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Quartic Horndeski-Cartan theories: a review on the stability of nonsingular cosmologies

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We briefly review recent progress in Horndeski theory (generalized Galileons) on a spacetime with torsion and the implications for the stability of the cosmological background against the graviton and the scalar mode. We highlight a critical modification of a well known NO-GO theorem that holds for nonsingular cosmological solutions in the torsionless theory. We show that these results trivially extend to a broader family of Lagrangians containing the contraction of the torsionful higher derivative terms - typical of Horndeski - with the fully antisymmetric tensor.

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Dedicatory

We dedicate this review to Professor Rubakov who sadly passed away on 19 of October 2022, while we were collaborating with S. Mironov to obtain the first results that we summarize below. MVV is indebted to Professor Rubakov for all his teachings in the roughly one year counting with his scientific advisory, specially his inspiring rigurosity in his daily approach to science.

1. Introduction

Horndeski theory is the most general modification of GR with a real scalar field in the action such that the equations of motion remain of second order [1–6]. Namely, on top of GR

$$\int \mathrm{d}^4x\,\sqrt{-g}\,R \tag{1}$$

consider four general functions G_2, G_3, G_4 and G_5 that depend on a scalar field ϕ and the term $X = -(1/2)\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, where $g_{\mu\nu}$ is the metric with signature $(-,+,+,+)$. Then, Horndeski theory can be written as [4]

$$\begin{aligned} S_H &= \int \mathrm{d}^4x\,\sqrt{-g}\,\Big(G_2 - G_3\,\nabla_\mu\nabla^\mu\phi + G_4(\phi,X)\,R + G_{4,X}\,\left((\nabla_\mu\nabla^\mu\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right) \\ &+ G_5\,G^{\mu\nu}\,\nabla_\mu\nabla_\nu\phi - \frac{1}{6}G_{5X}\,\Big((\nabla_\mu\nabla^\mu\phi)^3 - 3\,(\nabla_\mu\nabla^\mu\phi)\,(\nabla_\nu\nabla_\rho\phi)\,\nabla^\nu\nabla^\rho\phi \\ &+ 2\,(\nabla_\mu\nabla_\nu\phi)\,(\nabla^\nu\nabla^\rho\phi)\,\nabla^\mu\nabla_\rho\phi\Big)\Big)\,, \end{aligned} \tag{2}$$

where $G^{\mu\nu}$ is the Einstein tensor, ∇_μ is the covariant derivative computed with the Christoffel symbol $\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$ and $G_{4,X} = \partial G_4/\partial X$.

Let us notice the generality of (2), including the cosmological constant, minimally coupled scalars and non-minimally coupled, such as Brans-Dicke. Crucially, it is designed to avoid the Ostrogradsky ghost, and it allows - without obvious pathologies - to violate the Null Energy Condition (NEC), which is one of the assumptions for the Penrose - Hawking singularity theorems to hold (See for instance [7]). However, after many attempts to build all time stable, nonsingular cosmologies [8–15], it was finally established in the form of a No-Go theorem that under very general assumptions, gradient instabilities or ghosts will inevitably arise at some time in the entire evolution of the universe [9–11].

Besides a handful of special assumptions to avoid the instabilities in Horndeski theory such as "asymptotically strong gravity" or very specific models [11, 15–17], the alternative of Horndeski theory on spacetimes with torsion has been recently analyzed in [18–33]. In particular, it has been shown that in Horndeski-Cartan gravity (considering torsion in the second order, metric formalism), a similar No-Go theorem also holds (in up to the quartic case[1]) [19]: namely, the all-time *sub/luminality*, stability and nonsingularity of an spatially flat FLRW cosmology are mutually inconsistent, up to a few special cases.

That now there is a further characteristic incompatible with the stable and nonsingular cosmology, namely, *sub/ luminality*, suggests that in further generalizations of the theory the No-Go may

[1]We consider only the terms in (2) with G_2, G_3, G_4. We denote these theories as "up to quartic".

break altogether. Indeed, other violations of the No-Go have been recently reported in teleparallel Horndeski gravity [20], hence supporting the hope that a spacetime with torsion may help to cure the instability without other pathologies.

