\mu^0 \delta_\nu^0 \delta^3, \qquad [\varphi_{\mu\nu}, \ddot{\tilde{\phi}}'] = -4 i \phi_0^{-1} (\delta_\mu^0 \delta_\nu^i + \delta_\mu^i \delta_\nu^0) \partial_i \delta^3, \\
&[\varphi_{\mu\nu}, \dddot{\tilde{\phi}}'] = 4 i \phi_0^{-1} (-m^2 \eta_{\mu\nu} + 2 \delta_\mu^0 \delta_\nu^0 \Delta + \delta_\mu^i \delta_\nu^j \partial_i \partial_j) \delta^3, \\
&[\varphi_{\mu\nu}, \dot{A}'_\sigma] = 0, \qquad [\varphi_{\mu\nu}, \ddot{A}'_\sigma] = -i \frac{1}{2\gamma} \eta_{\mu\nu} \delta_\sigma^0 \delta^3, \\
&[\dot{\varphi}_{\mu\nu}, K'_{\sigma\tau}] = -i \frac{1}{\gamma} \Big[\eta_{\mu\nu} \eta_{\sigma\tau} - \eta_{\mu\sigma} \eta_{\nu\tau} - \eta_{\mu\tau} \eta_{\nu\sigma} + \eta_{\mu\nu} \delta_\sigma^0 \delta_\tau^0 + \frac{2}{3} \eta_{\sigma\tau} \delta_\mu^0 \delta_\nu^0 \\
&- (\eta_{\mu\sigma} \delta_\tau^0 + \eta_{\mu\tau} \delta_\sigma^0) \delta_\nu^0 - (\eta_{\nu\sigma} \delta_\tau^0 + \eta_{\nu\tau} \delta_\sigma^0) \delta_\mu^0 - \frac{4}{3} \delta_\mu^0 \delta_\nu^0 \delta_\sigma^0 \delta_\tau^0 \Big] \delta^3, \\
&[\varphi_{\mu\nu}, b'_\rho] = i \phi_0^{-2} (\delta_\mu^0 \eta_{\rho\nu} + \delta_\nu^0 \eta_{\rho\mu}) \delta^3, \qquad [\varphi_{\mu\nu}, \dot{b}'_\rho] = -i \phi_0^{-2} (\delta_\mu^i \eta_{\rho\nu} + \delta_\nu^i \eta_{\rho\mu}) \partial_i \delta^3, \\
&[\varphi_{\mu\nu}, b'] = 0, \qquad [\varphi_{\mu\nu}, \dot{b}'] = -2 i \frac{\alpha}{\gamma} \eta_{\mu\nu} \delta^3.
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
&[\dot{\tilde{\phi}}, \tilde{\phi}'] = i \delta^3, \qquad [\dot{\tilde{\phi}}, \ddot{\tilde{\phi}}'] = i (\Delta - 2m^2) \delta^3, \qquad [\ddot{\tilde{\phi}}, \tilde{\phi}'] = [\ddot{\tilde{\phi}}, \ddot{\tilde{\phi}}'] = 0, \\
&[\dddot{\tilde{\phi}}, \ddot{\tilde{\phi}}'] = i \Delta (\Delta - 4m^2) \delta^3, \\
&[\dot{\tilde{\phi}}, K'_{\sigma\tau}] = i \frac{\phi_0}{6\gamma} (\eta_{\sigma\tau} + \delta_\sigma^0 \delta_\tau^0) \delta^3, \qquad [\ddot{\tilde{\phi}}, K'_{\sigma\tau}] = i \frac{\phi_0}{6\gamma} (\delta_\sigma^0 \delta_\tau^i + \delta_\tau^0 \delta_\sigma^i) \partial_i \delta^3, \\
&[\dddot{\tilde{\phi}}, K'_{\sigma\tau}] = i \frac{\phi_0}{6\gamma} [(\eta_{\sigma\tau} + 2 \delta_\sigma^0 \delta_\tau^0) \Delta + \delta_\sigma^i \delta_\tau^j \partial_i \partial_j] \delta^3, \\
&[\dot{\tilde{\phi}}, A'_\sigma] = 0, \qquad [\ddot{\tilde{\phi}}, A'_\sigma] = [\tilde{\phi}, \ddot{A}_\sigma{}'] = i \frac{\phi_0}{4\gamma} \delta_\sigma^0 \delta^3, \\
&[\dddot{\tilde{\phi}}, A'_\sigma] = -i \frac{\phi_0}{4\gamma} \delta_\sigma^i \partial_i \delta^3, \qquad [\ddot{\tilde{\phi}}, \ddot{A}'_\sigma] = i \frac{\phi_0}{2\gamma} \delta_\sigma^0 \Delta \delta^3, \\
&[\dot{\tilde{\phi}}, b'] = -i \frac{\alpha}{\gamma} \phi_0 \delta^3, \qquad [\ddot{\tilde{\phi}}, \dot{b}'] = i \frac{\alpha}{\gamma} \phi_0 \Delta \delta^3.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&[\dot{K}_{\mu\nu}, K'_{\sigma\tau}] = i\frac{\phi_0^2}{12\gamma^2}\Big[-\frac{2}{3}\eta_{\mu\nu}\eta_{\sigma\tau} + \eta_{\mu\sigma}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\sigma} - \frac{2}{3}(\delta_\mu^0\delta_\nu^0\eta_{\sigma\tau} + \delta_\sigma^0\delta_\tau^0\eta_{\mu\nu}) \\
