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Anomalies and Dynamics in Strongly-Coupled Gauge Theories, New Criteria for Different Phases, and a Lesson from Supersymmetric Gauge Theories

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We review recent developments in our understanding of the dynamics of strongly-coupled chiral $SU(N)$ gauge theories in four dimensions, problems which are potentially important in our quest to go beyond the standard $SU(3)_{QCD} \times (SU(2) \times U(1))_{GWS}$ model of the fundamental interactions. The generalized symmetries and associated new 't Hooft anomaly-matching constraints allow us to exclude, in a wide class of chiral gauge theories, confining vacuum with full flavor symmetries supported by a set of color-singlet massless composite fermions. The color-flavor-locked dynamical Higgs phase, dynamical Abelianization or more general symmetry breaking phase, appear as plausible IR dynamics, depending on the massless matter fermions present. We revisit and discuss critically several well-known confinement criteria in the literature, for both chiral and vectorlike gauge theories, and propose tentative, new criteria for discriminating different phases. Finally, we review an idea which might sound rather surprising at first, but is indeed realized in some softly-broken supersymmetric theories, that confinement in QCD is a small deformation (in the IR end of the renormalization-group flow) of a strongly-coupled, nonlocal, nonAbelian conformal fixed point.

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1. Introduction

The main topic of this talk is the dynamics of strongly-coupled chiral gauge theories in four dimensions. This talk is based on several recent works [1]-[10] and on some earlier ones [11]-[14].

There are several good motivations for these efforts:

(1) We are living in a world with a nontrivial chiral structure (e.g., chiral DNA spirals);

(2) At a microscopic level, the standard model of the fundamental interactions $SU(3)_{QCD} \times SU(2)_L \times U(1)_Y$ gauge theory is a chiral theory;

(3) Possible GUT extensions of the standard model are all chiral.

Although the $SU(2)_L \times U(1)_Y$ Glashow-Weinberg-Salam model is a chiral gauge theory, it is weakly coupled and is well understood in perturbation theory. But it also means that it is at best a very good low-energy effective theory.

Surprisingly little is known today about the phases of strongly-coupled chiral gauge theories, after many years of studies [15]-[28]. This is to be contrasted with the case of vectorlike gauge theories, where we have a far better grasp of the strong gauge dynamics. Quantum Chromodynamics (QCD) has been under intense theoretical and experimental investigations over 50 years, with many solid results established. Also many exact results are known about the gauge dynamics of $\mathcal{N} = 2$ supersymmetric gauge theories (which are all vectorlike), since the discovery of the Seiberg-Witten solutions [29, 30] [31]-[35].

We propose thus the problem:

Understand better the dynamics of strongly-coupled (especially, chiral) gauge theories

as a challenge to all theoretical physicists. We take as our theoretical laboratories - or the battle ground - the following classes of $SU(N)$ gauge theories, with fermions in various (anomaly-free) representations:

(i) $\psi^{ij}\,,\eta^B_i\,, \quad (i, j, = 1, 2, \ldots, N\,, B = 1, 2, \ldots, N + 4)$, that is,

$$\square\!\square \oplus (N+4)\,\bar{\square} \tag{1}$$

plus possible p pairs of Dirac fermions in the fundamental representations (known as the generalized Bars-Yankielowicz (BY) models),

(ii) $\chi_{[ij]}\,,\tilde{\eta}^{Bj}\,, \quad B = 1, 2, \ldots, (N - 4)$,

$$\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix} \oplus (N-4)\,\bar{\square} \tag{2}$$

plus possible p pairs of Dirac fermions in the fundamental representations (called often as the generalized Georgi-Glashow (GG) models),

(iii) $\psi^{\{ij\}}\;, \chi_{[ij]}\;, \eta_i^A\;, \quad A = 1, 2, \ldots, 8\;,$

$$\square\!\square \oplus \overline{\begin{matrix}\square\\[-4pt]\square\end{matrix}} \oplus 8\,\bar{\square}\;. \tag{3}$$

(sometimes referred to as the $\psi\chi\eta$ model),

(iv) $\frac{N-4}{k}\,\psi^{\{ij\}} \oplus \frac{N+4}{k}\,\bar{\chi}_{[ij]}$

$$\frac{N-4}{k}\;\square\!\square \oplus \frac{N+4}{k}\;\overline{\begin{matrix}\square\\[-4pt]\square\end{matrix}}\;, \tag{4}$$

(v) ψ's in a self-conjugate, antisymmetric representation, (e.g., for $SU(6)$),

$$\begin{matrix}\square\\[-4pt]\square\\[-4pt]\square\end{matrix}\;, \tag{5}$$

(vi) $N_f \quad \eta \oplus \bar{\eta}$'s (QCD)

(vii) $N_f \quad \lambda$ (adjoint QCD),

and some others, to start with.

A well-known tool of the analysis - the 't Hooft anomaly matching conditions -, unfortunately is not sufficiently stringent [2],[17]-[26]. For this reason we appeal to a more powerful approach, mainly based on the generalized symmetries and associated anomalies [3]-[9].

2. Anomalies and dynamics: phases of chiral gauge theories

The main tool of the following analyses is the idea of the generalized symmetries [36]-[40]. The first step is to go from the conventional 0-form symmetries, acting on local fields, to k-form symmetries, acting on line, surface, etc. operators. The idea of 1-form symmetry is actually familiar from the example of the so-called center symmetry in $SU(N)$ Yang-Mills (YM) theory. It acts on the Polyakov loop,

$$e^{i\oint_\gamma A} \longrightarrow \Omega_N e^{i\oint_\gamma A}\;, \qquad \Omega_N = e^{2\pi i/N}\mathbf{1} \in \mathbb{Z}_N\;. \tag{6}$$

The vanishing (or nonvanishing) of the vacuum expectation value (VEV)

$$\langle e^{i\oint_\gamma A}\rangle \tag{7}$$

can be used as a criterion of the confinement (or Higgs) phase, respectively. Below we shall concentrate on the use of 1-form symmetries.

The second step is to consider the "gauging" of the 1-form discrete $\mathbb{Z}_N$ symmetry. The gauging of 1-form discrete $\mathbb{Z}_N$ symmetry proceeds by introducing the 2-form gauge fields $(B_c^{(2)}, B_c^{(1)})$,

$$NB_c^{(2)} = dB_c^{(1)}\;, \tag{8}$$

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and coupling them to the $SU(N)$ gauge fields a appropriately. This is done by embedding it into a $U(N)$ gauge field $\widetilde{a}$ as

$$\widetilde{a} = a + \frac{1}{N} B_{\rm c}^{(1)} \tag{9}$$

and requiring the invariance under the 1-form gauge transformation,

$$B_{\rm c}^{(2)} \to B_{\rm c}^{(2)} + \lambda_{\rm c} \ , \qquad B_{\rm c}^{(1)} \to B_{\rm c}^{(1)} + N\lambda_{\rm c} \ ,$$
$$\widetilde{a} \to \widetilde{a} + \lambda_{\rm c} \ , \tag{10}$$

where $\lambda_{\rm c}$ is the (1-form) gauge function such that

$$\oint \lambda_{\rm c} = \frac{2\pi\ell}{N} \ , \qquad (n \in \mathbb{Z}, \quad \ell \in \mathbb{Z}) \ . \tag{11}$$

The third important step is the idea of *color-flavor locked* 1-form discrete $\mathbb{Z}_N$ symmetry. Consider an $SU(N)$ gauge theory with a set of the massless matter Weyl fermions $\{\psi^k\}$. In general, the center $\mathbb{Z}_N$ symmetry is broken by the presence of the fermions (unless the fermions are all in the adjoint representation of $SU(N)$). However the situation changes if some global, nonanomalous $U(1)$ symmetries, $U_i(1), i = 1, 2, \ldots$, are there, which are gauged (in the usual sense by the introduction of external gauge fields A_i^{μ}). In those cases the color $\mathbb{Z}_N \subset SU(N)$ and the $U_i(1)$ transformations can compensate each other, restoring the symmetry.

