Generalization of the Bargmann-Wigner approach to constructing relativistic fields

I.L. Buchbinder,[a] **S.A. Fedoruk,**[a] **A.P. Isaev**[a,b,∗] **and M.A. Podoinitsyn**[a]

[a]*Bogoliubov Laboratory of Theoretical Physics, JINR,*
141980 Dubna, Moscow region, Russia

[a]*Physics Faculty, Lomonosov Moscow State University, Russia*
E-mail: buchbinder@theor.jinr.ru, fedoruk@theor.jinr.ru,
isaevap@theor.jinr.ru, mpod@theor.jinr.ru

We review the method for constructing local relativistic fields corresponding to the Bargmann-Wigner wave functions that describe the unitary irreducible representations of the $4D$ Poincaré group. The method is based on the use of the generalized Wigner operator connecting the wave functions of induced representations and local relativistic fields. Applications of this operator for constructing massive local relativistic fields as well as massless helicity local fields and massless local infinite spin fields are considered.

∗Speaker

1. Introduction

The Poincaré group is the mathematical expression of full relativistic symmetry. Therefore, the study of various aspects of the Poincaré group and its representations is closely related to describing the properties of fundamental objects in modern physics. It is sufficient to say that just irreducible unitary representations of the Poincaré group underly the classification of elementary particles in mass and spin.

The fundamentals of describing unitary irreducible representations of the $4D$ Poincaré group were laid down in the eminent paper by Wigner [1][1]. The approach to describing irreducible representations of the Poincaré group was further developed in the works by Wigner and Bargmann [4, 5], where a general construction of such representations was proposed.

As is well known, irreducible unitary representations of the Poincaré group are divided into two classes: massive and massless. Massless representations in turn are subdivided into helicity representations and representations with continuous (infinite) spin. It is generally believed that only helicity representations are physically acceptable. However, recently continuous (infinite) spin representations have attracted attention as well (see e.g. [6–9] and the references therein).

Following Wigner, all unitary irreducible representations of the Poincaré group can be constructed as induced ones from unitary irreducible representations of the test momentum stability subgroup (Wigner little group). As it is known, these representations are realized on the Bargmann-Wigner wave functions [4], [5]. However, in coordinate space, these functions are non-locally transformed under the action of the Lorentz group. Therefore, in their initial form, the Bargmann-Wigner wave functions can not be used to construct relativistic field theories which are formulated in terms of local fields. The problem of constructing bosonic relativistic fields on the basis of the Bargmann-Wigner wave function was discussed in our recent papers [10–12], where the method of deriving the corresponding relativistic fields was developed. We were used the ideas of papers [13, 14]. The central object of such a construction is the generalized Wigner operator that transforms the Bargmann-Wigner wave functions into local relativistic fields. The form of this operator is different for different classes of representations: it is a matrix acting on tensors for massive representations, an integral operator for representations with continuous (infinite) spin, and for helicity ones it is given in terms of distributions. In fact, this generalized Wigner operators provide the most general relativistic equations and expressions for local relativistic fields corresponding to unitary irreducible representations of the Poincaré group. The paper under consideration is a brief review of our approach.

The work is organized as follows. In Section 2, we recall the basic notion concerning the irreducible representations of the Poincaré group. Subsection 3.1 is devoted to describing the general construction of the induced Wigner representations. In Subsection 3.2, we discuss in detail the case of massive representations, their realization in terms of local relativistic fields and free equations of motion for these fields. Subsection 3.3 is devoted to the explicit construction of Bargmann-Wigner wave functions for massless representations with continuous (infinite) spin. In Subsection 3.4, the generalized Wigner operator is introduced and its properties are discussed. In Sections 4 and 5, we find an explicit form of the generalized Wigner operators in the case of massless representations with continuous (infinite) spin and in the case of helicity representations,

---

[1]Earlier, this subject, in a slightly less general form, was also studied by Majorana [2] and Dirac [3].

respectively. Then with the help of these operators we construct the corresponding local relativistic fields. We also demonstrate here that in the case of helicity representations, the approach under consideration automatically leads to a description of the corresponding relativistic fields in terms of potentials defined up to gauge transformations.

2. Little group and the fundamental Wigner operators

In this section we briefly describe the basic notation and conventions used in the paper.

We begin with the well known one-to-one correspondence between the set of coordinates $x_\mu \in \mathbb{R}^{1,3}$ and the set $\mathcal{H}$ of Hermitian matrices $X = x_\mu \sigma^\mu \in \mathcal{H}$, where $\sigma^0 = I_2$ is the 2×2 unit matrix and σ^i, $i = 1, 2, 3$ are the Pauli σ-matrices. The action of the group $SL(2, \mathbb{C})$ on the set $\mathcal{H}$

$$X \rightarrow X' = AXA^\dagger, \qquad X, X' \in \mathcal{H}, \quad A \in SL(2, \mathbb{C}) \tag{2.1}$$

leads to the following group homomorphism $SL(2, \mathbb{C}) \rightarrow SO^\uparrow(1, 3)$:

$$A \sigma^\mu A^\dagger = \sigma^\nu \Lambda_\nu{}^\mu(A), \qquad A \in SL(2, \mathbb{C}), \quad \Lambda_\mu{}^\nu(A) \in SO^\uparrow(1, 3). \tag{2.2}$$

Let $p \in \mathbb{R}^{1,3}$ be the 4-momentum of the relativistic particle and let $\mathring{p} \in \mathbb{R}^{1,3}$ be some singled out test 4-momentum. The momentum coordinates p_μ and $\mathring{p}_\mu$ are transformed one into another by the Wigner operator $A_{(p)}$ according to the relation

$$A_{(p)}(\mathring{p}\,\sigma)A_{(p)}^\dagger = (p\,\sigma) \quad \Rightarrow \quad \Lambda_\mu{}^\nu(A_{(p)})\,\mathring{p}_\nu = p_\mu, \tag{2.3}$$

where we have used the notation $(p\sigma) := p_\mu \sigma^\mu$. The transformations (2.3) are in fact the definition of the operators $A_{(p)} \in SL(2, \mathbb{C})$ which transform some test momentum $\mathring{p}$ into the arbitrary momentum p. It is well known that the above transformations preserve the square of the 4-momentum $p_\mu p^\mu$, i.e. $p_\mu p^\mu = \mathring{p}_\mu \mathring{p}^\mu$. Further we will consider the case of massive particles, where the 4-momentum is time-like, $p_\mu p^\mu = \boldsymbol{m}^2 \neq 0$, and the case of massless particles, when the 4-momentum is light-like, $p_\mu p^\mu = 0$.