This short review is dedicated to Horndeski-Cartan gravity. In section 2 we spell out the theory and introduce notation for torsion. In section 3 we introduce notation for the perturbative expansion about a spatially flat FLRW background. In section 4 we extend the class of Horndeski-Cartan theories and the results of [18, 19] to include quadratic terms in second (torsionful) derivatives of the scalar with antisymmetric contractions. In section 5 we review a possible classification of these theories according to the dynamics of the scalar mode. In section 6 we review the No-Go argument for all time stable, sub/luminal, nonsingular cosmologies. In section 7 we give the conclusions.

2. Up to quartic Horndeski-Cartan Lagrangians

In a spacetime with torsion multiple Horndeski Lagrangians are possible according to different contractions of the Lorentz indices. These possible contractions can be collected in a two-parameter family of Horndeski-Cartan Lagrangians, if one stays, for simplicity, in up to quartic case (namely, no G_5). All these new possible contractions boil down to the fact that two torsionful covariant derivatives ($\tilde{\nabla}$) acting on the Horndeski scalar do not commute.

Namely, writing the metric compatible derivative $\tilde{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}^\nu_{\mu\lambda} V^\lambda$ on any vector V^μ, where $\tilde{\Gamma}^\rho_{\mu\nu} \neq \tilde{\Gamma}^\rho_{\nu\mu}$, we have

$$\left[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu\right] \phi \neq 0\,, \tag{3}$$

and thus, we can add the following terms proportional to $G_{4,X}$ in the Horndeski action (2) (without G_5), by considering all possible contractions with the metric [18] and the fully antisymmetric tensor of the terms of the form $G_{4,X}\,(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi)^2$,

$$G_{4,X}\left(\left(\tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi\right)^2 + c\,\left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \tilde{\nabla}^\mu \tilde{\nabla}^\nu \phi + s\,\left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \tilde{\nabla}^\nu \tilde{\nabla}^\mu \phi + b\,\epsilon^{\mu\nu\rho\sigma}\left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \tilde{\nabla}_\rho \tilde{\nabla}_\sigma \phi\right), \tag{4}$$

with $c + s = -1$. The later condition leaves only two free parameters, say c and b. It guarantees that the terms in (4) reduce to the standard terms in (2) when we assume a torsionless, Christoffel connection. It can be readily verified that none of these terms introduces higher than two partial derivatives in any of the equations of motion, hence, they do not introduce Ostrogradsky ghosts, as in the torsionless Horndeski theory.

All in all, we can write the two-parameter family of up to quartic Horndeski-Cartan theories as,

$$\begin{aligned}\mathcal{S} \;=\; &\int \mathrm{d}^4x\,\sqrt{-g}\,\Big(G_2 - G_3\,\tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi + G_4(\phi, X)\tilde{R} + G_{4,X}\Big(\left(\tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi\right)^2 \\ &- \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \tilde{\nabla}^\nu \tilde{\nabla}^\mu \phi - c\,\left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right)\left[\tilde{\nabla}^\mu, \tilde{\nabla}^\nu\right]\phi + b\,\epsilon^{\mu\nu\rho\sigma}\left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \tilde{\nabla}_\rho \tilde{\nabla}_\sigma \phi\Big)\Big).\end{aligned} \tag{5}$$

The family of theories with general parameter c, but without b was already analyzed in [18, 19]. In this review we also show the triviality of the extension of previous results [18, 19] to include a general parameter b.

To write the action with explicit torsion, we express the torsionful connection in terms of the usual GR Christoffel symbol as $\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - K^\rho{}_{\mu\nu}$. $K^\rho{}_{\mu\nu}$ is named contortion tensor, which can be written in terms of the torsion tensor as $T^{\rho}{}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} - \tilde{\Gamma}^{\rho}_{\nu\mu}$. In this review we consider three fundamental fields: the metric, scalar and *contortion*. With our convention of torsionful covariant derivatives K is antisymmetric in the first and third indices, $K_{\mu\nu\sigma} = -K_{\sigma\nu\mu}$, such that it has 24 independent components [2].