As one encircles a closed loop L in spacetime, the fields transform as

$$\mathcal{P} e^{i \oint_L a} \to e^{\frac{2\pi i}{N}} \mathcal{P} e^{i \oint_L a} \ ; \quad \psi^k \to e^{\frac{2\pi i \mathcal{N}_k}{N}} \psi^k \ , \qquad \mathbb{Z}_N \subset SU(N) \ ; \tag{12}$$

$$\Pi_i \, e^{i \oint_L A_i} \to \left(e^{2\pi i \sum_{i,k} q_k^{(i)}} \right) \Pi_i \, e^{i \oint_L A_i} \ ; \qquad \psi^k \to e^{2\pi i \sum_{i,k} q_k^{(i)}} \psi^k \ , \qquad U_i(1) \ ; \tag{13}$$

where $a \equiv a_{\mu}^A t^A \, dx^{\mu}$ is the $SU(N)$ gauge field; $\mathcal{N}_k$ is the N-ality of the kth fermion, $q_k^{(i)}$ is the charge of ψ_k under $U_i(1)$. The factor $e^{i \oint_L A_i}$ is nothing but the Aharonov-Bohm phase for the i-th fermion.

We recall that the center symmetry is formally defined as a path-ordered sequence of local $SU(N)$ gauge transformations along the loop: the fermions must also be transformed in order to keep the action invariant. After encircling the loop and coming back to the original point, the gauge field is transformed by a nontrivial periodicity with a $\mathbb{Z}_N$ factor, dragging the fermions fields to transform as in (12). It would invalidate their periodic boundary condition (i.e., their uniqueness at each spacetime point). This is the reason why the presence of a fermion, such as the one in the fundamental representation, breaks the center symmetry itself [1].

When the conditions

$$\sum_i q_k^{(i)} = -\frac{\mathcal{N}_k}{N} \ , \qquad \forall k \tag{14}$$

are satisfied, however, a new, color-flavor locked center symmetry (12), (13), can be defined, accompanying the color $\mathbb{Z}_N$ center transformations with appropriate $U_i(1)$ gauge transformations.

[1] In the case of the Polyakov loop defined in the Euclidean spacetime, the fermions are required to satisfy antiperiodic boundary condition, but the conclusion is the same.

As the ordinary $\mathbb{Z}_N$ center transformation, such a color-flavor combined $\mathbb{Z}_N$ center symmetry is still just a *global 1-form symmetry.*

Finally, we put all these ideas together, and introduce the *gauging of the color-flavor locked 1-form symmetry* and studying possible topological obstructions in doing so (the generalized 't Hooft's anomalies) [36]-[46], [3]-[9]. As in the case of conventional gauging of 0-form symmetries, the idea of gauging is that of *identifying* the field configurations connected by the given symmetry transformations, and of eliminating the double counting. However, as one is now dealing with a 1-form symmetry, the associated gauge transformations are parametrized by a 1-form Abelian [2] gauge function, $\lambda = \lambda_\mu(x)dx^\mu$; see (10).

2.1 $(\mathbb{Z}_2)_F$ anomaly

One of the most significant results found is a $(\mathbb{Z}_2)_F$ anomaly [3]-[7],[9]: a quantum anomaly associated with the nontrivial classical fermion parity symmetry,

$$\psi_i \to -\psi_i \;, \qquad \forall i \;, \tag{15}$$

found in all BY and GG models (see (1), (2)), with even N and even p. We call these type I BY and/or GG models, and refer to all others (either N or p or both, odd) as type II models. In the standard quantization, (15) is conserved both classically, and quantum-mechanically, i.e., it is non anomalous. The standard calculation of the possible anomaly (say, à la Fujikawa), yields

$$\Delta S = \sum_i c_i \times \frac{1}{8\pi^2} \int_{\Sigma_4} d^4x \; \mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} \times (\pm\pi) = 2\pi\mathbb{Z} \;, \tag{16}$$

where

$$\sum_i c_i = 2\mathbb{Z} \neq 0 \;, \tag{17}$$

and the partition function is invariant. The fact that (15) is respected, in the type I models, by the instantons because the sum of the contributions from different fermions $\sum_i c_i$ is a *nonvanishing, but even, integer, and not because it is zero (as in the type II models)*, is fundamental.

In fact, in type I models, the symmetry group has a disconnected structure (before dividing out by a common $\mathbb{Z}_N$ factor). For instance, in the $\psi\eta$ models, it is

$$\frac{SU(N)_c \times SU(N+4) \times U(1)_{\psi\eta} \times \mathbb{Z}_2}{\mathbb{Z}_N} \;. \tag{18}$$

The calculation of the $\mathbb{Z}_N$ anomaly, by gauging the color-flavor locked 1-form $\mathbb{Z}_N$ symmetry gives, in all type I models, the result (a master formula)

$$\Delta S^{\text{Mixed anomaly}} = (\pm\pi) \cdot \sum_{\text{fermions}} \Big((d(R)\mathcal{N}(R)^2 - N \cdot D(R) \Big) \frac{1}{8\pi^2} \int_{\Sigma_4} \left(B_{\mathrm{c}}^{(2)}\right)^2 = \pm\pi \;, \tag{19}$$

where $d(R)$ is the dimension of the representation R, $\mathcal{N}(R)$ its N-ality, and $D(R)$ its Dynkin index. The partition function changes sign under (15): it is anomalous [4]-[7].

[2] Here we remember the crucial aspect of higher form symmetries: they are all Abelian. This is the reason why the color-flavor locked 1-form symmetries are possible.

fields	$SU(N)_{\rm c}$	$SU(N+4)$	$U(1)_{\psi\eta}$	$(\mathbb{Z}_2)_F$
ψ	$\square\!\square$	$(\cdot)$	$\frac{N+4}{2}$	$+1$
η	$\bar{\square}$	$\square$	$-\frac{N+2}{2}$	-1
$\mathcal{B}^{AB}$	$(\cdot)$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$-\frac{N}{2}$	-1

Table 1: The charges of UV and IR fermions with respect to the unbroken symmetry groups, in a putative confining, chirally symmetric vacuum, contemplated in some earlier literature. $\mathcal{B}^{AB}$ are the hypothetical massless baryons, $\sim \psi\eta^A\eta^B$. That they satisfy the conventional anomaly-matching conditions can be checked by studying simple arithmetic equations involving various triangle diagrams, $(SU(N+4))^3$, $(SU(N+4))^2-U(1)_{\psi\eta}$, $(U(1)_{\psi\eta})^3$, $U(1)_{\psi\eta}-Gravity^2$. See [5] for explicit exposition of these anomaly-matching arithmetics in all BY and GG models.

In the IR, the assumed massless composite fermions (see Table 1) cannot support such an anomaly, as they are color singlets and not coupled to the $B_{\rm c}^{(2)}$ field. It means that the confining, flavor symmetric vacuum contemplated in some earlier literature [16]-[19] cannot be dynamically realized in the infrared.

This conclusion was challenged by Tong and others [45]. The subtle point noted already in [4] is that the color-flavor locked $\mathbb{Z}_2$ transformation corresponds to the background fields winding simultaneously as (in the case of the $\psi\eta$ model) [3]

$$\oint B_{\rm c}^{(1)} = 2\pi \ , \qquad \oint A_0 = \pi \tag{20}$$

(the latter corresponding to (15)). But the latter implies a singular $\mathbb{Z}_2$ vortex configuration, a possible technical (ethical?) issue.

The authors of [45] propose, instead, to use a regular $U_{\psi\eta}(1)$ field A_0 and the consequent color-flavor locked $(B_{\rm c}^{(1)}, B_{\rm c}^{(2)})$ field such that

$$\oint A_0 = 2\pi \ , \qquad \oint B_{\rm c}^{(1)} = 4\pi \ , \tag{21}$$

i.e., with twice the 't Hooft flux. Hence there is no $\mathbb{Z}_2$ anomaly (see (8) and (19))!

The problem with this argument is that the flux (21) corresponds to a trivial element of $\mathbb{Z}_2$ holonomy group,

$$\psi_i \to \psi_i \ , \qquad \forall i \ : \tag{22}$$

a trivial (i.e., no) transformation. That this "transformation" is found to be nonanomalous is certainly a good news, but the significance of such a statement is not entirely clear [4].

The cure for this technical issue (the need to use the singular $\mathbb{Z}_2$ vortex like configuration) of the original work on the $(\mathbb{Z}_2)_F$ anomaly [4, 5, 7], can be found [9] by starting with a model with an extra Dirac-like pair of the fermions. Consider the $\psi\eta$ model, with an additional Dirac pair $q, \tilde{q}$ fermions, and a singlet scalar field ϕ, with the Yukawa coupling (let us call it the "X-ray model),

$$\Delta L = g\,\phi\,q\,\tilde{q} + h.c.\,, \qquad \langle\phi\rangle = v \gg \Lambda_{\psi\eta} \ . \tag{23}$$

[3] A_0 is the background gauge field for the anomaly-free $U_{\psi\eta}(1)$ global symmetry group.