Among the transformations of the Lorentz group $SL(2, \mathbb{C})$ there are those that preserve the test momentum $\mathring{p}$, i.e.:

$$h(\mathring{p}\,\sigma)h^\dagger = (\mathring{p}\,\sigma). \tag{2.4}$$

Such matrices $h \in SL(2, \mathbb{C})$ form the Wigner little group G_0 of the test momentum $\mathring{p} \in \mathbb{R}^{1,3}$.

As follows from (2.3) and (2.4), the Wigner operators $A_{(p)}$ are defined up to right multiplication by the elements of G_0 and parametrize the coset space $SL(2, \mathbb{C})/G_0$. According to (2.2)–(2.3), the left action of the group $SL(2, \mathbb{C})$ on this coset is defined by

$$A\,A_{(p)} = A_{(\Lambda p)}\,h_{A,p}, \qquad h_{A,p} \in G_0. \tag{2.5}$$

This relation, rewritten as

$$h_{A,p} = A_{(\Lambda p)}^{-1} A\,A_{(p)} \quad \Rightarrow \quad h_{A,\Lambda^{-1}p} = A_{(p)}^{-1} A\,A_{(\Lambda^{-1}p)}, \tag{2.6}$$

determines the element $h_{A,p}$ of the Wigner little group G_0, where the labels A, p in the notation $h_{A,p}$ indicate its dependence on $A \in SL(2, \mathbb{C})$ and the 4-momentum p.

Unitary irreducible representations of the Poincaré group are induced by unitary irreducible representations of the Wigner little group G_0, the structure of which is determined by the mass parameter of the particle being considered, i.e., massive particle with $\boldsymbol{m} \neq 0$ or massless one with $\boldsymbol{m} = 0$.

• Massive case. The test momentum $\mathring{p}$ can be chosen as the rest particle momentum

$$||\mathring{p}_\mu|| = (\mathring{p}_0, \mathring{p}_1, \mathring{p}_2, \mathring{p}_3) = (\boldsymbol{m}, 0, 0, 0) \tag{2.7}$$

and the Wigner little group is a unitary group: $G_0 \simeq \mathrm{SU}(2)$. In this case, formula (2.6) can also be written in the form

$$h_{A,p} = (h_{A,p})^{-1\dagger} = A^\dagger_{(\Lambda p)} A^{-1\dagger} A^{-1\dagger}_{(p)}. \tag{2.8}$$

• Massless case. The test momentum $\mathring{p} \in \mathbb{R}^{1,3}$ can be fixed as the light-cone particle momentum

$$||\mathring{p}_\mu|| = (\mathring{p}_0, \mathring{p}_1, \mathring{p}_2, \mathring{p}_3) = (E, 0, 0, E), \qquad (\mathring{p}\,\sigma) = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.9}$$

The Wigner little group is formed by the matrices

$$h = \begin{pmatrix} 1 & b_1 + \mathrm{i}b_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\theta/2} & 0 \\ 0 & e^{-\mathrm{i}\theta/2} \end{pmatrix}, \tag{2.10}$$

where $\theta \in [0, 2\pi)$ and $\vec{b} = (b_1, b_2) \in \mathbb{R}^2$, and is isomorphic to the group of motions of two-dimensional Euclidean space $G_0 \simeq \mathrm{ISO}(2)$.

3. Generalized Wigner operators and relativistic fields

Irreducible representations of the Poincaré group as well as its covering group are classified according to the eigenvalues of the corresponding Casimir operators. The Poincaré algebra in four dimensions has two Casimir operators: the squared momentum $\hat{P}^2$ and the squared Pauli-Lubanski vector $\hat{W}^2$. For the cases under consideration, all possible unitary irreducible representations are determined by the following eigenvalues of these operators [1, 4, 5].

• Massive representations.
On the spaces of these representations, the following conditions are fulfilled:

$$\hat{P}^2 = \boldsymbol{m}^2, \qquad \hat{W}^2 = -\boldsymbol{m}^2 j(j+1), \tag{3.1}$$

where $\boldsymbol{m} > 0$ is the real parameter describing the particle mass, and $j \in \mathbb{Z}_{\geq 0}/2$ is the particle spin.

• Massless representations.
In these representations, the Casimir operators are equal to:

$$\hat{P}^2 = 0, \qquad \hat{W}^2 = -\boldsymbol{\mu}^2, \tag{3.2}$$

where $\boldsymbol{\mu} \in \mathbb{R}_{\geq 0}$ is the mass dimension parameter. Depending on the value of this parameter, the massless representations belong to one of the two classes, i.e. they are subdivided into

– either the helicity representations with $\mu = 0$;

– or the continuous (infinite) spin representations with $\mu > 0$.

Note also that the parameter μ, which defines the massless continuous spin representations (3.2), corresponds to the radius of the circle in the auxiliary two-dimensional Euclidean space that arises when considering the small group (2.10). In addition, we emphasize that the continuous spin representation contains an infinite number of massless states with all possible integer or half-integer helicities[2]. The name "continuous spin representation" was developed historically after its introduction in [1, 4, 5] and is associated with the possibility of describing this discrete set as a Fourier series with respect to the angular variable which is similar to the angle θ in (2.10). To avoid any misunderstandings, we also use the name "infinite spin representations" for such representations.