3. Notation for the linearized analysis about the FLRW background

As a first approximation to the dynamics of Horndeski-Cartan theories a linearized analysis about a spatially flat FLRW background was performed in [18].

In brief, the decomposition of the perturbations into irreducible components under the small rotation group reads, for the perturbed metric

$$\mathrm{d}s^2 = \left(\eta_{\mu\nu} + \delta g_{\mu\nu}\right) \mathrm{d}x^\mu \, \mathrm{d}x^\nu \tag{6}$$

where

$$\eta_{\mu\nu} \mathrm{d}x^\mu \, \mathrm{d}x^\nu = a^2(\eta) \left(-\mathrm{d}\eta^2 + \delta_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j\right) \tag{7}$$

is a spatially flat FLRW background metric, η is conformal time, and we denote spatial indices with latin letters such as $i = 1, 2, 3$ and space-time indices with greek letters, such as $\mu = 0, 1, 2, 3$. The metric perturbation is written as

$$\begin{aligned} \delta g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \;\; &= \;\; a^2(\eta)\Big(- 2\,\alpha \, \mathrm{d}\eta^2 + 2\left(\partial_i B + S_i\right) \mathrm{d}\eta \, \mathrm{d}x^i \\ &+ \;\; \left(-2\,\psi\,\delta_{ij} + 2\,\partial_i\partial_j E + \partial_i F_j + \partial_j F_i + 2\,h_{ij}\right) \mathrm{d}x^i \, \mathrm{d}x^j\Big)\,, \end{aligned} \tag{8}$$

with α, B, ψ, E scalar perturbations, S_i, F_i transverse vector perturbations, and h_{ij}, a symmetric, traceless and transverse tensor perturbation.

The contortion perturbation, can be decomposed as eight scalars denoted as $C^{(n)}$ with $n = 1, \ldots, 8$, six (two-component) transverse vectors denoted as $V_i^{(m)}$ with $m = 1, \ldots, 6$ and two (two-component) traceless, symmetric, transverse tensors $T_{ij}^{(1)}$, $T_{ij}^{(2)}$. In the scalar sector

$$\begin{aligned} \delta K_{i00}^{\text{scalar}} &= \partial_i C^{(1)} \\ \delta K_{ij0}^{\text{scalar}} &= \partial_i\partial_j C^{(2)} + \delta_{ij} C^{(3)} + \epsilon_{ijk}\partial_k C^{(4)} \\ \delta K_{i0k}^{\text{scalar}} &= \epsilon_{ikj}\partial_j C^{(5)} \\ \delta K_{ijk}^{\text{scalar}} &= \left(\delta_{ij}\partial_k - \delta_{kj}\partial_i\right) C^{(6)} + \epsilon_{ikl}\partial_l\partial_j C^{(7)} + \left(\epsilon_{ijl}\partial_l\partial_k - \epsilon_{kjl}\partial_l\partial_i\right) C^{(8)}\,, \end{aligned} \tag{9}$$

in the vector sector

$$\begin{aligned} \delta K_{i00}^{\text{vector}} &= V_i^{(1)} \\ \delta K_{ij0}^{\text{vector}} &= \partial_i V_j^{(2)} + \partial_j V_i^{(3)} \\ \delta K_{i0k}^{\text{vector}} &= \partial_i V_k^{(4)} - \partial_k V_i^{(4)} \\ \delta K_{ijk}^{\text{vector}} &= \delta_{ij} V_k^{(5)} - \delta_{kj} V_i^{(5)} + \partial_j\partial_i V_k^{(6)} - \partial_j\partial_k V_i^{(6)}\,, \end{aligned} \tag{10}$$

[2]This amounts to introduce torsion in the second order formalism, because the equations for the metric remain of second order. Other approaches in the context of Horndeski have been recently analyzed in [20–33]

and in the tensor sector

$$\begin{aligned} \delta K^{\text{tensor}}_{ij0} &= T^{(1)}_{ij} \\ \delta K^{\text{tensor}}_{ijk} &= \partial_i T^{(2)}_{jk} - \partial_k T^{(2)}_{ji}\,, \end{aligned} \tag{11}$$