[4] Another observation made in [45] is that (15) is a part of the proper Lorentz group. Again, this is true and is well known, but is not a point which can be used to try to invalidate the argument of [4]: see Sec. 2.2 bellow.

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	$SU(N)_c$	$SU(N+4)$	$U(1)_{\psi\eta}$	$U(1)_V$	$U_0(1)$	$\tilde{U}(1)$
ψ	$\square\!\square$	$(\cdot)$	$\frac{N+4}{2}$	0	1	$\frac{N+4}{2}$
η	$\bar{\square}$	$\square$	$-\frac{N+2}{2}$	0	-1	$-\frac{N+2}{2}$
q	$\square$	$(\cdot)$	0	1	1	$\frac{N+2}{2}$
$\tilde{q}$	$\bar{\square}$	$(\cdot)$	0	-1	1	$-\frac{N+2}{2}$
ϕ	$(\cdot)$	$(\cdot)$	0	0	-2	0

Table 2: The fields and charges of the X-ray model with respect to the nonanomalous symmetries.

Before the VEV of ϕ forms (i.e., in the UV) the model has a nonanomalousymmetry

$$\frac{SU(N)_c \times SU(N+4) \times \tilde{U}(1) \times U_0(1)}{\mathbb{Z}_N} \tag{24}$$

where the charges of the fermions are listed in Table 2. Note that this time the color-flavor locked $\mathbb{Z}_N$ symmetry is in the intersection among the nonanomalous, *continuous*, symmetry groups,

$$SU(N_c) \cap \{U(1)_{\psi\eta} \times U_0(1)\} \tag{25}$$

hence no difficulties arise in introducing the dynamical and background gauge fields associated with these symmetry factors (cfr. (18)).

The calculation of the mixed anomalies [9] yields

(i) $\tilde{U}(1) - \left(B_{\rm c}^{(2)}\right)^2$ anomaly:

$$\delta S_{\delta\alpha} = \frac{\tilde{C}}{8\pi^2}\int_{\Sigma_4} (B_{\rm c}^{(2)})^2\,\delta\alpha\ , \qquad \tilde{C} = -\frac{N^3(N+3)}{2} \neq 0\ . \tag{26}$$

The $\tilde{U}(1)$ symmetry is broken (i.e., gets anomalous) by the generalized 1-form gauging of the $\mathbb{Z}_N$.

(ii) $A_0 - (B_{\rm c}^{(2)})^2$ anomaly:

An analogous calculation leads to the $U_0(1)$ anomaly due to the 1-form gauging of the $\mathbb{Z}_N$ symmetry,

$$\delta S_{\delta\alpha_0} = \frac{C_0}{8\pi^2}\int_{\Sigma_4} (B_{\rm c}^{(2)})^2\,\delta\alpha_0\ , \qquad C_0 = N^2(N+3)\ . \tag{27}$$

This appears to imply that the $U_0(1)$ symmetry is also broken by the 1-form gauging of the $\mathbb{Z}_N$ symmetry. However, the scalar VEV $\langle\phi\rangle = v$ breaks spontaneously the $U_0(1)$ symmetry to $\mathbb{Z}_2$ (see Table 2). It means that, in contrast to (26), the generic variation $\delta\alpha_0$ cannot be used in (27) to examine the generalized anomaly-matching check. For that purpose, we can use only the nonanomalous [5] and unbroken symmetry operation, i.e., variations corresponding to a nontrivial $\mathbb{Z}_2$ transformation $\delta\alpha_0 = \pm\pi$ (see Table 2). Taking into account the nontrivial 't Hooft flux

$$\frac{1}{8\pi^2}\int_{\Sigma_4} (B_{\rm c}^{(2)})^2 = \frac{n}{N^2}\ , \qquad n \in \mathbb{Z}\ , \tag{28}$$

[5] In the sense of the standard strong anomaly.

and the crucial coefficient of the anomaly, $C_0 = N^2(N+3)$, it is seen that the partition function changes sign, for even N. We thus reproduce exactly the $\mathbb{Z}_2$ anomaly first found in [4]. This anomaly cannot be reproduced (matched) by the low-energy, massless baryons, as they are not coupled to $B_c^{(2)}$.

To conclude, the mixed anomaly $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ means that confinement and the full global chiral symmetries (no condensates) are not compatible, in type I BY and GG models: one or both must be abandoned. The dynamical Higgs phase discussed below seems to be fully consistent.

2.2 Dynamical Higgs phase

That the conventional 't Hooft anomaly-matching condition is also consistent with a dynamical Higgs phase characterized by certain bifermion condensates in the BY and GG models is well known. For instance, in the $\psi\eta$ model, a possible bifermion condensate is [17]

$$\langle \psi^{\{ij\}} \eta_j^A \rangle = C\, \delta^{iA}\ , \quad i, A = 1, 2, \ldots, N\ , \tag{29}$$

which breaks the color and flavor symmetries as

$$G \to G' = SU(N)_{CF} \times SU(4)_F \times U'(1)\ . \tag{30}$$

The low-energy theory is described by a set of massless composite fermions ("baryons") and of massless (Nambu-Goldstone) bosons. The baryons listed in Table 77 saturate the conventional 't Hooft anomaly-matching conditions with respect to the unbroken symmetry group.

	fields	$SU(N)_{\rm cf}$	$SU(4)_{\rm f}$	$U(1)'$	$(\mathbb{Z}_2)_F$
UV	ψ	$\square\!\square$	$\frac{N(N+1)}{2}\cdot(\cdot)$	$N+4$	1
	η^{A_1}	$\overline{\square\!\square} \oplus \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$N^2\ \cdot\ (\cdot)$	$-(N+4)$	-1
	η^{A_2}	$4\cdot\bar{\square}$	$N\ \cdot\ \square$	$-\frac{N+4}{2}$	-1
IR	$\mathcal{B}^{[A_1B_1]}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\frac{N(N-1)}{2}\cdot(\cdot)$	$-(N+4)$	-1
	$\mathcal{B}^{[A_1B_2]}$	$4\cdot\bar{\square}$	$N\ \cdot\ \square$	$-\frac{N+4}{2}$	-1

Table 3: The charges of the UV and IR fermions with respect to the unbroken symmetry groups, in the color-flavor locked, dynamical Higgs phase of the $\psi\eta$ model. A_1 or B_1 stand for $1, 2, \ldots, N$, A_2 or B_2 the rest of the flavor indices, $N+1, \ldots, N+4$.

Unlike the case of the confining, chiral symmetric vacuum (Table 1), in the dynamical Higgs phase here the conventional 't Hooft anomaly-matching is totally obvious, as after the Dirac pair of fermions get massive and decouple, the set of the remaining massless fermions are identical in UV and in IR. (See Table 3.)

And unlike the case of the confining, chiral symmetric vacuum (Table 1), here in the dynamical Higgs phase there is no difficulty in the matching of the 1-form, mixed 't Hooft anomalies, $(\mathbb{Z}_2)_F - (\mathbb{Z}_N)^2$. The vacuum breaks spontaneously both $SU(N)_{\rm c}$ and $U(1)_{\psi\eta}$. The color-flavor locked 1-form $\mathbb{Z}_N$ is broken in the IR.

A subtle, possibly confusing point is that the 0-form $(\mathbb{Z}_2)_F$ symmetry itself does not need to be, and indeed is not, spontaneously broken, as all bifermion condensates are invariant under (15). In fact, as the fermion parity coincides with an angle 2π space rotation, a spontaneous breaking of $(\mathbb{Z}_2)_F$ would have been a disaster: the spontaneous breaking of the Lorentz invariance. Which does not occur.

In this respect, even though the mixed anomaly $(\mathbb{Z}_2)_F - [\mathbb{Z}_N]^2$ found in [4] and in [5, 7],[9], may look at first sight similar to the mixed anomaly $CP - [\mathbb{Z}_N]^2$ in the pure $SU(N)$ Yang-Mills theory at $\theta = \pi$ [40], the *way* the mixed anomaly manifests itself at low energies is different. In the latter case, the new anomaly is consistent with, or implies, the phenomenon of the double vacuum degeneracy and the consequent spontaneous CP breaking à la Dashen [47] [6].

2.3 Strong anomaly and phases

In all anomaly-matching argument à la 't Hooft, conventional or generalized, we consider only the symmetries which are anomaly-free, i.e., which are not broken by the nonperturbative, strong-interaction effects. The breaking of the axial $U_A(1)$ symmetry in QCD, broken by the instantons, is a famous example of such a "strong anomaly". Actually, the strong anomaly should not be considered as a simple loss of a symmetry, but as a particular manifestation of a classical symmetry through the strong dynamics. Recently it was shown [5] that the consideration of the strong anomaly gives rather a clear indication about the possible phase of a wide class (BY and GG) of chiral gauge theories.