&+\delta_\mu^0\delta_\sigma^0\eta_{\nu\tau} + \delta_\mu^0\delta_\tau^0\eta_{\nu\sigma} + \delta_\nu^0\delta_\sigma^0\eta_{\mu\tau} + \delta_\nu^0\delta_\tau^0\eta_{\mu\sigma} + \frac{4}{3}\delta_\mu^0\delta_\nu^0\delta_\sigma^0\delta_\tau^0\Big]\delta^3, \\
&[K_{\mu\nu}, \dot{A}'_\sigma] = [K_{\mu\nu}, \ddot{A}'_\sigma] = 0, \\
&[K_{\mu\nu}, \beta'_\sigma] = i(\delta_\mu^0\eta_{\sigma\nu} + \delta_\nu^0\eta_{\sigma\mu} + \delta_\mu^0\delta_\nu^0\delta_\sigma^0)\delta^3, \\
&[K_{\mu\nu}, \dot{\beta}'_\sigma] = i(\delta_\mu^0\delta_\nu^0\delta_\sigma^i + \delta_\mu^0\delta_\nu^i\delta_\sigma^0 + \delta_\mu^i\delta_\nu^0\delta_\sigma^0 + \eta_{\mu\sigma}\delta_\nu^i + \eta_{\nu\sigma}\delta_\mu^i)\partial_i\delta^3, \\
&[K_{\mu\nu}, \dot{\beta}'_\sigma] = i(\delta_\mu^0\delta_\nu^0\delta_\sigma^i + \delta_\mu^0\delta_\nu^i\delta_\sigma^0 + \delta_\mu^i\delta_\nu^0\delta_\sigma^0 + \eta_{\mu\sigma}\delta_\nu^i + \eta_{\nu\sigma}\delta_\mu^i)\partial_i\delta^3, \\
&[K_{\mu\nu}, \ddot{\beta}'_\sigma] = i[(\delta_\mu^0\eta_{\nu\sigma} + \delta_\nu^0\eta_{\mu\sigma} + 2\delta_\mu^0\delta_\nu^0\delta_\sigma^0)\Delta + (\delta_\mu^0\delta_\nu^i\delta_\sigma^j + \delta_\mu^i\delta_\nu^0\delta_\sigma^j \\
&+\delta_\mu^i\delta_\nu^j\delta_\sigma^0)\partial_i\partial_j]\delta^3, \\
&[K_{\mu\nu}, \dddot{\beta}'_\sigma] = i\{[\delta_\mu^i\eta_{\nu\sigma} + \delta_\nu^i\eta_{\mu\sigma} + 2(\delta_\mu^0\delta_\nu^0\delta_\sigma^i + \delta_\mu^0\delta_\nu^i\delta_\sigma^0 + \delta_\mu^i\delta_\nu^0\delta_\sigma^0)]\Delta \\
&+\delta_\mu^i\delta_\nu^j\delta_\sigma^k\partial_j\partial_k\}\partial_i\delta^3.
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
&[\dot{A}_\mu, A'_\sigma] = i\frac{1}{4\alpha}(\eta_{\mu\sigma} + \delta_\mu^0\delta_\sigma^0)\delta^3, \qquad [\ddot{A}_\mu, A'_\sigma] = i\frac{1}{4\alpha}(\delta_\mu^0\delta_\sigma^i + \delta_\mu^i\delta_\sigma^0)\partial_i\delta^3, \\
&[\ddot{A}_\mu, \dot{A}'_\sigma] = -i\frac{1}{4\alpha}[(\eta_{\mu\sigma} + 2\delta_\mu^0\delta_\sigma^0)\Delta + \delta_\mu^i\delta_\sigma^j\partial_i\partial_j]\delta^3, \\
&[A_\mu, b'] = -i\frac{1}{2}\delta_\mu^0\delta^3, \qquad [\dot{A}_\mu, b'] = -i\frac{1}{2}\delta_\mu^i\partial_i\delta^3, \qquad [\dot{A}_\mu, \dot{b}'] = i\frac{1}{2}\delta_\mu^0\Delta\delta^3, \\
&[A_\mu, \dot{\beta}'_\sigma] = -i\Big(\eta_{\mu\sigma} + \frac{1}{2}\delta_\mu^0\delta_\sigma^0\Big)\delta^3, \qquad [A_\mu, \ddot{\beta}'_\sigma] = i\frac{1}{2}(\delta_\mu^0\delta_\sigma^i + \delta_\mu^i\delta_\sigma^0)\partial_i\delta^3, \\
&[A_\mu, \dddot{\beta}'_\sigma] = -i\Big[(\eta_{\mu\sigma} + \delta_\mu^0\delta_\sigma^0)\Delta + \frac{1}{2}\delta_\mu^i\delta_\sigma^j\partial_i\partial_j\Big]\delta^3, \\
&[\dot{A}_\mu, \dddot{\beta}'_\sigma] = -i(\delta_\mu^0\delta_\sigma^i + \delta_\mu^i\delta_\sigma^0)\partial_i\Delta\delta^3.
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
&\{\dot{\tilde{\zeta}}_\mu, \tilde{\zeta}'_\sigma\} = -\Big(\eta_{\mu\sigma} + \frac{1}{2}\delta_\mu^0\delta_\sigma^0\Big)\delta^3, \qquad \{\ddot{\tilde{\zeta}}_\mu, \tilde{\zeta}'_\sigma\} = -\frac{1}{2}(\delta_\mu^0\delta_\sigma^i + \delta_\mu^i\delta_\sigma^0)\partial_i\delta^3, \\
&\{c^\mu, \dot{\bar{c}}'_\sigma\} = \phi_0^{-2}\delta_\sigma^\mu\delta^3, \qquad \{c, \dot{\bar{c}}'\} = -\frac{\alpha}{\gamma}\delta^3.
\end{aligned} \tag{A.5}
$$

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