3.1 Bargmann-Wigner wave functions

The main method in constructing unitary irreducible representations U of the Poincaré covering group ISL$(2, \mathbb{C})$ is based on the Wigner scheme [1, 4, 5]. Within this scheme, all unitary irreducible representations (UIR's) are realized on the Bargmann-Wigner wave functions $\Phi_M(p)$, which depend on the four-momentum p_μ on the mass shell $p^\mu p_\mu = m^2$ and have an additional (multi)index M that runs over a finite or infinite number of values. This index is related to the unitary irreducible representation of the Wigner little group G_0.

Representation $U(A)$ of the element $A \in SL(2, \mathbb{C})$ acts on the Bargnmann-Wigner wave function as follows:

$$\Phi'_M(p) := [U(A)\Phi]_M(p) = \mathcal{D}_{MN}(h_{A,\Lambda^{-1}p})\Phi_N(\Lambda^{-1}p), \tag{3.3}$$

where the matrix Λ is related to A by (2.2), $\mathcal{D}_{MN}$ is UIR of the Wigner little group G_0 and the element $h_{A,\Lambda^{-1}p}$ is determined by relation (2.6). One can check directly that $U(A\,B) = U(A)\,U(B)$. The representation (3.3) of the Poincaré group ISL$(2, \mathbb{C})$, constructed in this way, is induced by the unitary representation of the Wigner little group G_0.

However, the transformation law (3.3) becomes non-local for the space-time field (field in coordinate representation) obtained from the Bargmann-Wigner wave function $\Phi_M(p)$ using the Fourier transform. To find local relativistic fields, it is necessary to modify the Bargmann-Wigner wave function before performing the Fourier transform. In the next subsection we will briefly describe how it can be done for the case of massive representations (for a detailed discussion of them, see [12, 13, 15]).

3.2 Massive representations

In the massive case, UIR of the Wigner little group $G_0 = SU(2)$ is characterized by the spin quantum number $j \in \mathbb{Z}_{\geq 0}/2$. This representation is realized on the Bargmann-Wigner wave functions $\Phi_{a_1 \cdots a_{2j}}(p) = \Phi_{(a_1 \cdots a_{2j})}(p)$ which are the SU(2)-spinors of rank $2j$, where $a = 1, 2$ is the SU(2)-index and the parentheses $(a_1 \ldots a_{2j})$ mean total symmetrization over all indices. In

---

[2]For these representations helicities are not invariants under the action of the Lorentz group SL$(2, \mathbb{C})$.

this case, the transformation law (3.3) is written as

$$\Phi'_{a_1...a_{2j}}(p) := [U(A)\Phi]_{a_1...a_{2j}}(p) = (h_{A,\Lambda^{-1}p})_{a_1}^{\ b_1} \cdots (h_{A,\Lambda^{-1}p})_{a_{2j}}^{\ b_{2j}}\ \Phi_{b_1...b_{2j}}(\Lambda^{-1}p)\,. \quad (3.4)$$

Let us move from the Bargmann-Wigner wave functions $\Phi_{a_1\cdots a_{2j}}(p)$ to the new wave functions whose transformation law, in contrast to the transformation law (3.4), uses the matrices which do not depend on the argument p. These new wave functions are the multicomponent fields labeled by the indices q, r and defined as follows:

$$\Psi^{(q,r)\,\dot\beta_1...\dot\beta_r}_{\quad\ \alpha_1...\alpha_q}(p) = (A_{(p)})_{\alpha_1}^{\ b_1} \cdots (A_{(p)})_{\alpha_q}^{\ b_q}\,(\tilde A^{-1\dagger}_{(p)})^{\dot\beta_1 b_{q+1}} \cdots (\tilde A^{-1\dagger}_{(p)})^{\dot\beta_r b_{2j}}\Phi_{b_1...b_{2j}}(p)\,, \quad (3.5)$$

where $q = 0, 1, \ldots, 2j$, $r = 2j - q$ and the following notation[3] $\tilde A^{-1\dagger}_{(p)} := A^{-1\dagger}_{(p)}\tilde\sigma_0$ is used. The constructed fields (3.5) are symmetric with respect to indices of the same type: $\Psi^{(q,r)\,\dot\beta_1...\dot\beta_r}_{\quad\ \alpha_1...\alpha_q}(p) = \Psi^{(q,r)\,(\dot\beta_1...\dot\beta_r)}_{\quad\ (\alpha_1...\alpha_q)}(p)$. An important property of the fields (3.5) is their transformation law

$$[U(A)\Psi^{(q,r)}]^{\dot\beta_1...\dot\beta_r}_{\alpha_1...\alpha_q}(p) = A_{\alpha_1}^{\ \gamma_1}\cdots A_{\alpha_q}^{\ \gamma_q}\,(A^{\dagger-1})^{\dot\beta_1}_{\ \dot\kappa_1}\cdots(A^{\dagger-1})^{\dot\beta_r}_{\ \dot\kappa_r}\ \Psi^{(q,r)\,\dot\kappa_1...\dot\kappa_r}_{\quad\ \gamma_1...\gamma_q}(\Lambda^{-1}\cdot p)\,, \quad (3.6)$$

which is a direct consequence of the transformation law (3.4) and relations (2.6), (2.8). Here $A \in \mathrm{SL}(2,\mathbb{C})$. Since the transformation matrix in (3.6) does not depend on the 4-momentum p, the constructed fields $\Psi^{(q,r)\,\dot\beta_1...\dot\beta_r}_{\quad\ \alpha_1...\alpha_q}(p)$ are the standard local spin-tensor fields in the momentum representation. The operators $A^{\otimes q}_{(p)}(\tilde A^{-1\dagger}_{(p)})^{\otimes r}$, $r + q = 2j$, which transform the Bargmann-Wigner wave functions $\Phi_{a_1\cdots a_{2j}}(p)$ into local relativistic fields $\Psi^{(q,r)\,\dot\beta_1...\dot\beta_r}_{\quad\ \alpha_1...\alpha_q}(p)$, are called the spin j Wigner operators.