All in all, the components of contortion perturbation are

$$\delta K_{i\mu\nu} = \delta K^{\text{scalar}}_{i\mu\nu} + \delta K^{\text{vector}}_{i\mu\nu} + \delta K^{\text{tensor}}_{i\mu\nu}\,. \tag{12}$$

On the other hand, the non-vanishing components of the background contortion tensor on the FLRW background are

$$\begin{aligned} {}^0K_{0jk} &= x(\eta)\delta_{jk} \\ {}^0K_{ijk} &= y(\eta)\epsilon_{ijk}\,, \end{aligned} \tag{13}$$

such that the perturbed contortion with all indices down is

$$K_{\mu\nu\sigma} = {}^0K_{\mu\nu\sigma} + \delta K_{\mu\nu\sigma} \tag{14}$$

Finally, the perturbed scalar field $\phi(x)$ is written as $\phi(\eta) + \Pi$ where Π is a spacetime dependent scalar field perturbation and $\phi(\eta)$ is the background scalar field. $\phi(\eta)$ can be distinguished from the perturbed scalar $\phi(x)$ by the context.

It is important to notice that the equations of motion for the background fields x, y, ϕ, a fix $y = 0$ as the only nontrivial background solution. They also fix the remaining torsion background x in terms of ϕ and a.

4. Triviality of the b term in the Horndeski-Cartan theories about the spatially flat FLRW background

The theories considered in [18, 19] correspond to the action (5) without parameter b, which corresponds to all terms of the form $G_{4,X}\,(\tilde\nabla_\mu\tilde\nabla_\nu\phi)^2$ contracted with the metric. Below we extend the results of [18, 19] to all c and b, namely to all terms of the form $G_{4,X}\,(\tilde\nabla_\mu\tilde\nabla_\nu\phi)^2$ contracted with the metric and the fully antisymmetric tensor, by proving the latter are trivial on our cosmological background. Let us note, however, that in principle one could expect nontrivial modifications to the dynamics of perturbations (12) by the latter terms, because $\tilde\nabla_\mu\tilde\nabla_\nu\phi$ has an antisymmetric part.

It suffices to show that the linearized dynamics of (5) does not depend on the parameter b. This is easy to see for the tensor and scalar sector: let us consider the last term of the action (5) $\int \mathrm{d}^4x\,\sqrt{-g}\,b\,G_{4,X}\,\epsilon^{\mu\nu\rho\sigma}\,(\tilde\nabla_\mu\tilde\nabla_\nu\phi)\,\tilde\nabla_\rho\tilde\nabla_\sigma\phi$. Writing contortion explicitly,

$$\int \mathrm{d}^4x\,\sqrt{-g}\,b\,G_{4,X}\,\epsilon^{\mu\nu\rho\sigma}\,K_{\lambda'\mu\nu}\,K_{\gamma'\rho\sigma}\,g^{\lambda'\lambda}\,g^{\gamma'\gamma}\,(\tilde\nabla_\lambda\phi)\tilde\nabla_\gamma\phi\,. \tag{15}$$

- For the tensor sector, since the scalar perturbation Π does not mix with the tensor perturbations the Lorentz indices λ and γ must be 0. Now, necessarily $g^{\lambda'0}$, $g^{\gamma'0}$ need to be the background component η^{00} in all perturbative expressions of (15), because otherwise, if they were to contribute with a perturbation δg^{00} or δg^{i0} in a quadratic expansion, they would be either α,

$\partial^i B$ or S^i, which do not mix with the symmetric, transverse and traceless tensor perturbations. Thus, (15) is proportional to $K_{0\mu\nu}\, K_{0\rho\sigma}$ which, according to equations (13) and (11) for either backgrounds or perturbations, are symmetric in μ, ν and ρ, σ, and contracted with the antisymmetric tensor vanishes all b (15) contribution to the tensor sector.

