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Real-time dynamics of the Schwinger model via variational quantum algorithms

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In this article we investigate the real-time dynamics in the $(1+1)$-dimensional $U(1)$ gauge theory called the Schwinger model by using variational quantum algorithms. Specifically, we first prepare the ground state of the Hamiltonian without external electric field via the variational quantum eigensolver, and then perform real-time evolution under the Hamiltonian in the presence of the external field using the variational quantum simulation method. The same ansatz is used for both algorithms which reduces the overall depth of the quantum circuit. We test our protocol by using a noiseless statevector simulator and confirm that results from the quantum algorithms are consistent with those obtained by exact diagonalization. This article summarizes our previous work [1].

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1. Introduction

In recent years, there has been an increasing interest in digital quantum simulation in the context of high-energy physics since the seminal work by Jordan, Lee, and Preskill [2]. In particular, by using quantum simulation, we can investigate real-time dynamics, models with a topological term, and those with a finite chemical potential, which cannot be treated naively by the conventional Monte-Carlo method. As a prototype of $(3+1)$-dimensional non-Abelian gauge theories (e.g., QCD), the $(1+1)$-dimensional $U(1)$ gauge theory known as the Schwinger model [3] is frequently used for benchmarking quantum simulation algorithms (see e.g. [4–7]). The Schwinger model is simple enough to implement but still exhibits non-trivial phenomena, such as confinement and charge screening. We can also include a topological term in the Lagrangian which cannot be studied via the Monte-Carlo method naively.

In this work, we focus on the Schwinger model with a topological term to study the effects of an external electric field. Specifically, we first prepare the ground state in the absence of the external field, and then suddenly turn on the external field (the quantum quench) to perform real-time evolution under the Hamiltonian in the presence of the external field. Two variational quantum algorithms are used for this purpose: the variational quantum eigensover (VQE) for ground state preparation, and the variational quantum simulation (VQS) method proposed by Li and Benjamin [8] for real-time evolution.

This article is organized as follows. In Section 2, we introduce the Schwinger model and define the physical states and observables of interest. Section 3 explains the variational quantum algorithms used for ground state preparation and real-time evolution. The results from our numerical simulation are presented in Section 4. Finally, we summarize our work and discuss possible future directions in Section 5.

2. Model

The continuum Lagrangian of the Schwinger model with a topological term is given by

$$\mathcal{L}_{\text{con}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\bar{\psi}\gamma^{\mu}(\partial_{\mu} + \mathrm{i}gA_{\mu} - m)\psi + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}\,. \tag{1}$$

where the third term is a topological term. We can obtain the lattice Hamiltonian with lattice spacing a by using the Kogut-Susskind formalism [9] as follows.

$$H = J\sum_{n=0}^{N-2}(L_n + q)^2 - \mathrm{i}w\sum_{n=0}^{N-2}\left(\chi_n^{\dagger}U_n\chi_{n+1} - \chi_{n+1}^{\dagger}U_n^{\dagger}\chi_n\right) + m\sum_{n=0}^{N-1}(-1)^n\chi_n^{\dagger}\chi_n\,, \tag{2}$$

where $w = 1/(2a)$, $J = g^2a/2$ and $q = \theta/(2\pi)$. To have a nonzero value q corresponds to the presence of the external electric field. Besides, χ_n is a staggered fermion, and L_n, U_n are link variables corresponding to the gauge degrees of freedom. These lattice variables satisfy the following commutation relations.

$$\{\chi_n^{\dagger}, \chi_m\} = \delta_{mn}\,, \quad \{\chi_n, \chi_m\} = 0\,, \quad [U_n, L_m] = \delta_{mn}U_n\,.$$

Furthermore, the physical states must satisfy Gauss's law

$$L_n - L_{n-1} = \chi_n^\dagger \chi_n - \frac{1-(-1)^n}{2}\,, \tag{3}$$

to guarantee gauge symmetry. By fixing $L_{-1} = 0, U_n = 1$ and solving these constraints, we can fully eliminate the gauge degrees of freedom from the model. Then the resulting fermionic Hamiltonian can be easily mapped to a spin system via the Jordan-Wigner transformation [10] given as

$$H = J\sum_{n=0}^{N-2}\left[\sum_{i=0}^{n}\frac{Z_i+(-1)^i}{2}+q\right]^2 + \frac{w}{2}\sum_{n=0}^{N-2}\left[X_nX_{n+1}+Y_nY_{n+1}\right] + \frac{m}{2}\sum_{n=0}^{N-1}(-1)^n Z_n\,, \tag{4}$$

up to an irrelevant constant.