The fields $\Psi^{(q,r)\,\dot\beta_1...\dot\beta_r}_{\quad\ \alpha_1...\alpha_q}(p)$ at fixed spin $j = (q + r)/2$ are not independent of mass-shell: they automatically satisfy the Dirac-Pauli-Fierz equations

$$\begin{aligned}
(p\,\tilde\sigma)^{\dot\gamma_1\alpha_1}\Psi^{(q,r)\,\dot\beta_1...\dot\beta_r}_{\quad\ \alpha_1...\alpha_q}(p) &= \boldsymbol{m}\,\Psi^{(q-1,r+1)\,\dot\gamma_1\dot\beta_1...\dot\beta_r}_{\quad\ \ \alpha_2...\alpha_q}(p)\,, \quad (r = 0,\ldots,2j-1)\,, \\
(p\,\sigma)_{\gamma_1\dot\beta_1}\Psi^{(q,r)\,\dot\beta_1...\dot\beta_r}_{\quad\ \alpha_1...\alpha_q}(p) &= \boldsymbol{m}\,\Psi^{(q+1,r-1)\,\dot\beta_2...\dot\beta_r}_{\quad\ \ \gamma_1\alpha_1...\alpha_q}(p)\,, \quad (r = 1,\ldots,2j)\,.
\end{aligned} \quad (3.7)$$

which follow from (2.3) and (2.7). In particular, for $j = \frac{1}{2}$, eqs. (3.7) are two Weyl projections of the spin-$\frac{1}{2}$ Dirac equation. These and other aspects of such a description of massive particle states are given in detail in [12, 15]. Other methods of deducing relativistic field equations from the first principles were also considered in [16].

3.3 Massless representations

UIR's of the group ISO(2), being the Wigner little group G_0 in the massless case, are well known (see e.g. [17, 18]). They are characterized by a real nonnegative parameter ρ and realized on the functions $\Phi(\varphi)$ defined on the circle of the radius ρ on the plane parameterized by the vector coordinates $\vec t_\varphi = ((t_\varphi)_1, (t_\varphi)_2) = (\rho\cos\varphi, \rho\sin\varphi)$. Induced UIR's of the Poincaré group

---

[3]In addition to the relativistic Pauli matrices $(\sigma^\mu)_{\alpha\dot\beta}$ we also use the matrices $(\tilde\sigma^\mu)^{\dot\alpha\beta} = (\tilde\sigma^0, \tilde\sigma^i) = (\sigma^0, -\sigma^i)$.

ISL$(2, \mathbb{C})$ are realized on the wave functions $\Phi(p, \varphi)$, which also depend on the 4-momentum p_μ. In the massless case, the formula (3.3) of SL$(2, \mathbb{C})$ takes the following form:

$$
\begin{aligned}
\Phi'(p, \varphi) &:= [U(A)\Phi](p, \varphi) = \int_0^{2\pi} d\varphi' \, \mathcal{D}_{\varphi\varphi'}(\theta_{A, \Lambda^{-1}p}, \vec{b}_{A, \Lambda^{-1}p}) \, \Phi(\Lambda^{-1}p, \varphi') \\
&\equiv e^{-i\vec{b}_{A, \Lambda^{-1}p} \cdot \vec{t}_\varphi} \, \Phi(\Lambda^{-1}p, \varphi - \theta_{A, \Lambda^{-1}p}),
\end{aligned}
\tag{3.8}
$$

where the operator $\mathcal{D}_{\varphi\varphi'}(\theta_{A, \Lambda^{-1}p}, \vec{b}_{A, \Lambda^{-1}p})$ is the representative of the little group element $h_{A, \Lambda^{-1}p} \in$ ISO(2) in unitary representation (see e.g. [17]). The dependence of the element $h \in$ ISO(2) (parametrized by the pair $(\theta, \vec{b})$ as in (2.10)) on A, p was discussed in [10]. The matrix element $\mathcal{D}_{\varphi\varphi'}(\theta, \vec{b})$ has the following explicit form:

$$
\mathcal{D}_{\varphi\varphi'}(\theta, \vec{b}) = e^{-i\vec{b}\cdot\vec{t}_\varphi} \delta(\varphi - \varphi' - \theta),
\tag{3.9}
$$

where $\delta(\varphi)$ is the periodic Dirac δ-function on a circle and $\vec{b} \cdot \vec{t}_\varphi = \rho \, (b_1 \cos\varphi + b_2 \sin\varphi)$.

There is another equivalent formulation of the induced massless Poincaré group representations (see e.g. [6], [10]), which is obtained after expanding the Bargmann-Wigner wave function $\Phi(p, \varphi)$ into a Fourier series:

$$
\Phi(p, \varphi) = \sum_{n=-\infty}^{\infty} \Phi_n(p) e^{in\varphi}.
\tag{3.10}
$$

In the space of functions $\Phi_n(p)$, the transformation law (3.8) has the form:

$$
[U(A)\Phi]_n(p) = \sum_{m=-\infty}^{\infty} \mathcal{D}_{nm}(\theta_{A, \Lambda^{-1}p}, \vec{b}_{A, \Lambda^{-1}p}) \, \Phi_m(\Lambda^{-1}p),
\tag{3.11}
$$

where the matrix elements $\mathcal{D}_{nm}(\theta, \vec{b})$ of the little group ISO(2) are

$$
\mathcal{D}_{nm}(\theta, \vec{b}) = (-ie^{i\beta})^{m-n} \, e^{-im\theta} J_{(m-n)}(b\rho).
\tag{3.12}
$$

Here the real numbers β and b are the polar coordinates of the 2-vector $\vec{b} = (b_1, b_2) = b\,(\cos\beta, \sin\beta)$ and $J_{(n)}(x)$ are the Bessel functions of integer order n.