- For the scalar sector let us consider the unitary gauge in which Π and E are set to zero. Then, again the Lorentz indices λ and γ are 0 in all contributions to the quadratic expansion of (15). Now, *at least one of the contortion fields in (15) must be a perturbation in the quadratic expression $O(\delta^2)$ of (15)*: Indeed, if we assume that both contortion components are backgrounds, close inspection shows that since the only nonvanishing torsion background is symmetric $-{}^0K_{kj0} = {}^0K_{0jk} = x\,\delta_{jk}$ (recall that ${}^0K_{ijk} = \epsilon_{ijk}\, y = 0\,,\ {}^0K_{i00} = {}^0K_{i0k} = 0$) these two contortion backgrounds vanish contracted with the antisymmetric tensor (e.g. $\epsilon^{\mu 0\rho 0}\, ({}^0K_{\lambda'\mu 0})\, ({}^0K_{\gamma'\rho 0}) = 0$).

  If in the quadratic expression $O(\delta^2)$ both perturbations are of contortion, then necessarily $\lambda' = \gamma' = 0$. Namely, the term looks like $\epsilon^{\mu\nu\rho\sigma}\, \delta K^{scalar}_{0\mu\nu}\, \delta K^{scalar}_{0\rho\sigma}$. Now, in such a term only the first two lines in (9) can contribute, but both of the perturbations cannot be $\propto C^{(1)}$ because both indices ρ and μ cannot be simultaneously 0. Hence, from the second line in (9) antisymmetry indicates that at least one of the torsion perturbations must be $C^{(4)}$.

  On the other hand, if there is at most one perturbation of contortion, both λ' and γ' cannot be simultaneously different from 0. Indeed, without lost of generality, the only remaining type of term is $\epsilon^{\mu\nu\rho\sigma}\, (\delta K^{\rm scalar}_{\lambda'\mu\nu})\, ({}^0K_{\gamma'\rho\sigma})\, g^{\lambda' 0}\, g^{\gamma' 0}\, \dot\phi^2$, from which only the $\gamma' \neq 0$, $\lambda' = 0$ contribution is nonzero. Namely, with $\gamma' = i$ necessarily $\sigma = 0$ in the torsion background, giving $\epsilon^{\mu\nu j 0}\, (\delta K^{\rm scalar}_{0\mu\nu})\, ({}^0K_{ij0})\, \eta^{00}\, \delta g^{i0}\, \dot\phi^2$. Meaning again by the second line in (9) and antisymmetry that these type of terms are also proportional to $C^{(4)}$. All in all, in the scalar sector the quadratic expression $O(\delta^2)$ of (15) is proportional to $C^{(4)}$.

  Now, as described in detail in [19], the Lagrange multiplier $C^{(5)}$ imposes $C^{(4)} \equiv 0$ as a constraint - an equation which is not modified by the b terms (15) by the same argument above (namely, there is no $b\, C^{(5)}$ type term that could modify the constraint equation)-, hence vanishing all b terms (15) contribution to the scalar sector.

- The vector sector is not our main interest in this note. However, a direct computation shows that the b contribution vanishes on-shell by using the constraints $V_i^{(2)} = V_i^{(3)}$ (as described in detail in [18]), imposed by the Lagrange multipliers $V_i^{(4)}$.

In conclusion, all results in [18, 19] follow without modification irrespective of the b term.

5. Classification of the family of Horndeski-Cartan theories

The action (5) can be brought to the following form on the spatially flat FLRW background (See [18] for a detailed derivation)

$$
\begin{aligned}
S \;=\; & \int \mathrm{d}\eta\, \mathrm{d}^3x\, a^4 \left[\frac{1}{2\,a^2} \left(\mathcal{G}_\tau \left(\dot h_{ij}\right)^2 - \mathcal{F}_\tau (\partial_k\, h_{ij})^2 \right) \right. \\
& \left. +\ \frac{1}{a^2} \left(\dot\psi \left(\mathcal{G}_{S\mathrm{I}} - c\, \frac{1}{a^2}\, \mathcal{G}_{S\mathrm{II}}\, \partial_i \partial_i \right) \dot\psi - \mathcal{F}_S (\partial_i \psi)^2 \right) \right], \qquad (16)
\end{aligned}
$$

Table 1: Summary of tensor and scalar modes classified according to the parameter c of the theory. This holds for all b.
† A No-Go theorem holds in this case for all-time stable, non-singular and subluminal cosmology.
*This is a result that holds only in high momentum and provided the assumption of a *healthy* graviton, namely, a stable, non ghost and subluminal graviton.
** The no ghost, stability and subluminality conditions are satisfied if $G_4 > -2\,X\,G_{4,X} > 0$.