Next, let us specify the quantum states that we will simulate. We first prepare the ground state $|\Psi_{\mathrm{GS}}(q=0)\rangle$ in the absence of the external field. Then the external field q is turned on suddenly at $t = 0$. We perform real-time evolution under the Hamiltonian with a nonzero q to have $|\Psi(t)\rangle = e^{-\mathrm{i}H_{q\neq 0}t}\,|\Psi_{\mathrm{GS}}(q=0)\rangle$ at each time step.

Finally, we evaluate the following three observables using the obtained states $|\Psi(t)\rangle$. The first one is the electric field whose spin description is given by

$$\mathcal{E}(t) = \frac{g}{2N}\sum_{n=0}^{N-1}\sum_{k=0}^{n}\langle Z_k\rangle_t + \frac{g}{2N}\sum_{n=0}^{N-1}\sum_{k=0}^{n}(-)^k + gq\,, \tag{5}$$

where $\langle\bullet\rangle_t := \langle\Psi(t)|\bullet|\Psi(t)\rangle$. The second observable is the chiral condensate written as

$$\Sigma(t) = \frac{ag}{N}\sum_{n=0}^{N-1}(-)^n\langle Z_n\rangle_t\,, \tag{6}$$

in the spin representation. This gives an approximate metric for the particle density when the value of mass is not too large. The final observable we will consider is the $U(1)$ charge defined by

$$Q = \frac{1}{N}\sum_{n=0}^{N-1}\langle Z_n\rangle_t\,, \tag{7}$$

which must be conserved under the evolution under the exact Hamiltonian (4).

3. Method

This section explains the two variational algorithms we will use for the simulation. Our protocol is divided into two parts. The first part is state preparation, which we implement using VQE. The second part is real-time evolution under the Hamiltonian with a nonzero q. We perform this simulation via the VQS method [8, 11].

First, we introduce a parameterized state ansatz $|\psi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\rangle$ which we will utilize for both the state preparation and real-time evolution. Specifically, we repeat the same structure L times to construct the ansatz as

$$|\psi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\rangle = U_{L-1}\cdots U_0 V_{\mathrm{init}}\,|0\rangle\,. \tag{8}$$

Each layer is given by

$$V_{\text{init}} = \prod_{n:\text{even}} X_n \,, \tag{9}$$

$$\begin{aligned} U_l(\boldsymbol{\alpha}_l, \boldsymbol{\beta}_l, \boldsymbol{\gamma}_l) = &\prod_{n=0}^{N-1} u_n^{(Z)}(\gamma_{l,n}) \prod_{n:\text{odd}} u_n^{(ZZ)}(\beta_{l,n}) \prod_{n:\text{even}} u_n^{(ZZ)}(\beta_{l,n}) \\ &\times \prod_{n:\text{odd}} u_n^{(XY)}(\alpha_{l,n}) \prod_{n:\text{even}} u_n^{(XY)}(\alpha_{l,n}) \,, \end{aligned} \tag{10}$$

where

$$u_n^{(Z)}(\gamma_{l,n}) = \exp\left[\mathrm{i}\frac{\gamma_{l,n}}{2} Z_n\right] , \tag{11}$$

$$u_n^{(ZZ)}(\beta_{l,n}) = \exp\left[\mathrm{i}\frac{\beta_{l,n}}{2} Z_n Z_{n+1}\right] , \tag{12}$$

$$u_n^{(XY)}(\alpha_{l,n}) = \exp\left[\mathrm{i}\frac{\alpha_{l,n}}{2} \frac{X_n X_{n+1} + Y_n Y_{n+1}}{2}\right] . \tag{13}$$

This ansatz, called the Hamiltonian variational ansatz (HVA) [12–14], mimics the Suzuki-Trotter decomposition of real-time/adiabatic time evolution and is thus expected to be useful for both state preparation and real-time evolution. This decomposition also preserves $U(1)$ symmetry which has to be preserved under the exact time evolution via the Hamiltonian H. Below, we will denote the whole set of parameters by $\boldsymbol{\lambda} := (\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, \boldsymbol{\gamma}_0, \cdots, \boldsymbol{\alpha}_{L-1}, \boldsymbol{\beta}_{L-1}, \boldsymbol{\gamma}_{L-1})$.

The VQE is a hybrid algorithm, where a quantum circuit evaluates the expectation value of the target Hamiltonian (4) in the ansatz

$$C(\boldsymbol{\lambda}) = \langle \psi(\boldsymbol{\lambda}) | H_{q=0} | \psi(\boldsymbol{\lambda}) \rangle \,, \tag{14}$$

and then the classical optimizer tries to minimize the cost function $C(\boldsymbol{\lambda})$. We denote the optimized parameters as $\boldsymbol{\lambda}_{\text{opt}}$.