The induced unitary Wigner representations (3.8) and (3.11) of the Poincaré group ISL$(2, \mathbb{C})$ are irreducible for $\rho \neq 0$ and infinite-dimensional. The corresponding representations $\mathcal{D}$ of ISO(2) written for the discrete bases (3.10)-(3.12) are decomposed into a direct sum of one-dimensional irreducible representations of the subgroup SO$(2) \subset$ ISO(2). The action of the group ISO(2) in the representation $\mathcal{D}$ in the limit $\rho \to 0$ is reduced to the action of its subgroup SO(2). Thus, in the limit $\rho \to 0$, the representation $\mathcal{D}$ of ISO(2) becomes reducible and is decomposed into a sum of infinite number of one dimensional UIR's. In the limit $\rho \to 0$ expression (3.11) is used in the description of the ISL$(2, \mathbb{C})$ helicity representations with finite numbers of relativistic spin states (see Section 5 below).

To construct a local relativistic field corresponding to the Bargmann-Wigner wave function $\Phi(p, \varphi)$, we use a generalization of the method applied in Subsection 3.2 in the case of massive representations.

3.4 Generalized Wigner operators in the infinite spin case

In the massive case, relation (3.5) between the local relativistic fields and the Bargmann-Wigner wave functions is served by the Wigner operators $A_{(p)}^{\otimes q}(\tilde{A}_{(p)}^{-1\dagger})^{\otimes r}$. In the general case of massless representations including infinite-dimensional representations of continuous spin, it is required to use an infinite-dimensional generalization of this correspondence.

The main role in this correspondence is played by the Wigner operator $A_{(p)}$ whose matrices parameterize the coset-space $\mathrm{SL}(2,\mathbb{C})/G_0$. After restoring the indices, the left action (2.5) of the element $A \in \mathrm{SL}(2,\mathbb{C})$ on $\mathrm{SL}(2,\mathbb{C})/G_0$ is written as

$$A_\alpha{}^\beta \, (A_{(p)})_\beta{}^c = (A_{(\Lambda p)})_\alpha{}^b \, (h_{A,p})_b{}^c \,, \tag{3.13}$$

where the matrix $(A_{(p)})_\alpha{}^a$ of the Wigner operator has the $\mathrm{SL}(2,\mathbb{C})$-index α and G_0-index a (in particular, for the massive case, we have $G_0 = \mathrm{SU}(2)$). Thus, the operators $A_{(p)}^{\otimes q}(\tilde{A}_{(p)}^{-1\dagger})^{\otimes r}$ play the role of a bridge that converts the little subgroup indices $a, b, ...$ of G_0-type into the relativistic indices $\alpha, \beta, ...$ of $\mathrm{SL}(2,\mathbb{C})$-type.

An analogue of such a transformation for the Bargmann-Wigner wave function $\Phi(p,\varphi)$ introduced in (3.8) for the massless case is given by the following formula:

$$\Psi(p,y) \;=\; \int\limits_0^{2\pi} d\varphi \, \mathcal{A}(p,y,\varphi) \, \Phi(p,\varphi) \,, \tag{3.14}$$

where additional variables y denote a certain set of variables that play the role of the relativistic $\mathrm{SL}(2,\mathbb{C})$-type indices while φ is of the $\mathrm{ISO}(2)$-type (G_0-type) variable. The transform (3.14) can be thought as an infinite-dimensional version of the transition (3.5): it converts the index φ of the Wigner little group $\mathrm{ISO}(2)$ into some (as yet undefined) relativistic index y. We call the integral operator with the kernel $\mathcal{A}(p,y,\varphi)$ in the right-hand side of (3.14) *the generalized Wigner operator*.

In the next section, the $\mathrm{SL}(2,\mathbb{C})$-type variables y are taken to be the components of the Lorentz vector $\eta^\mu \in \mathbb{R}^{1,3}$, or the components of a pair of Weyl spinors u^α, $\bar{u}^{\dot\alpha}$. The requirement of the locality of the $\mathrm{SL}(2,\mathbb{C})$ transformations of the relativistic fields $\Psi(p,\eta)$, or $\Psi(p,u,\bar{u})$, leads to the definition of the explicit form of the generalized Wigner operators in (3.14) for these two cases.

4. Relativistic fields with infinite spin

4.1 Additional vector variables

Let us consider variables y in (3.14) as additional vector coordinates $\eta_\mu \in \mathbb{R}^{1,3}$ of the fields. The field $\Psi(p,\eta)$ must have the local transformation law under the action of $A \in \mathrm{SL}(2,\mathbb{C})$ (independent of p), as it was in (3.6). In the case when, a relativistic field is characterized by its dependence on the auxiliary vector variables η_μ instead of external indices, this transformation law takes the form

$$\Psi'(p,\eta) \;:=\; [U(A)\Psi](p,\eta) \;=\; \Psi(\Lambda^{-1}p, \Lambda^{-1}\eta) \,, \tag{4.1}$$

where the matrices A and Λ are related by (2.2) and momentum p is on the mass shell $p^2 = 0$. Taking into account (3.8) and (3.14), the condition (4.1) leads to the equation for the kernel $\mathcal{A}(p,\eta,\varphi)$:

$$\mathcal{A}(\Lambda^{-1}p, \Lambda^{-1}\eta, \varphi) \;=\; \int d\varphi' \mathcal{A}(p,\eta,\varphi') \, \mathcal{D}_{\varphi'\varphi}(\theta_{A,\Lambda^{-1}p}, \vec{b}_{A,\Lambda^{-1}p}) \,. \tag{4.2}$$

This equation on the generalized Wigner operator is an infinite-dimensional counterpart of the matrix coset transformation (3.13).

As shown in [10, 19, 20], the infinitesimal form of a relation (4.2) is represented as three differential equations for the kernel $\mathcal{A}(p, \eta, \varphi)$, which can be solved exactly. There are two types of solutions of these equations, which can be called non-singular and singular [10, 19, 20].