	$c < 0$	$c = 0^{\dagger}$	$0 < c \leq 2$	$c > 2$
Scalar mode	Non wave-like dispersion relation. *Not a ghost**.	*Wave-like* dispersion relation†	Non wave-like dispersion relation. *A ghost**.	Non wave-like dispersion relation.
Graviton	Is massless**	Is massless†,**	Is massless**	
Vector sector	Non dynamical.			

where h_{ij} is the graviton and ψ is a spacetime dependent scalar perturbation. $\mathcal{G}_\tau$, $\mathcal{F}_\tau$, $\mathcal{G}_{SI}$, $\mathcal{G}_{SII}$, $\mathcal{F}_S$ are a combination of the G_A lagrangian functions and their derivatives expressed in terms of $a(\eta)$, $\phi(\eta)$ (The relevant expressions are given in the discussions below. For more details see [19]).

The key take aways from (16) are:

i) There is one tensor perturbation, the massless graviton. Its dynamics is the same for all c.

ii) The vector sector is non dynamical

iii) There is one scalar perturbation. Its dynamics *depends* on the parameter c of the theory. In fact, the scalar mode of these theories has different stability, ghost/ no-ghost and super/ sub/ luminality properties depending on the parameter c of the theory (See table 1).

iv) Importantly, the theory with parameter $c = 0$ is special because it is the only one in which the scalar mode propagates with a relativistic dispersion relation, as can be directly seen from the action (16). **Thus, the analysis below is restricted to the theory with** $c = 0$.

Let us notice that the first and third aspects are somewhat unexpected, because there is a kinetic mixing of contortion with the scalar field in the action (5), which would naively signal potentially more DOFs than in the torsionless Horndeski theory. This issue may suggest that symmetries or accidental symmetries are at play (see a related discussion in [18]).

6. The search for all-time stable, nonsingular and sub/ luminal cosmologies: a NO-GO for up-to quartic Horndeski-Cartan theories

Avoiding the singularity theorems of Penrose and Hawking has been one of the main motivations to explore Horndeski theory. The latter can support nonsingular solutions because it can violate the Null Energy Condition (NEC) (See [7] for a review).

In particular, looking for a cosmological solution in Horndeski-Cartan theory without initial singularity

- **I)** we require that there is a lower bound on the scale factor $a(\eta) > b_1 > 0$, with the bounce happening when $\dot{H} > 0$ holds.

Besides, we also aim for theories where the small perturbations about the cosmological background - namely, the graviton h_{ij} and the scalar mode ψ in (16) - do not suffer gradient instabilities and remain sub/ luminal. Let us explicitly state these conditions: to start, let us consider only the theory with parameter $c = 0$, because only this class has a scalar mode with a relativistic dispersion relation.

Now, let us consider the speed squared of the graviton c_g^2 and the scalar c_s^2

$$c_g^2 = \frac{\mathcal{F}_\tau}{\mathcal{G}_\tau}\,, \quad c_s^2 = \frac{\mathcal{F}_S}{\mathcal{G}_{SI}}\,, \tag{17}$$

where

$$\mathcal{G}_\tau = 2\frac{G_4^2}{G_4 + 2\,X\,G_{4,X}}\,, \quad \mathcal{F}_\tau = 2\,G_4\,, \quad \mathcal{G}_{SI} = 3\,\mathcal{G}_\tau + \frac{\mathcal{G}_\tau^2\,\Sigma}{\Theta^2}\,, \quad \mathcal{F}_S = \frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{a\,\mathcal{G}_\tau\,T}{\Theta}\right) - \mathcal{F}_\tau\,. \tag{18}$$