Next, we use the VQS method for the real-time evolution. The time evolution governed by Schrödinger's equation is translated into the evolution equation of the parameters via McLachlan's variational principle [15] given by

$$\sum_{i,j} M_{ij} \dot{\lambda}_j = V_j \,, \tag{15}$$

where

$$M_{ij} = 2\,\mathrm{Re}\left[A_{ij}\right] + 2C_i^{(0)} C_j^{(0)} \,, \tag{16}$$

$$V_i = 2\,\mathrm{Im}\left[C_i\right] + 2\mathrm{i}C_i^{(0)} \langle H \rangle_\psi \,, \tag{17}$$

with

$$A_{ij} = \frac{\partial \langle \psi |}{\partial \lambda_i} \frac{\partial |\psi\rangle}{\partial \lambda_j} \,, \quad C_i = \frac{\partial \langle \psi |}{\partial \lambda_i} H |\psi\rangle \,, \tag{18}$$

$$C_i^{(0)} = \frac{\partial \langle \psi |}{\partial \lambda_i} |\psi\rangle \,, \quad \langle H \rangle_\psi = \langle \psi | H | \psi \rangle \,. \tag{19}$$

Note that H is the Hamiltonian with a nonzero q for the quench dynamics. We evaluate coefficients M_{ij}, V_i by using a quantum circuit given in [8, 11] and update parameters according to Eq. (15) by using a classical computer. Since we use the same ansatz for VQE and VQS, we simply set $\boldsymbol{\lambda}(t = 0) = \boldsymbol{\lambda}_{\text{opt}}$ to implement the quench dynamics. This reduces the overall circuit depth significantly.

4. Results

In this section, we compare the results from the variational quantum algorithms with those from exact diagonalization (ED) to test our protocol. The quantum circuits are implemented by using a noiseless statevector simulator called Qulacs [16] and ED results are obtained from QuSpin [17].

4.1 State preparation via VQE

First, we present the results of the state preparation in the absence of the electric field q. We investigate the accuracy with $ag = m/g = 1$ fixed and the number of depths L increasing. A metric of accuracy is defined by $r(E) := (E_{\text{max}} - E_{\text{VQE}})/(E_{\text{max}} - E_{\text{min}})$, where $E_{\text{max/min}}$ are the highest/lowest eigenvalues of the Hamiltonian H obtained by ED. Fig 1 shows the metric of accuracy

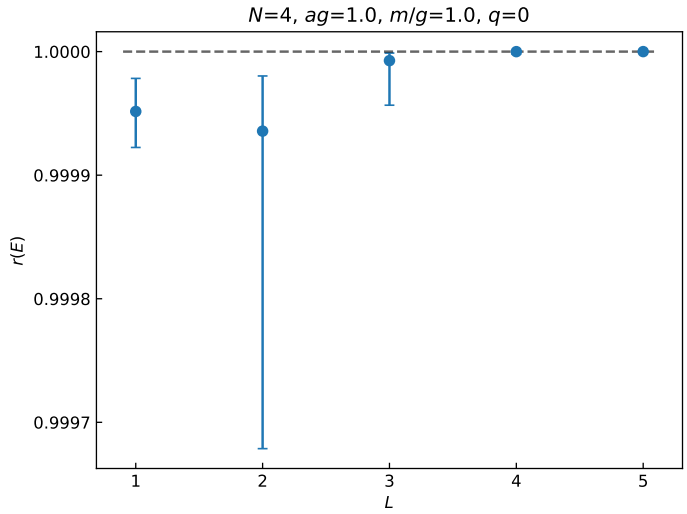

Figure 1: Ground state preparation via VQE: a metric of accuracy $r(E) := (E_{\text{max}} - E_{\text{VQE}})/(E_{\text{max}} - E_{\text{min}})$ for various L. Dots/error bars show the median and 25-75 percentiles of 20 samples.

for each layer L. We repeat experiments 20 times starting from different random initializations. The dots represent the median for those experiments and error bars show the 25-75 percentiles. This figure shows that one can obtain the ground state with high accuracy $r(E) \geq 0.99$ for all L and accuracy improves drastically for $L \geq 4$.