- Non-singular solution.
  The operator $\mathcal{A}(p, \eta, \varphi)$ that satisfies (4.2) has the following explicit form:

$$\mathcal{A}(p, \eta, \varphi) = e^{i\mu\,\eta\cdot\varepsilon_{(1)}(\varphi)/(\eta\cdot p)}\, f(\eta\cdot\eta, \eta\cdot p)\,, \tag{4.3}$$

  where $f(\eta\cdot\eta, \eta\cdot p)$ is an arbitrary function of two variables $\eta\cdot\eta := \eta^\mu \eta_\mu$ and $\eta\cdot p := \eta^\mu p_\mu$. The dimensional constant μ is related to the dimensionless constant ρ by the relation

$$\mu = E\rho\,, \tag{4.4}$$

  where E is defined by nonzero components of the massless test momentum $\overset{\circ}{p}$ in (2.9). In (4.3), the vector $\varepsilon^\mu_{(1)}(\varphi) = \varepsilon^\mu_{(1)} \cos\varphi - \varepsilon^\mu_{(2)} \sin\varphi$ is constructed from the polarization vectors $\varepsilon_{(1)}$ and $\varepsilon_{(2)}$ which are determined by the conditions $\varepsilon_{(1)}\cdot\varepsilon_{(1)} = \varepsilon_{(2)}\cdot\varepsilon_{(2)} = -1$, $p\cdot\varepsilon_{(1)} = p\cdot\varepsilon_{(2)} = \varepsilon_{(1)}\cdot\varepsilon_{(2)} = 0$.

- Singular solution.
  In this case, the solution of (4.2) can be written as follows:

$$\mathcal{A}(p, \eta, \varphi) = \delta(\eta\cdot p)\,\delta(\eta\cdot\varepsilon_{(2)}(\varphi))\, e^{i\mu\,\eta\cdot\varepsilon/(\eta\cdot\varepsilon_{(1)}(\varphi))}\, g(\eta\cdot\varepsilon_{(1)}(\varphi))\,, \tag{4.5}$$

  where $g(\eta\cdot\varepsilon_{(1)}(\varphi))$ is an arbitrary function and $\varepsilon^\mu_{(2)}(\varphi) = \varepsilon^\mu_{(1)} \sin\varphi + \varepsilon^\mu_{(2)} \cos\varphi$. In (4.5), the vector ε^μ obeys the conditions $\varepsilon\cdot p = 1$, $\varepsilon\cdot\varepsilon = \varepsilon\cdot\varepsilon_{(1)} = \varepsilon\cdot\varepsilon_{(1)} = 0$ and forms tetrads with the vectors $p, \varepsilon_{(1)}, \varepsilon_{(2)}$.

Two expressions (4.3) and (4.5) for the generalized Wigner operators produce two types of relativistic fields $\Psi(p, \eta)$, according to the transform (3.14), where one can use the expansion (3.10) for the Bargmann-Wigner wave function. By construction, the fields $\Psi(p, \eta)$ describe physical states with the eigenvalues (3.2) of the Casimir operators. Indeed, the first eigenvalue in (3.2) is evident. The second eigenvalue was checked in [10], where we used the explicit formula for the square of the Pauli-Lubanski operator $\hat{W}^2$ acting in the space of the fields $\Psi(p, \eta)$. All these statements are fulfilled for both non-singular and singular solutions.

4.2 Additional spinor variables

Let us now consider the case when the components of the commuting Weyl spinor u_α, $\bar{u}_{\dot\alpha} = (u_\alpha)^\dagger$, $\alpha, \dot\alpha = 1, 2$ are taken as additional variables y in (3.14). The local relativistic field $\Psi(p, u, \bar{u})$ constructed by using the generalized Wigner operator in the procedure described above was found in [10] and has the following form:

$$\Psi(\pi, \bar{\pi}, u, \bar{u}) = \int_0^{2\pi} d\varphi\, \exp\left\{-i\,\mu\left(\frac{u^\alpha \lambda_\alpha}{u^\beta \pi_\beta}\, e^{-i\varphi} + \frac{\bar{u}^{\dot\alpha} \bar{\lambda}_{\dot\alpha}}{\bar{u}^{\dot\beta} \bar{\pi}_{\dot\beta}}\, e^{i\varphi}\right)\right\} \mathrm{f}(u^\gamma \pi_\gamma\, e^{\frac{i}{2}\varphi}, \bar{u}^{\dot\gamma} \bar{\pi}_{\dot\gamma}\, e^{-\frac{i}{2}\varphi})\, \Phi(p, \varphi)\,,$$

$$\tag{4.6}$$

where f is an arbitrary function of two conjugated scalar variables. In (4.6), the commuting Weyl spinor π_α is actually a twistor which defines massless 4-momentum through the Cartan-Penrose relation

$$(p\,\sigma)_{\alpha\dot\beta} = \left(A_{(p)}\,(\overset{\circ}{p}\,\sigma)\,A^\dagger_{(p)}\right)_{\alpha\dot\beta} = \pi_\alpha\bar\pi_{\dot\beta}\,. \tag{4.7}$$

Another commuting Weyl spinor λ_α is determined by the normalization condition $\pi^\alpha\lambda_\alpha = 1$, that is, a pair of spinors $\pi_\alpha,\lambda_\alpha$ forming the basis in the 2-component spinor space. Thus, for the choice (2.9), one can take the parametrization: $A_{(p)} = \begin{pmatrix} \pi_1/\sqrt{2E}, & \lambda_1\sqrt{2E} \\ \pi_2/\sqrt{2E}, & \lambda_2\sqrt{2E} \end{pmatrix}$.

The square of the Pauli-Lubanski vector $\hat{W}^2$ on the space of the fields $\Psi(\pi,\bar\pi,u,\bar u)$ has the following representation (see e.g. [7]):

$$\hat{W}^2 = (u^\alpha\pi_\alpha\bar\pi_{\dot\alpha}\bar u^{\dot\alpha})\left(\frac{\partial}{\partial\bar u^{\dot\beta}}\,\bar\pi^{\dot\beta}\pi^\beta\frac{\partial}{\partial u^\beta}\right)\,. \tag{4.8}$$

By making use of the explicit form (4.6) for $\Psi(\pi,\bar\pi,u,\bar u)$, it can be shown that

$$\hat{W}^2\,\Psi(\pi,\bar\pi,u,\bar u) = -\boldsymbol{\mu}^2\,\Psi(\pi,\bar\pi,u,\bar u)\,. \tag{4.9}$$

Therefore, the UIR's of the group ISL$(2,\mathbb{C})$ with infinite spin are realized on the field $\Psi(\pi,\bar\pi,u,\bar u)$. For more details about the equations of motion for the field $\Psi(\pi,\bar\pi,u,\bar u)$ see [10, 21, 22].