In QCD (e.g., with $N_F = 2$), the global flavor $SU_L(2) \times SU_R(2) \times U_V(1) \times U_A(1)$ symmetries are broken by the biquark condensate,

$$\langle U \rangle = \langle \bar{\psi_R} \psi_L \rangle \sim \Lambda \neq 0 \tag{31}$$

to $SU_V(2) \times U_V(1)$. For small quark masses, there must be four light Nambu-Goldstone bosons, but in Nature we observe only three pions (of $SU_A(2)$ breaking). Where is the fourth NG boson? A possible fourth NG boson, η, has actually mass

$$m_\eta \gg m_\pi \tag{32}$$

(the $U(1)$ problem).

The basic answer is given by 't Hooft: the axial $U(1)$ current satisfies an anomalous divergence equation

$$\partial^\mu J^{(A)}_\mu = N_f \, \frac{g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \, , \qquad \text{with} \quad \int d^4x \, \frac{g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} = \mathbb{Z} \, , \tag{33}$$

where $\mathbb{Z}$ is the integer instanton number.

An efficient way to represent the strong anomaly effects is that of writing a low-energy effective action, containing the term reproducing the correct $U_A(1)$ symmetry breaking [52]-[55],

$$L = L_0 + \hat{L} \, , \qquad \hat{L} = \frac{i}{2} \, q(x) \, \log \det U/U^\dagger + \frac{N}{a_0 F_\pi^2} q(x)^2 \, ; \tag{34}$$

[6] That this occurs in $SU(N)$ Yang-Mills theory at $\theta = \pi$ has been known for some time, from the QCD Effective Lagrangian analysis [48, 49] and also from soft supersymmetry breaking perturbation [50, 51] of the exact Seiberg-Witten solutions [29, 30] of pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Still, it is remarkable that the same result is reproduced by a symmetry consideration, based on the generalized mixed-anomaly matching requirement.

where L_0 is the standard chiral Lagrangian describing the massless pions,

$$U = U_0\, e^{i\pi^a(x)t^a/F_\pi} \tag{35}$$

and

$$q(x) = \frac{g^2}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \ . \tag{36}$$

Such an effective Lagrangian correctly reproduces the anomalous $U_A(1)$ variation,

$$\Delta S = 2N_f\, \alpha \int d^4x \frac{g^2}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \ , \qquad \psi_{L,R} \to e^{\pm i\alpha}\psi_{L,R} \ , \tag{37}$$

and would give a mass $\propto \Lambda$ to m_η.

The question is: does a (multi-valued) logarithmic function make sense as an effective Lagrangian?

The answer is: yes, it does, if U acquires a nonvanishing VEV, and if (34) is regarded as a function of the pion field $\pi(x)$, i.e., as an expansion,

$$U = \langle U \rangle e^{i\pi^a(x)t^a/F_\pi} = \langle U \rangle (1 + i\pi^a(x)t^a/F_\pi + \ldots) \ . \tag{38}$$

Now, the idea is to invert the logic, and say that an effective action (34) with the logarithmic, anomaly term, *requires* that the chiral composite $U \sim \bar{\psi}_R \psi_L$ to get a nonvanishing VEV, (31). In other words, *the strong anomaly implies the spontaneous symmetry breaking* of the chiral $SU_L(2) \times SU_R(2) \times U_V(1) \times U_A(1)$ symmetry to the diagonal, vector subgroup, $SU_V(2) \times U_V(1)$.

We now apply the same idea to chiral gauge theories, where the form of the strong anomaly is known but not the dynamical, infrared phase. For concreteness, let us take the "$\chi\eta$ model" (see (2)), with fermions,

$$\yng(1,1) \oplus (N-4)\,\bar{\yng(1)} \ . \tag{39}$$

The form of the strong anomaly is known:

$$\frac{i}{2} q(x) \log(\chi\eta)^{N-4}\chi\chi + \text{h.c.} \ , \tag{40}$$

($q(x)$ is the topological density defined in (36)) where

$$(\chi\eta)^{N-4}\chi\chi \equiv \epsilon_{i_1 i_2 \ldots i_N}\epsilon_{m_1 m_2 \ldots m_{N-4}} (\chi\eta)^{i_1 m_1}(\chi\eta)^{i_2 m_2} \ldots (\chi\eta)^{i_{N-4} m_{N-4}} \chi^{i_{N-3} i_{N-2}} \chi^{i_{N-1} i_N} \ . \tag{41}$$

Now such a strong-anomaly effective action implies

$$\langle \chi\eta \rangle \neq 0 \ , \quad \langle \chi\chi \rangle \neq 0 \ : \tag{42}$$

i.e., the system is in the dynamical Higgs phase [6, 56].

Note that the confining, chirally symmetric phase, with no condensates, and the massless "baryons", $\mathcal{B} \sim \chi\eta\eta$, as the only infrared degrees of freedom, fails the "matching" of the strong anomaly. The strong anomaly effective action above cannot be written in terms of $\mathcal{B}'s$, as the fermion zero mode counting (in the instanton background) fails.

It can be shown that the strong-anomaly consideration favors the dynamical Higgs phase, against the confining symmetric phase with no condensate formation, in all BY and GG models [6], in agreement with the indication coming from the studies of the generalized anomaly matching (especially, the $\mathbb{Z}_2$ anomalies) studied in Sec. 2.1.

Such an agreement is not really a coincidence. Both are consequences of taking into account appropriately the effects of the strong anomalies (i.e., topologically nontrivial, nonperturbative effects of the strong $SU(N)$ gauge dynamics).

2.4 Dynamical Abelianization

Another interesting result found concerns the "$\psi\chi\eta$" (and some other) model, with fermions,

$$\psi^{\{ij\}} \,, \qquad \chi_{[ij]} \,, \qquad \eta_i^A \,, \qquad A = 1, 2, \ldots 8 \,, \tag{43}$$

or

$$\square\!\square \oplus \bar{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \oplus 8 \times \bar{\square} \,. \tag{44}$$

It is asymptotically free, the first coefficient of the beta function being,

$$b_0 = \frac{1}{3}\left[11N - (N+2) - (N-2) - 8\right] = \frac{9N-8}{3} \,. \tag{45}$$

The model has a global $SU(8)$ symmetry as well as two $U(1)$ symmetries,

$$\tilde{U}(1): \qquad \psi \to e^{2i\alpha}\psi \,, \quad \chi \to e^{-2i\alpha}\chi \,, \quad \eta \to e^{-i\alpha}\eta \,, \tag{46}$$

and

$$U(1)_{\psi\chi}: \qquad \psi \to e^{i\frac{N-2}{N^*}\beta}\psi \,, \qquad \chi \to e^{-i\frac{N+2}{N^*}\beta}\chi \,, \qquad \eta \to \eta \,, \tag{47}$$

where

$$N^* = GCD(N+2, N-2) \quad \text{and} \quad \alpha,\ \beta \in (0, 2\pi) \,. \tag{48}$$

The problem is to understand how these symmetries and what kind of phase, are realized at low energies.

We adopt the following strategy: our initial investigation of this model [1, 2] has shown that the conventional 't Hooft anomaly matching algorithm allows, among few others, the dynamical Abelanization. The idea is to study whether the consequences of this dynamical assumption expected in the infrared are consistent with the indications of possible generalized 't Hooft anomalies, in the ultraviolet. These provide stronger constraints than the conventional anomaly-matching algorithm.

We assume that bifermion condensates in the adjoint representation

$$\langle \psi^{ik}\chi_{kj} \rangle = \Lambda^3 \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_N \end{pmatrix}_j^{\,i} \,, \qquad \langle \psi^{ij}\eta_j^A \rangle = 0 \,, \tag{49}$$

$$c_n \in \mathbb{C} \,, \qquad \sum_n c_n = 0 \,, \qquad c_m - c_n \neq 0 \,, \ \ m \neq n \,, \tag{50}$$

(with no other particular relations among c_j's) form in the infrared, and induce the symmetry breaking

$$SU(N) \to U(1)^{N-1} \ . \tag{51}$$

a phenomenon well-known in the $\mathcal{N} = 2$ supersymmetric gauge theories [7]. In [6] a detailed study was made of all aspects of symmetry realization in low energies, including the effects of the strong anomalies. In particular, analysis of the consistency with the gauging (and the associated 't Hooft anomalies) of the 1-form color-flavor locked $\mathbb{Z}_N$ symmetry (see Table 4 below) give convincing evidence that the assumption of the dynamical Abelianization is a correct one. (See also [57]).