4.2 Quench dynamics via VQS

Next, we perform the real-time evolution via the VQS method in the presence of the electric field $q = 2$. We first evaluate the coefficients M_{ij} and V_i at each time step and obtain the time-

dependence of parameters $\boldsymbol{\lambda}(t)$. By using these parameters, we evaluate the dynamics of three physical observables explained in Section 3. First of all, we observe that the $U(1)$ charge is

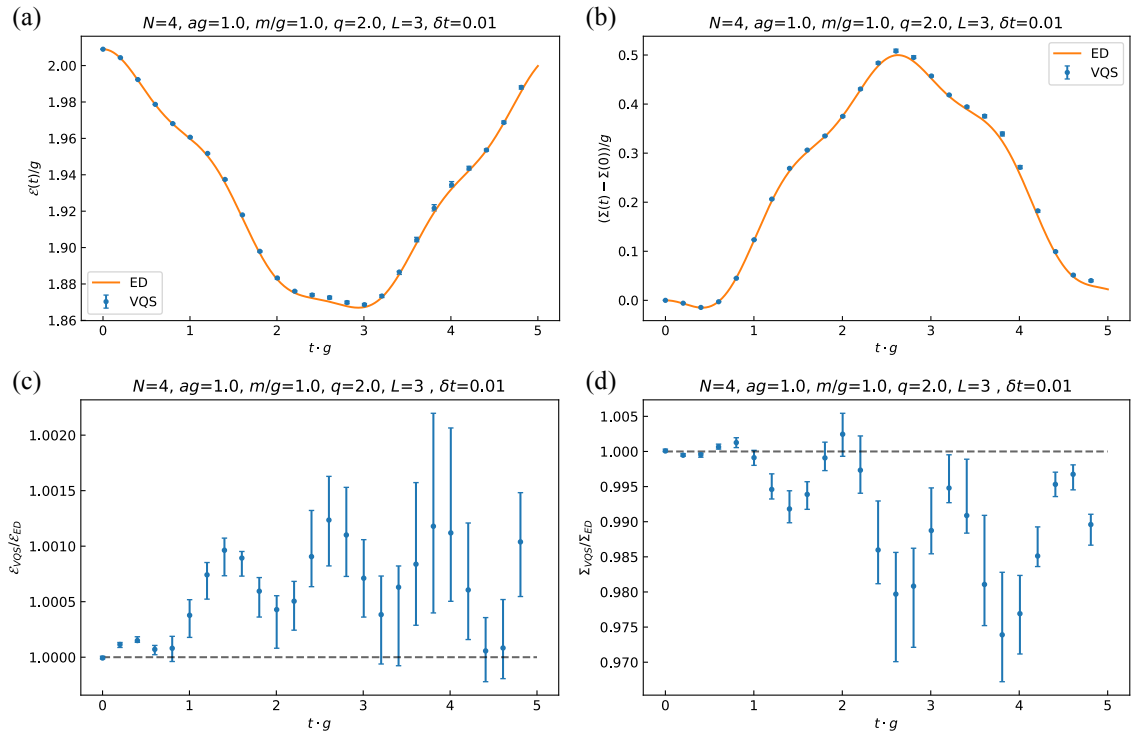

Figure 2: Real-time dynamics of physical observables for $N = 4, a \cdot g = 1.0, m/g = 1.0, q = 2.0$ with $L = 3$ and $\delta t = 0.01$. Dots/error bars show the median and 25-75 percentiles of 20 samples: (a) electric field, (b) chiral condensate, (c), (d) ratio between the values of observables obtained from ED and VQS

preserved under the VQS evolution, which is expected from the choice of our ansatz. Fig 2 shows the other two observables. One can see that the VQS results are consistent with those obtained from ED up to a few percent errors. The dependence of systematic errors on the expressivity of the ansatz (i.e., depth L) and the time increment $\delta t = T_{\text{max}}/N_{\text{step}}$, where T_{max} is the maximal time for the simulation and N_{step} is the number of steps, are also investigated in [1].

We can observe that for $t \cdot g \lesssim 3$ the magnitude of the electric field decreases while the value of the chiral condensation increases. A possible interpretation for this is that the electric field gives its energy to the fermions and fermions are pair-created as a result.

5. Summary and discussion

In this work, we investigated the quench dynamics in the Schwinger model by using the two variational quantum algorithms; the VQE and VQS. We checked our proposed protocol by comparing the results from VQE and VQS with those from ED and found that they are consistent. One can interpret these results as a fermion pair-creation due to the external electric field.

There are many possible future directions to be addressed. First of all, investigating the effects of a finite number of shots and quantum noise would be useful to understand how feasible it would be to implement this approach on real quantum devices. In particular, it would be important to

understand how required resources (e.g., the number of gates, the number of shots) to achieve a certain accuracy scale with the number of qubits and the lattice spacing a. It would be also interesting to study the systematic error coming from the choice of an ansatz. Finally, it would be important to consider an extension to higher-dimensional and/or non-Abelian gauge theories.

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