$$\Theta = \frac{4\,\mathcal{G}_\tau^2\,\theta}{a\,\mathcal{F}_\tau^4}\,, \quad \Sigma = \frac{2\,\mathcal{G}_\tau^3\,\sigma}{a\,\mathcal{F}_\tau^6}\,, \quad T = \mathcal{F}_\tau\,(c_g^2 - 2)\,, \tag{19}$$

where θ and σ are complicated expressions that depend on $\phi(\eta)$, $a(\eta)$ background fields, whose explicit form is not essential for the discussion below (See [19] for the explicit expressions).

With these definitions, the conditions to have no ghosts are

- **II.a)** $\mathcal{G}_\tau > 0\,, \mathcal{G}_{SI} > 0\,,$

so that the kinetic terms contribute with a positive energy. To avoid gradient instabilities we also require

- **II.b)** $\mathcal{F}_\tau > 0\,, \mathcal{F}_S > 0\,,$

which, let us note, is clearly necessary to have real speeds of propagation provided **II.a**.

It turns out that it is very difficult to find solutions in Horndeski theory that satisfy **I), II.a/ b)**, even if they violate the NEC. This has been known as a No-Go theorem, with very few ways to avoid it [8–15]. In fact, the only ways to satisfy **I), II.a/ b)** require either to restrict to a specific class of theories defined by the equation $\Theta \equiv 0$ [17] or, if one remains in the case of general theories, one needs unconventional assumptions, such as $\mathcal{F}_\tau \to 0$ asymptotically [11, 16]. The latter means asymptotically strong gravity, although recent advances have shown that there is in fact no issue with strong coupling [34].

If one wishes, however, to avoid these assumptions one could for instance explore Teleparallel Horndeski [20, 21] or Beyond Horndeski or Horndeski-Cartan theories, as we will show below. In the latter, there is a new interesting link between nonsingular, stable and no-ghosty solutions with the speed of the graviton. In brief, it was shown in [18] that

it is not possible to obtain a nonsingular FLRW cosmology that is always free of gradient instabilities against the scalar perturbation and an always sub/ luminal graviton. However, it was also shown that a potentially, arbitrarily short violation of the sub/ luminality of the graviton at any time can allow nonsingular and stable (first-order) solutions.

Let us review the argument. It follows the same line of reasoning as in [9–11]: to state it let us also require all-time sub/ luminal modes

- **III)** $c_g^2 \leq 1\,, c_s^2 \leq 1\,,$

let us avoid the *apparent* strong gravity condition by requiring

- **IV)** a lower bound $\mathcal{F}_\tau(\eta) > b_2 > 0$ as $\eta \to \pm\infty$,

and finally, let us require that

- **V)** Θ vanishes at most a finite amount of times.

By **(V)** the theory **is not defined by the equation** $\Theta \equiv 0$, and thus we remain in the generic Horndeski-Cartan theories only with the condition $c = 0$.

The argument: Let us notice that

A) by **(I) - (III)**

$$N =: \frac{a\,\mathcal{G}_\tau\,\mathcal{F}_\tau\,(c_g^2 - 2)}{\Theta} \neq 0\,,$$

because Θ is a regular (finite) function of $H,\ \phi$,

B) by **(I), (IV)** and the third inequality in **(II)**, N is monotonous increasing