	$\tilde{U}(1)$	$U(1)_{\psi\chi}$	$(\mathbb{Z}_{N+2})_\psi$	$(\mathbb{Z}_{N-2})_\chi$	$SU(8)_\eta$	$\mathbb{Z}_{N^*}$	$\mathbb{Z}_{4/N^*}$
Mixed Anomalies	✓	X	X	X	✓	✓	✓
Dyn. Abel.	✓	X	X	X	✓	✓	✓

Table 4: Dynamical Abelianization postulate of the present work is confronted with the implications of the mixed anomalies. ✓ for a conserved symmetry, X for a broken symmetry. The discrete $\mathbb{Z}_{N^*}$ symmetry is defined in [6].

2.5 More general dynamical symmetry breaking patterns

There are many systems other than the $\psi\chi\eta$ model (43) in which a bifermion condensate in the adjoint representation can form. An interesting class of models are those (see (4)) with fermions in second-rank tensor representations,

$$\tfrac{N-4}{k}\,\psi^{\{ij\}} \oplus \tfrac{N+4}{k}\,\bar{\chi}_{[ij]} \ . \tag{52}$$

It is plausible that a bifermion condensate

$$\langle \psi\chi \rangle \neq 0 \tag{53}$$

forms, but it may not necessarily imply the dynamical Abelianization (as we assumed for the $\psi\chi\eta$ model). The condensate can have the form [10],

$$\langle \psi\chi \rangle = \text{diag.}\,(c_1 \mathbf{1}_{n_1}, c_2 \mathbf{1}_{n_2}, \ldots) \ , \qquad \sum_i c_i n_i = 0 \ , \tag{54}$$

leading to a dynamical color symmetry breaking,

$$SU(N) \to SU(n_1) \times SU(n_2) \times \cdots \prod_k U_k(1) \ . \tag{55}$$

In particular, we wish to know whether some of these nonAbelian subgroups can be infrared-free, i.e., can survive in the infrared. Such a question is relevant because the standard model of the fundamental interactions is based on a gauge theory having precisely the gauge group of this type.

[7]In contrast to $\mathcal{N} = 2$ susy theories, here the scalar in the adjoint representation $\phi \sim \psi\chi$ appears as a dynamical, composite field.

2.5.1 $N = 5$, $k = 1$ model

For instance, consider the model (52) with $N = 5$, $k = 1$, with fermions

$$\square\square \oplus 9 \cdot \overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} \;. \tag{56}$$

A possible pattern is

$$G_c = SU(5) \longrightarrow SU(3) \times SU(2) \times U(1) \;. \tag{57}$$

The global symmetry is broken as

$$G_F = SU(9) \times U_0(1) \rightarrow SU(8) \times U_0(1)' \;. \tag{58}$$

$U_0(1)'$ is the combination

$$e^{i\alpha \begin{pmatrix} \mathbf{1}_8/8 & 0 \\ 0 & -1 \end{pmatrix}} \in SU(9) \quad \text{with} \quad e^{i\beta Q_0} \;, \tag{59}$$

that is, with

$$-\alpha + \beta\left(\frac{9}{N+2} - \frac{1}{N-2}\right) = 0 \;, \qquad \alpha = 4\,\frac{2N-5}{N^2-4}\,\beta \;. \tag{60}$$

The fermions, decomposed in the representations of the unbroken subgroup are shown in Table 5. The fermions participating in the condensate, $\psi^{iJ}\chi^9_{Ji}$, become massive, and leave the

	fields	$SU(3)$	$SU(2)$	$U(1)$	$SU(9)$	$U_0(1)$
UV	ψ^{ij}	$\square\square$	$(\cdot)$	4	$(\cdot)$	$\frac{9}{N+2}$
	ψ^{iJ}	$\square$	$\square$	-1	$(\cdot)$	$\frac{9}{N+2}$
	ψ^{JK}	$(\cdot)$	$\square\square$	-6	$(\cdot)$	$\frac{9}{N+2}$
	χ^A_{ij}	$\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \square$	$(\cdot)$	-4	$\square$	$-\frac{1}{N-2}$
	χ^A_{iJ}	$\bar{\square}$	$\square$	1	$\square$	$-\frac{1}{N-2}$
	χ^A_{JK}	$(\cdot)$	$(\cdot)$	6	$\square$	$-\frac{1}{N-2}$

Table 5: $\psi\chi$ model, $N = 5$, $k = 1$. $A = 1, 2, \ldots, 9$, $i, j = 1, 2, 3$; $J, K = 4, 5$.

massless fermions in Table 6. The "low-energy" $SU(3)$ group turns out to be asymptotically free (evolve to stronger interactions in the IR, leading to further dynamical symmetry breaking or confinement), whereas the $SU(2)$ is infrared free: are still asymptotically free:

$$\beta_{SU(3)} = 11 \cdot 3 - 5 - 9 \cdot 1 - 8 \cdot 1 \cdot 2 > 0 \;, \tag{61}$$

$$\beta_{SU(2)} = 11 \cdot 2 - 4 - 8 \cdot 3 \cdot 1 < 0 \;. \tag{62}$$

This could be interesting, in principle. After all, the standard $SU(3) \times SU(2) \times U(1)$ model contains also an asymptotically-free sub-gauge group $SU(3)$ and the infrared-free $SU(2) \times U(1)$ part. Unfortunately the matter content is not quite the same, although one can notice intriguing similarities.

	fields	$SU(3)$	$SU(2)$	$U(1)$	$SU(8)$	$U_0(1)'$
IR	ψ^{ij}	$\square\!\square$	$(\cdot)$	4	$(\cdot)$	$2(N-2)$
	ψ^{JK}	$(\cdot)$	$\square\!\square$	-6	$(\cdot)$	$2(N-2)$
	χ^9_{ij}	$\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \square$	$(\cdot)$	-4	$(\cdot)$	$-2(N-2)$
	χ^B_{ij}	$\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \square$	$(\cdot)$	-4	$\square$	-1
	χ^B_{iJ}	$\bar{\square}$	$\square$	1	$\square$	-1
	χ^9_{JK}	$(\cdot)$	$(\cdot)$	6	$(\cdot)$	$-2(N-2)$
	χ^B_{JK}	$(\cdot)$	$(\cdot)$	6	$\square$	-1
	$\phi^B \sim \Re(\psi^{iJ}\chi^B_{Ji})$	$(\cdot)$	$(\cdot)$	$(\cdot)$	$\square$	$2N-5$
	$\pi^B \sim \Im(\psi^{iJ}\chi^B_{Ji})$	$(\cdot)$	$(\cdot)$	$(\cdot)$	$\square$	$2N-5$
	$\pi^9 \sim \Im(\psi^{iJ}\chi^9_{Ji})$	$(\cdot)$	$(\cdot)$	$(\cdot)$	$(\cdot)$	0

Table 6: Massless fermions and Goldstone bosons in the infrared, in $N = 5$, $k = 1$, $\psi\chi$ model. $B = 1, 2, \ldots, 8$.

2.5.2 Infrared nonAbelian gauge groups

More generally, we ask whether or not the subgroup

$$SU(N) \longrightarrow SU(n) \times \ldots \tag{63}$$

in (2.5.2) can remain infrared-free (weakly coupled) at low energies, in a given model. By decomposing the fermions (52) as direct sums of the irreps of the $SU(n)$ subgroup, the (first coefficient the) beta function of $SU(n)$ can be seen to be

$$\begin{aligned}\beta(SU(n)) &= 11 \cdot n - \frac{N-4}{k} \cdot (n+2) - \frac{N+4}{k} \cdot (n-2) - \frac{8}{k} \cdot (N-n) \cdot 1 \\ &= \frac{16 + 8n + 11kn - 8N - 2nN}{k} \,. \end{aligned} \tag{64}$$

The change of sign happens at

$$n^* = \frac{8N - 16}{8 + 11k - 2N} \tag{65}$$

so the integer part of n^*, $[n^*]$ is the biggest n that is IR free. Clearly if $[n^*] = 1$ there are no non-abelian IR free (gauge) symmetry breaking patterns. Let us discuss a few cases.