5. Relativistic gauge fields

Another important example of massless UIR's of the Poincaré covering group is massless helicity representations. Such representations are physically motivated and wildly used in higher spin field theory (see, e.g., the review [23]).

Helicity representations are induced from UIR's of the subgroup SO(2) $\subset$ ISO(2), or from UIR's of ISO(2) with trivial realizations of the 2-dimensional translations. All UIR's of the SO(2) group are one-dimensional and characterized by an integer or half-integer number corresponding to particle helicity. In what follows we will be interested only in representations with integer helicities and demonstrate that the approach of the generalized Wigner operators [11] reproduces well-known expressions for massless fields of higher helicities. One of the advantages of this method is obtaining a group theoretical description of massless helicity representations in terms of field potentials defined up to gauge transformations.

Helicity representations are induced by the UIR's of the Wigner little group $ISO(2)$ at $\boldsymbol{\rho} = 0$. In this limit, the transformation low (3.11) takes the form

$$[U(A)\Phi]_n(p) = e^{-in\theta_{A,\Lambda^{-1}p}}\,\Phi_n(\Lambda^{-1}p)\,, \qquad \forall\,n\in\mathbb{Z}\,, \tag{5.1}$$

where the property $J_{m-n}(0) = \delta_{m,n}$ for the Bessel functions was used. Thus, at $\boldsymbol{\rho} = 0$ the unitary representations of the Poincaré group are decomposed into a direct sum of UIR's induced from the one-dimensional UIR's of $SO(2)$ which are described by the functions $\Phi_n(p)$. Here the number n corresponds to particle helicity.

Relativistic local fields are constructed from the Wigner wave functions $\Phi_n(p)$ by using the transform (3.14) written in the discrete basis (3.10). It turns out that to construct a massless

relativistic field $\Psi_n(p)$ with spin $n > 0$, two Bargmann-Wigner wave functions with opposite helicities n and $-n$ should be used. In fact, this corresponds to the construction of a massless irreducible representation of the extended Poincaré group, which includes discrete transformations P and T (see e.g. [24, 25]). Thus, in accordance with (3.14), the local field of the helicity n particle has the form[4]

$$\Psi_n(p, \eta) = \mathcal{A}(p, \eta, n)\Phi_n(p) + \mathcal{A}(p, \eta, -n)\Phi_{-n}(p), \quad n \in \mathbb{Z}_{>0}. \tag{5.2}$$

The explicit form of the generalized Wigner operator $\mathcal{A}(p, \eta, n)$ in the discrete basis for the case $\rho = 0$ was found in [11] and has the form

$$\mathcal{A}(p, \eta, n) = \begin{cases} \delta(\eta \cdot p)\,(\varepsilon_{(+)} \cdot \eta)^n & \text{with } n > 0, \\[2mm] \delta(\eta \cdot p)\,(\varepsilon_{(-)} \cdot \eta)^{-n} & \text{with } n < 0, \end{cases} \tag{5.3}$$

where $\varepsilon_{(\pm)} = \varepsilon_{(2)} \pm i\varepsilon_{(1)}$. Then, the substitution of (5.3) into (5.2) gives the expression for the massless field of the helicity n:

$$\Psi_n(p, \eta) = \delta(\eta \cdot p)\, F_n(p, \eta), \tag{5.4}$$

where

$$F_n(p, \eta) = F_n^{(+)}(p, \eta) + F_n^{(-)}(p, \eta), \qquad F_n^{(\pm)}(p, \eta) = (\varepsilon_{(\pm)} \cdot \eta)^n\, \Phi_{\pm n}(p). \tag{5.5}$$

The explicit form of the functions (5.5) reproduces automatically the equations of motion of the field $F_n(p, \eta)$:

$$p^2 F_n(p, \eta) = 0, \quad \left(p \cdot \frac{\partial}{\partial \eta}\right) F_n(p, \eta) = 0, \quad \left(\frac{\partial}{\partial \eta} \cdot \frac{\partial}{\partial \eta}\right) F_n(p, \eta) = 0, \tag{5.6}$$

$$\left(\eta \cdot \frac{\partial}{\partial \eta}\right) F_n(p, \eta) = n\, F_n(p, \eta). \tag{5.7}$$

where the properties of polarization vectors were used. The last equation determines the degree of homogeneity for the field $F_n(p, \eta)$ in the variables η. In addition, the presence in the definition of $\Psi_n(p, \eta)$ of the field $F_n(p, \eta)$ together with the δ-function $\delta(\eta \cdot p)$ leads to the following equivalence relation:

$$F_n(p, \eta) \sim F_n(p, \eta) + (p \cdot \eta)\, \epsilon_{n-1}(p, \eta), \tag{5.8}$$

where $\epsilon_{n-1}(p, \eta)$ are homogeneous functions in η of degree $(n-1)$, such that they satisfy equations (5.6). Relation (5.8) is essentially a gauge transformation with the parameters $\epsilon_{n-1}(p, \eta)$. Thus, the construction (5.4)-(5.5) reproduces the description of gauge fields with all integer spins $n > 0$ in terms of distributions (see [6, 26]). A method similar to that described in this section but using additional spinor variables was discussed in [27].

---

[4]The case $n = 0$ is considered separately.

6. Conclusion

Let us briefly formulate the results.

In this paper, we have presented the main aspects of constructing a field realization of the $4D$ Poincaré group UIR's. We considered all possible physically interesting cases of UIR's including massive and massless representations. The basic fundamentals of our approach were the Wigner method for constructing induced representations. At the same time, the key point was to find a relation between the Bargmann-Wigner wave functions, on which induced representations are implemented, and relativistic fields which are locally transformed under the Poincaré group.