$$\frac{d\,N}{d\,\eta} > a^2\,\mathcal{F}_\tau > 0,$$

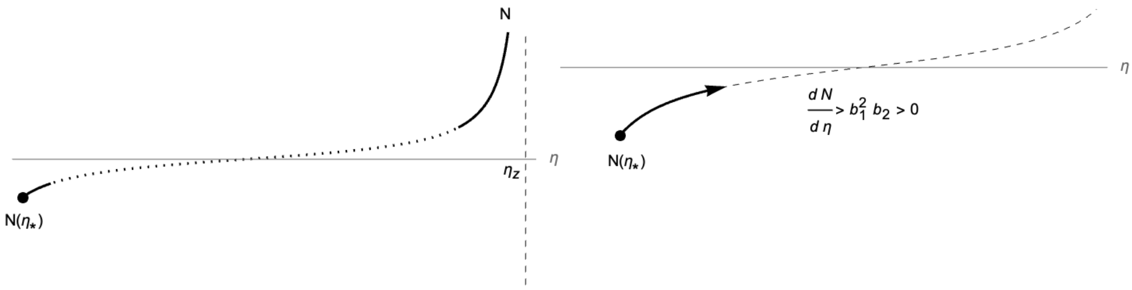
and in fact, there is a lower, positive bound on its slope $\frac{d\,N}{d\,\eta} > b_1^2\,b_2 > 0$ as $\eta \to \pm\infty$. As a consequence, we know the behavior of N around the zeros of Θ: denoting with η_z any zero of Θ, then $N(\eta) \to \infty$ as $\eta \to \eta_z^-$ (η approaches η_z by the *left*) and $N(\eta) \to -\infty$ as $\eta \to \eta_z^+$.

One can somewhat easily see that conclusions (A) and (B) are contradictory. Let us see, if we take a value of $N < 0$ at some time η_* and we move forward in time, two things can happen: (i) either one finds an η_z (a zero of Θ) where $\eta_* < \eta_z$, or (ii) one can move towards $\eta \to +\infty$ without finding an η_z, namely with N continuous in the interval $(\eta_*, +\infty)$. Now, in the former possibility (i), we know by (B) that $N(\eta)$ is positive as $\eta \to \eta_z^-$, thus, provided our starting point $N(\eta_*) < 0$, it means that $N(\eta)$ has already vanished before reaching η_z, namely, for some η with $\eta_* < \eta < \eta_z$ (because N is continuous in that interval), hence violating (A). This situation is depicted in Figure 1a.

In the latter possibility (ii), we also know by (B) that N increases as time grows and that N cannot have an horizontal asymptote as $\eta \to +\infty$ (recall $\frac{d\,N}{d\,\eta} > b_1^2\,b_2 > 0$ as $\eta \to \pm\infty$), hence provided our starting point $N(\eta_*) < 0$, and that N is continuous in the time interval (η_*, ∞) (without η_z), N will eventually cross zero, again violating (A). This situation is depicted in Figure 1b.

Similarly, if we take a value of $N > 0$ at some time η_* and we move backwards in time, we find by an analogous argument that (A) does not hold at all times.

In fact, **(III)** can be relaxed to $c_g^2 < 2$ and the argument still holds. But, an almost everywhere subluminal graviton turning to $c_g^2 > 2$ during an arbitrarily short interval at any time, is enough to avoid this No-Go argument. Indeed, an explicit example was built in [19].



(a) With $\eta_* < \eta_z$.

(b) No η_z in the time interval (η_*, ∞).

Figure 1: Behavior of $N(\eta)$ with one and no zeros of Θ (denoted as η_z) for initial value $N(\eta_*) < 0$.

7. Conclusions

We summarized recent advances in Horndeski-Cartan gravity (Galileons with torsion in the second order formalism) [18, 19]. We showed that these results trivially extend to include quadratic terms in second (torsionful) derivatives of the scalar with antisymmetric contractions.

In particular, a possible classification of these theories according to the dynamics of the scalar mode, presented in [18], and the No-Go argument for all time stable, sub/luminal, nonsingular cosmologies [19] hold even after including the former type of terms. Namely, (up to quartic) Horndeski-Cartan gravity can support all-time linearly stable nonsingular solutions if there exists at an arbitrary time a superluminal phase for the graviton and by at least an amount $c_g \geq \sqrt{2}\,c$. We argued that this modification of the usual No-Go, as well as other recent indications of its violation in teleparallel Horndeski [20] may suggest that further generalizations may lead to all time stability without any classical pathologies. Indeed, whether a short-lived superluminal phase for the graviton is a serious pathology either for the classical theory (*e.g.* closed timelike curves) and for potential UV completions remains to be investigated, specially provided that the perturbations propagate in a background that spontaneously breaks Lorentz invariance [35–37].

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