- $k = 1$, $N = 5$, $n^* = \frac{8}{3}$, so $[n^*] = 2$. Indeed

$$\beta(SU(2)) = -6 \,. \tag{66}$$

The possible IR free breaking is

$$SU(5) \longrightarrow SU(2) \times SU(2) \times U(1)^2 \,. \tag{67}$$

- $k = 1$, $N = 6$, $n^* = \frac{32}{7}$, so $[n^*] = 4$, and

$$\beta(SU(4)) = -4 \; . \tag{68}$$

A possible IR free breaking mode is (see Fig. 1)

$$SU(6) \to SU(4) \times SU(2) \times U(1) \; . \tag{69}$$

In the following, the massless fermions which remain in the infrared are shown in quiver graphs, instead of using tables such as Table 6. Circles with a number inside n represent a gauge group $SU(n)$, squares with a number m represent a global symmetry $SU(m)$ group, fermions are lines connecting the groups, arrows on the line indicate if is fundamental (ingoing) or antifundamental (outgoing), lines omitted for $SU(2)$; little "o" or "x" within a line indicates if they belong to symmetric or anti-symmetric tensor representation.

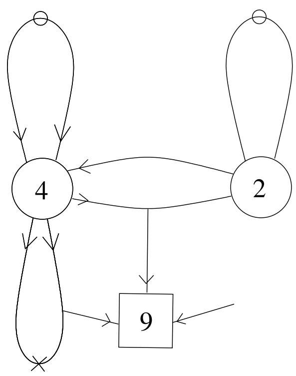

Figure 1: Diagram corresponding to (69).

Another possible IR free breaking mode is (see Fig. 2)

$$SU(6) \to SU(3) \times SU(3) \times U(1) \; . \tag{70}$$

- $k = 2$, $N = 6$, $n^* = \frac{16}{9}$. Since $[n^*] = 1$ there are no non-abelian IR free symmetry breaking patterns. Abelianization is the only IR-free possibility.

- $k = 2$, $N = 8$, $n^* = 3 + \frac{3}{7}$ and

$$\beta(SU(3)) = -3 \; . \tag{71}$$

Possible IR free breaking patterns are

$$SU(8) \to SU(3) \times SU(3) \times SU(2) \times U(1)^2 \tag{72}$$

or

$$SU(8) \to SU(2)^4 \times U(1)^3 \; . \tag{73}$$

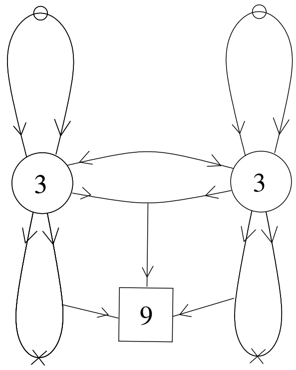

Figure 2: Quiver diagram corresponding to (70)

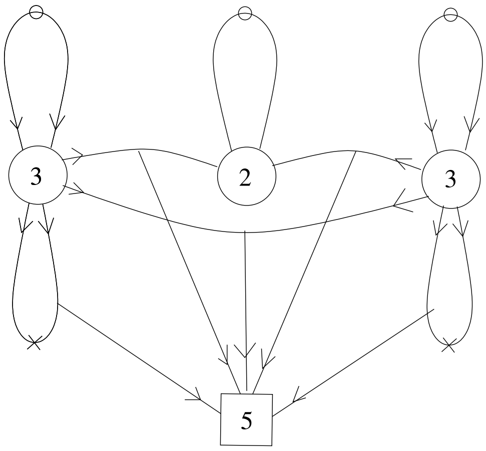

Figure 3: Diagram corresponding to (72)

- $k = 2$, $N = 10$, $n^* = \frac{32}{5}$, so $[n^*] = 6$, and indeed

$$\beta(SU(6)) = -2 \ . \tag{74}$$

Possible IR free breaking modes are

$$SU(10) \to SU(6) \times SU(4) \times U(1) \tag{75}$$

and

$$SU(10) \to SU(5) \times SU(5) \times U(1) \ , \tag{76}$$

and so on.
It is left for further study to understand which is the correct phase of each model.

3. Old and New criteria for confinement and other phases

These efforts to understand the dynamics and phases of chiral gauge theories in four dimensions reviewed above, remind us of the well debated confinement (or Higgs) criteria, in particular in the

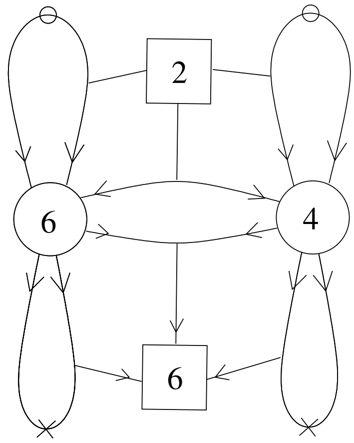

Figure 4: Diagram corresponding to (75)

context of pure Yang-Mills theory or of QCD, and urge us to revisit these ideas with more critical eyes.

There are three well-known "confinement criteria". (A): The original idea that a colored particle cannot be freely propagating: they are confined inside a color-singlet composite object. (B): Criteria which use Wilson loop or Polyakov loop; and (C): The dual superconductivity idea by 't Hooft. As will be seen, each of them has some issues.

(A) The original color-confinement idea that colored particles (e.g., quarks) cannot be freely propagating, and permanently confined inside a color-singlet composite states (e.g., hadrons).

This concept, which seems to be well-defined, appears to be somewhat problematic, when applied to some chiral gauge theories. Namely, gauge non-invariant operators or states could well be gauge-invariant ones, just written in a particular gauge.

A noteworthy and well known example is the Weinberg-Salam $SU(2) \times U(1)$ theory. It is usually stated that the $SU(2) \times U(1)$ gauge group is spontaneously broken by the Higgs VEV

$$\langle \phi \rangle = \begin{pmatrix} \mathrm{v} \\ 0 \end{pmatrix} , \qquad \mathrm{v} = 256 \, \mathrm{GeV} , \tag{77}$$

and the neutrino and the lefthanded electron are the upper and lower components of the fermion lefthanded doublet,

$$\psi_L = \begin{pmatrix} \nu \\ e_L \end{pmatrix} . \tag{78}$$

A more appropriate way to think about these is that (77) really means that

$$\langle \sum_{i=2}^{2} \phi_i^\dagger \phi^i \rangle \neq 0 \ : \tag{79}$$

and that the neutrino and electron are described by gauge invariant composite fields

$$\nu \sim \phi_i^\dagger \psi_L^i , \qquad e_L \sim \epsilon_{ij} \phi^i \psi_L^j . \tag{80}$$

The familiar expressions (77) and (78) are just formulae valid in the particular (and arbitrary) gauge chosen, (77).

Does it mean that there are no distinctions between the confinement and Higgs phases? The answer is: there are. The weakly coupled, $SU(2) \times U(1)$ theory in broken, Higgs phase, cannot be understood as a strongly coupled $SU(2) \times U(1)$ theory in confinement phase [63]. Analogous remarks can be made in certain classes of chiral gauge theories studied above. In particular, in BY and GG models which are likely to be in a dynamical Higgs phase, there is no difficulty in rewriting the condensate such as (29) or the NG boson of the system, in a gauge-invariant fashion. Nevertheless, the dynamical Higgs phase is clearly distinct from the putative, confining, symmetric phase [4, 5, 7].

(B) Another well-known criterion uses the Wilson loop

$$W_\gamma = \mathrm{Tr}\,\{\mathcal{P} e^{i\oint_\gamma A_\mu dx^\mu}\}\ , \tag{81}$$

or the Polyakov loop in Euclidian spacetime,

$$P(\mathbf{r}) = \frac{1}{N}\mathrm{Tr}\{\mathcal{T} e^{i\int_0^\beta d\tau A_0(\mathbf{r},\tau)}\}\ , \tag{82}$$

where $\mathcal{P}$ and $\mathcal{T}$ represent the path-ordered or time-ordered exponentials. Wilson's criterion is

$$\lim_{\gamma\to\infty}\langle W_\gamma\rangle = \begin{cases} e^{-A} & \text{confinement phase,} \\ e^{-L} & \text{Higgs phase :} \end{cases} \tag{83}$$

i.e., the area law - a linearly rising potential between two test charges - indicates confinement phase.

The Yang-Mills theory is invariant under the center symmetry transformation of the Polyakov loop,

$$P(\mathbf{r}) \to \mathbb{Z}_N\, P(\mathbf{r})\ . \tag{84}$$

The unbroken center symmetry

$$\lim_{\beta\to\infty}|\langle P(\mathbf{r})\rangle| = 0\ , \tag{85}$$

can be used as a criterion of confinement phase (an infinite free energy for an isolated quark).