It is shown that such a connection is provided by the generalized Wigner operator, which explicitly relates the Bargmann-Wigner wave functions and local relativistic fields. In the massive case, this operator is described by the finite-dimensional matrix and the local fields obtained are automatically satisfied by the well-known Dirac-Pauli-Fierz equations. In the massless case including both helicity and infinite spin representations, the generalized Wigner operator is either an infinite-dimensional matrix or the kernel of the integral transform (3.14). Infinite spin fields are defined in the space with additional commuting coordinates which can be vectorial or spinorial. In the case of vectorial additional coordinates, we reproduce the infinite spin fields studied in [19, 20]. In the case of spinor additional variables, we obtain twistor infinite spin fields [21, 22]. In the case of massless helicity representations, the generalized Wigner operator contains the δ-function that allows us to introduce local fields in form of potentials which are defined up to gauge transformations.

References

[1] E.P. Wigner, *On Unitary Representations of the Inhomogeneous Lorentz Group*, Annals Math. **40** (1939) 149.

[2] E. Majorana, *Teoria relativistica di particelle con momento intrinseco arbitrario*, Nuovo Cimento **9** (1932) 335.

[3] P.A.M. Dirac, *Relativistic wave equations*, Proc. Roy. Soc. Lond. **A155** (1936) 447.

[4] E.P. Wigner, *Relativistische Wellengleichungen*, Z. Physik **124** (1948) 665.

[5] V. Bargmann, E.P. Wigner, *Group Theoretical Discussion of Relativistic Wave Equations*, Proc. Nat. Acad. Sci. **34** (1948) 211.

[6] X. Bekaert, E.D. Skvortsov, *Elementary particles with continuous spin*, Int. J. Mod. Phys. **A32** (2017) 1730019, `arXiv:1708.01030 [hep-th]`.

[7] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, V.A. Krykhtin, *Towards Lagrangian construction for infinite half-integer spin field*, Nucl. Phys. **B958** (2020) 115114, `arXiv:2005.07085 [hep-th]`.

[8] P. Schuster, N. Toro, *Interactions of Particles with "Continuous Spin" Fields*, `arXiv: 2303.04816 [hep-th]`

[9] P. Schuster, N. Toro, *Quantum Electrodynamics Mediated by a Photon with Generalized (Continuous) Spin*, JHEP 04 (2023) 010, `arXiv: 2308.16218 [hep-th]`

[10] I.L Buchbinder, A.P. Isaev, M.A. Podoinitsyn, S.A. Fedoruk, *Generalization of the Bargmann-Wigner construction for infinite spin fields*, Theor. and Math. Phys. **216:1** (2023) 973–999, `arXiv:2303.11852 [hep-th]`.

[11] I.L Buchbinder, S.A. Fedoruk, A.P. Isaev, M.A. Podoinitsyn, *Generalized Wigner operators and relativistic gauge fields*, Phys. Part. Nuclei Lett. **20** (2023) 605–612, `arXiv:2304.05945 [hep-th]`.

[12] A.P. Isaev, M.A. Podoinitsyn, *Two-spinor description of massive particles and relativistic spin projection operators,* Nucl. Phys. B **929** (2018) 452, arXiv:1712.00833 [hep-th].

[13] S. Weinberg, *Feynman rules for any spin*, Phys. Rev. **133** (5B) (1964) B1318.

[14] S. Weinberg, *Feynman rules for any spin 2. Massless particles*, Phys. Rev. **134** (1964) B882.

[15] A.P. Isaev, M.A. Podoinitsyn, *Behrends–Fronsdal spin projection operator in space-time with arbitrary dimension*, Springer Proc. Math. Stat. **263** (2017) 137-148.

[16] X. Bekaert, N. Boulanger, *The unitary representations of the Poincare group in any spacetime dimension*, SciPost Physics Lecture Notes (2021): 030.

[17] N.Y. Vilenkin, *Special functions and the theory of group representations,* American Mathematical Soc., 1978.

[18] D.P. Zhelobenko, A.I. Shtern, *Representations of Lie groups,* Nauka, Moscow, 1983 (in Russian).

[19] P. Schuster, N. Toro, *On the theory of continuous-spin particles: wavefunctions and soft-factor scattering amplitudes*, JHEP 09 (2013) 104, `arXiv:1302.1198 [hep-th]`.

[20] P. Schuster, N. Toro, *On the theory of continuous-spin particles: helicity correspondence in radiation and forces*, JHEP **09** (2013) 105, `arXiv:1302.1577 [hep-th]`.

[21] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, A. Rusnak, *Model of massless relativistic particle with continuous spin and its twistorial description*, JHEP **1807** (2018) 031, `arXiv:1805.09706 [hep-th]`.

[22] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, *Twistorial and space-time descriptions of massless infinite spin (super)particles and fields*, Nucl. Phys. B **945** (2019) 114660, `arXiv:1903.07947[hep-th]`.

[23] X. Bekaert, S. Cnockaert, C. Iazeolla, M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, Proceedings of the 1st Solvay Workshop on Higher Spin Gauge Theories, 12- 14 May 2004. Brussels, Belgium, Eds. R. Argurio, G. Barnich, G. Bonelli, M. Grigoriev, Int. Solvay Institutes, 2006, 132-197, `arXiv:hep-th/0503128`.

[24] D.M. Gitman, A.L. Shelepin, *Fields on the Poincare group: arbitrary spin description and relativistic wave equations*, Int. J. Theor. Phys. **40** (2001) 603-684, `arXiv:hep-th/0003146`.

[25] I.L. Buchbinder, D.M. Gitman, A.L. Shelepin, *Discrete symmetries as automorphisms of the proper Poincare group*, Int. J. Theor. Phys. **41** (2002) 753-790, `arXiv:hep-th/0010035`.

[26] X. Bekaert, J. Mourad, *The continuous spin limit of higher spin field equations*, JHEP **0601** (2006) 115, `arXiv:hep-th/0509092`.

[27] S. Fedoruk, V.G. Zima, *Covariant quantization of d = 4 Brink-Schwarz superparticle with Lorentz harmonics,* Theor. Math. Phys. **102** (1995) 305–322