The lattice simulation indicates that the $SU(N)$ Yang-Mills theory is indeed in confinement phase, according to these criteria.

The problem with this criterion is that there is nothing to confine in pure YM theory (!). As soon as massless quarks are introduced, center symmetry is broken. Also, the confining string between two test charges is broken by the spontaneous production of quark-antiquark pairs from the vacuum (vacuum polarization), and the area law is lost. The perimeter law ensues. Thus neither (83) nor (85) can be used to discriminate the infrared phases (Higgs or confining) of quantum chromodynamics (QCD) with massless quarks.

(C) Confinement as a dual Meissner effect ('t Hooft). By considering the subgroup of the color $SU(3)$,

$$U(1)^2 \subset SU(3) \,, \tag{86}$$

the charges of a particle (elementary or solitonic) can take electric and magnetic quantum numbers

$$(n_1, n_2; m_1, m_2) \,, \qquad n_i, m_i \in \mathbb{Z} \,, \quad i = 1, 2 \,. \tag{87}$$

The corresponding $U_i(1)$ electric and magnetic coupling strengths are

$$n_1 e_1 \,, m_1 g_1 \,, n_2 e_2 \,, m_2 g_2 \,, \tag{88}$$

where the elementary electric and magnetic charges obey Dirac's quantization condition,

$$e_1\, g_1 \in \mathbb{Z}/2 \,, \qquad e_2\, g_2 \in \mathbb{Z}/2 \,. \tag{89}$$

Now define the "Dirac unit" between two particles

$$\mathcal{D}^{1,2} \equiv \sum_{i=1}^{2} (n_i^{(1)} m_i^{(2)} - n_i^{(2)} m_i^{(1)}) \,. \tag{90}$$

Then 't Hooft's criterion [58] is the following. If the field of particle 1 with charges (87) condenses

$$\langle M^{(1)} \rangle \neq 0 \,, \tag{91}$$

then all particles 2 carrying charges with nonvanishing Dirac unit with respect to particle 1,

$$\mathcal{D}^{1,2} \neq 0 \,, \quad \mathrm{Mod}(3) \,, \tag{92}$$

are confined.

For instance, the condensation of the magnetic monopole of $U_1(1)$,

$$\langle M_{0,0;1,0} \rangle \neq 0 \,, \tag{93}$$

implies that the quark with charge $Q_{(1,0;0,0)}$ is confined (dual Meissner effect).

The criterion (C) is also problematic. In formulating the confinement criterion in terms of the $U(1)^2 \subset SU(3)$ charges, *one has made an implicit, dynamical assumption* that these Abelian (electric, magnetic or dionic) degrees of freedom describe the physics in the infrared. In other words, one assumes dynamical Abelianization, analogous to what happens in $\mathcal{N} = 2$ supersymmetric gauge theories or in the chiral $\psi\chi\eta$ model we discussed in Sec. 2.4. However, in the standard QCD there are no elementary or (plausible) composite scalar fields in the adjoint representation [8], in contrast to these other systems. In such a situation dynamical Abelianization of the system is unlikely. Besides, there are no phenomenological indications in favor of an Abelian $U(1)^2$ infrared effective theory for QCD.

[8]Actually there are bifermion $\psi_L \bar{\psi}_R$ or bi-gluon $G_{\mu\nu}G^{\mu\nu}$ composites which may act as scalar fields in the adjoint representation. However the corresponding composite scalars in the color-singlet representation are in a much more strongly attractive channels, and indeed those are believed to form condensates in the real-world QCD.

It is possible that confinement in QCD is explained by a dual superconductor mechanism, but without Abelianization. But it means that the infrared physics involves nonAbelian monopoles and their quantum dynamics, a notoriously subtle problem. See e.g. [59] for a review and for references to earlier literature.

Quantum-mechanical properties of Abelian or nonAbelian monopoles and dyons, their dynamics and their possible roles in confinement and symmetry breaking, have, on the other hand, been largely clarified by the ground-breaking discovery of the exact Seiberg-Witten solutions of $\mathcal{N} = 2$ supersymmetric gauge theories [29]-[35]. Unfortunately, it turns out that it is rather difficult to make reliable predictions about ordinary (i.e., nonsupersymmetric) theories, by using the knowledge gained in the context of $\mathcal{N} = 2$ (or $\mathcal{N} = 1$) supersymmetric theories. In general, one expects various phase transitions, when the $\mathcal{N} = 2$ or $\mathcal{N} = 1$ susy breaking terms are added in the action, and are tuned to be larger than the dynamical scales $\Lambda_{\mathcal{N}=2}$ or $\Lambda_{\mathcal{N}=1}$ of supersymmetric theories.

Under these circumstances, the best thing one can do could be to try to learn the kind of physics phenomenon which is dynamically realizable, and which could be underlining the confinement in QCD, rather than attempting to deform the $\mathcal{N} = 2$ or $\mathcal{N} = 1$ QCD in some concrete manner (see [60–62] for recent efforts), hoping to get something which looks similar to the standard QCD.

We shall indeed take the first attitude, and discuss below an idea, realized in susy gauge theories, and might be underlying the physics of real-world QCD, that confinement is a deformation of the RG flow towards a nonAbelian strongly-coupled conformal fixed point (Sec. 4).

But before that, let us discuss new simple criteria for "color confinement", Higgs phase, etc.

3.1 New criteria (tentative) for different phases

The difficulties in the familiar "confinement criteria" (A) - (C) reviewed above indeed tempt us to propose the following, new criteria for different phases of chiral or vectorlike $SU(N)$ gauge theories in four dimensions.

Let us however keep note of a lesson, first of all, that the studies in different classes of chiral gauge theories discussed in Sec. 2, all based on the same $SU(N)$ Yang-Mills theory, taught us. It is the fact that the infrared dynamics (and the phase) of an asymptotically-free $SU(N)$ gauge theory *is not determined by that of the pure $SU(N)$ Yang-Mills theory on which it is based, but by the dynamics involving the massless matter fields,* which in turn depend critically on their representation. Thus an argument that a chiral gauge theory (e.g., the BY model) should necessarily confine, as the $SU(N)$ Yang-Mills theory is in confinement phase, is logically unfounded.

Our proposal is that the different phases are characterized by the number and types of various colored Nambu-Goldstone bosons the system produces. They may be generated by the condensation of either elementary or composite scalar fields. We exclude below those systems which are infrared-free $SU(N)$ theory (many fermions are present so that the interactions are weak at low energies. $SU(N)$ gauge bosons survive in the IR as asymptotic states - Coulomb phase. We exclude also those which flow into conformal fixed-point theories (however, this last class of models may have a subtle and important relation to confinement, see below).

(a) The system is in the confinement phase if it produces no colored NG bosons.

This is the case for YM, QCD, supersymmetric QCD and susy YM. Note that the phase of the standard QCD with massless fermions is classified as confinement according to this new criterion, whereas the old criterion (B) fails.

(b) (Dynamical) Higgs phase, when the system produces $N^2 - 1$ colored NG bosons.

This occurs (likely) in the chiral BY and GG models studied in [3]-[7], and in the Glashow-Weinberg-Salam electroweak theory.

(c) Dynamical Abelianization (or Coulomb phase) occurs when there are $N(N-1)$ colored NG bosons.

This was shown likely to be the correct phase of the $\psi\chi\eta$ model [8, 57], and is known to be realized in most $\mathcal{N} = 2$ supersymmetric gauge theories [29]-[35].

(d) Other groups other than (a)-(c) above, of colored NG bosons.

The system could flow into infrared effective theory containing some residual (infrared-free) nonAbelian gauge dynamics. The first attempts investigating these possibilities in chiral theories are reported in Sec. 2.5.2 above. In the context of $\mathcal{N} = 2$ supersymmetric gauge theories, this type of infrared-fixed-point theories are well-known (e.g., the r-vacua of $\mathcal{N} = 2$ SQCD [64, 65].)

These new criteria clarify to some extent the ideas about possible different phases occurring in various types of strongly-coupled gauge theories in four dimensions, chiral or vectorlike. Still, confinement in QCD with massless quarks certainly requires a better explanation than the earlier criteria (A)-(C) reviewed in Sec. 3, and a more detailed understanding of the mechanism than (a).

Below, we discuss an idea on the confinement in QCD, which might sound somewhat extraordinary, but is indeed realized in some softly-broken supersymmetric theories.

4. A lesson from supersymmetric gauge theories

As already said, it remains an unsolved problem to make reliable predictions about the dynamics of any specific (non-supersymmetric) gauge theory such as QCD, based on the knowledge about supersymmetric gauge theories.

But the result of studies on supersymmetric gauge theories during the last 50 years does teach us what sort of dynamical phenomena are possible in strongly-coupled nonAbelian gauge theories, how they depend on the gauge group and on the massless matter contents, and how they work concretely. *It has given solid understanding of the nonperturbative effects involving instantons, magnetic monopoles, dualities and interacting (super) conformal infrared fixed points (SCFT).*

From the point of view of the renormalization-group flow, confinement can be understood as a deformation (deviation) by some relevant operator (which might be present already in the UV theory or produced dynamically), from a trajectory leading to an infrared fixed point. See Fig. 5. The relevant infrared fixed point theory might be Abelian (Abelianization), or nonAbelian but local and weakly coupled, or a strongly-coupled, nonAbelian, nonlocal SCFT. The example of the first type of the RG flow is 't Hooft's dual Meissner effect model (assumption) of confinement [58]. Confinement à la 't Hooft shown by Seiberg and Witten by an $\mathcal{N} = 1$ perturbation of the "monopole point" of $\mathcal{N} = 2$ susy $SU(2)$ gauge theories [29, 30], is an explicitly realization.

The second type of the RG flow is the one into nonAbelian but weakly-coupled infrared-free low-energy systems (hence, again, "trivial" conformal fixed points), and confinement caused by a relevant perturbation. A known example is the so-called r-vacua of $\mathcal{N} = 2$ supersymmetric quantum chromodynamics (SQCD), leading to dual Meissner effects, involving both Abelian and nonAbelian monopoles, upon $\mathcal{N} = 1$ adjoint-scalar mass perturbation [64, 65].

Perhaps the most intriguing type of the RG flow, from the point of view of understanding confinement in real-world QCD, is the one which would point towards a strongly-coupled, nonAbelian, nonlocal conformal fixed points [11–14]. Though they are the most difficult ones to analyze in general, as they involve nonlocal, nonAbelian systems with strongly-coupled monopoles, dyons and quarks (meaning that the system has no Lagrangian description), some remarkable developments (Gaiotto-Seiberg-Tachikawa duality [66]) allow us to analyze the system explicitly, and to prove confinement, upon appropriate $\mathcal{N} = 1$ perturbations. Color confinement in the true sense (a) (i.e., without Abelianization) is indeed realized in these models, as has been shown in [12–14]. See Fig. 6.

4.1 A final reflection

The RG flow of an asymptotically-free gauge theory with massless matter fields is directed towards an infrared-fixed point theory. When a relevant perturbation is present (either in the UV theory, such as a mass term, or produced by the system dynamically, in the form of composite scalar fields) the RG flow may get deviated at the IR end of the trajectory (Fig.5), leading to a confining vacuum. A conformal theory (CFT), a scale invariant theory, and confinement (generation of mass scale and breaking of dilatation symmetry) might look at first sight diametrically opposite, conceptually. How can they be close to (or deformed into) each other?

There are at least two precise senses in which they can indeed be "close to each other". The first is that one of them may go into another when the parameters of the theory is varied (such as the number of the flavors, or the mass of the matter fermions). Namely they can be close to each other in the space of theories. In the standard $SU(3)$ QCD with N_F massless quark flavors, confining and conformal vacua are believed to be separated by an (unknown) critical flavor number N_F^*.

More significantly, we learn from the analysis of the supersymmetric theories that *the same degrees of freedom (monopoles, dyons and quarks) describe both the infrared fixed-point CFT and the nearby (perturbed by some relevant operators) vacuum in confinement phase.*

In the real-world $SU(3)$ QCD, with two nearly massless quarks, we may exclude Abelian or nonAbelian infrared-free phases, for the lack of any phenomenological evidence. Among the different RG flows (Fig. 5), then, the only one which seems plausible is the one towards the confining vacuum lying near a nonAbelian, strongly-coupled nonlocal conformal fixed point. It is possible that a phenomenon very similar to the one studied in [11]-[14] is indeed realized in the standard QCD, even though our ability of exhibiting the dynamical details is for the moment limited to the cases of $\mathcal{N} = 2$ supersymmetric cousins.

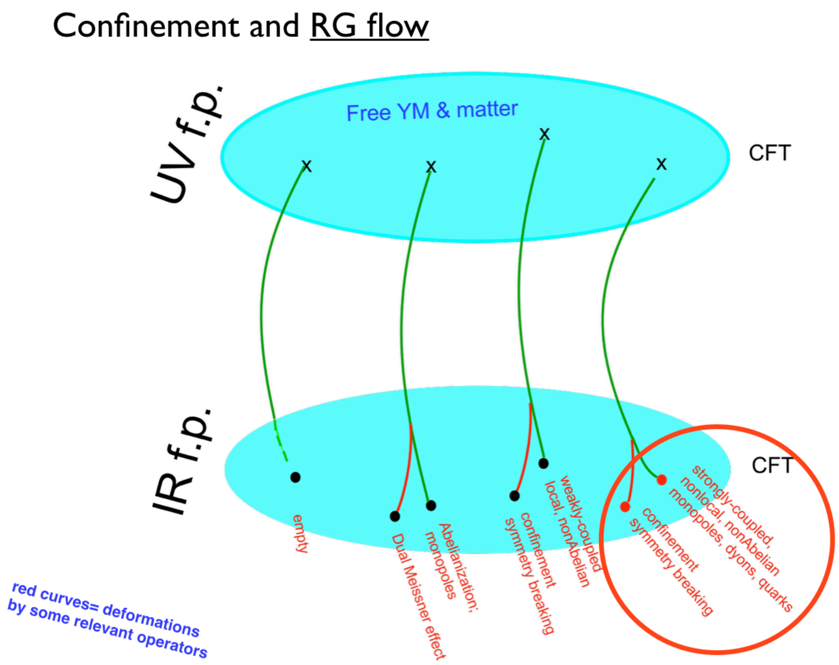

Figure 5: Various types of renormalization group (RG) flows. The IR fixed points can be trivial, Abelian free theories, nonAbelian but local theories, or a strongly-coupled, nonAbelian, and nonlocal CFT. Different types of confinement can occur, accordingly.

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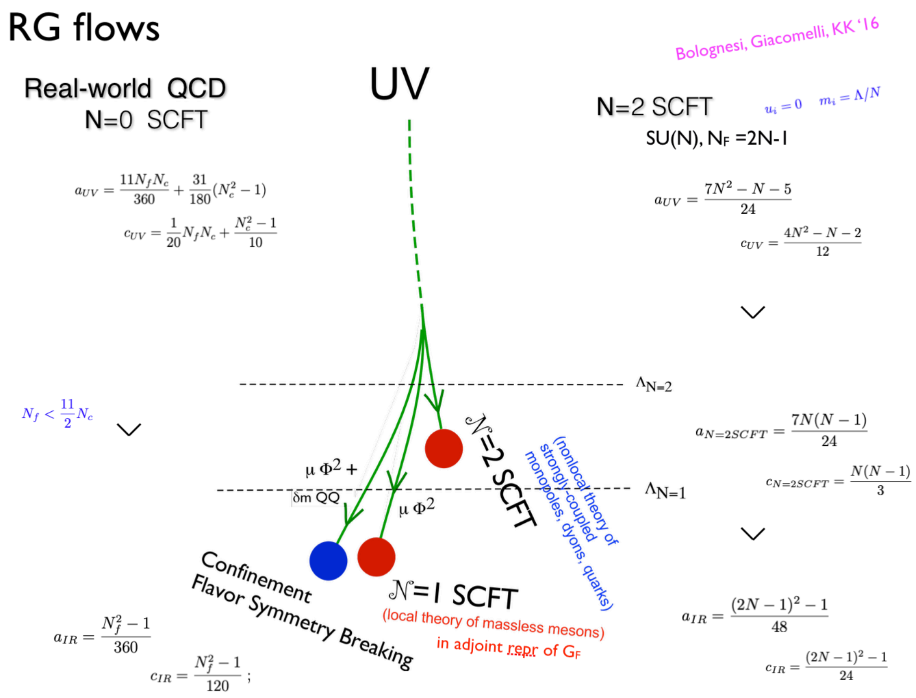

Figure 6: An illustration of a strongly-coupled, nonAbelian CFT occurring in $\mathcal{N} = 2$ SQCD with $N_f = 2N_c - 1$, deformed into a confining vacuum with relevant, $\mathcal{N} = 1$ perturbations. As a side remark, the a theorem (showing the consistency of the RG flow [67]) is illustrated in this model and in the standard, real-world QCD.

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