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The chiral condensate at large N using volume reduction

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We present a lattice calculation of the large-N limit of the QCD chiral condensate obtained from twisted volume-reduced models.

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1. Introduction

The study of the non-perturbative properties of SU(N) gauge theories in the large-N limit is an interesting theoretical issue, which has also several phenomenological implications for hadron physics, mainly related to the Witten–Veneziano mechanism. In particular, for this mechanism to hold it is necessary that the confined, chirally-broken phase characterizing the low-energy regime of $N = 3$ real-world QCD survives up to $N = \infty$ [1, 2] (see also [3]). As it is well known, the spontaneous breaking of chiral symmetry implies a non-zero value of the quark condensate $\langle\bar{\psi}\psi\rangle \equiv -\Sigma$. To calculate this quantity from first principles, lattice numerical simulations are a natural non-perturbative tool.

If on one hand the lattice computation of the chiral condensate has been extensively performed in $N = 3$ QCD adopting a variety of lattice discretizations and number of fermion species [4–11], on the other hand the large-N behavior of this observable has been much less investigated. As a matter of fact, the few lattice large-N studies which could be retrieved so far either do not present continuum-extrapolated results [12, 13], or just provide a preliminary study of the large-N limit [14].

This proceeding reports on the main findings of our recent paper [15] (see also [16]), where a reliable computation of the chiral condensate Σ of large-N QCD is performed from the low-lying spectrum of the Dirac operator, with controlled chiral and continuum limits. Perfectly agreeing results for Σ are obtained from the chiral behavior of the pion mass as a function of the quark mass.

Our computation is performed within the framework of the the Twisted Eguchi–Kawai (TEK) model [17, 18], and exploits large-N volume independence [19]. In a few words, our reduced model [17–20] allows to reach values of N of the order of $\sim O(100)$ by regularizing the theory on a lattice which is completely reduced to just a single point [21–33]. One of the main advantages of this approach with respect to standard ones [13, 14, 34–44] is that it permits to completely avoid the necessity of doing a large-N extrapolation from small values of N, as it allows to work directly in the large-N theory.

This proceeding is organized as follows: in Sec. 2 we briefly summarize our numerical setup; in Sec. 3 we present and discuss the main results of [15]; finally in Sec. 4 we draw our conclusions.

2. Numerical setup

2.1 Lattice discretization

Since in QCD with fixed number of fermion species quark loops are sub-leading in $1/N$ compared to gluon ones, the quenched approximation becomes exact in the large-N limit. For this reason, we here consider the pure-gauge TEK model, which involves $d = 4$ SU(N) matrices as dynamical degrees of freedom. The Wilson TEK action is:

$$S_{\mathrm{W}}^{(\mathrm{TEK})}[U] = -Nb \sum_{\nu\neq\mu} z_{\nu\mu} \mathrm{Tr}\left\{U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right\}. \quad (1)$$

Here $1/b$ is the bare 't Hooft coupling, U_μ are the SU(N) link variables, the number of colors is taken to be a perfect square $N = L^2$, and $z_{\nu\mu} = z_{\mu\nu}^* = \exp\left\{i\frac{2\pi k(N)}{\sqrt{N}}\right\}$ $(\nu > \mu)$ is the twist factor, with $k(N) \in \mathbb{N}$ co-prime with $\sqrt{N} \in \mathbb{N}$. This action can be thought of as the reduction on a single site of an ordinary Yang–Mills theory discretized on a torus with twisted periodic boundary conditions specified by $z_{\nu\mu}$, and ample evidence has been collected so far showing that this model correctly reproduces the behavior of the large-N SU(N) Yang–Mills theory [22, 24, 25, 28, 29, 31]. Concerning Monte Carlo updating algorithms, we employed the over-relaxation algorithm described in detail in Ref. [27].

Even if sea fermions are quenched, we are still free to consider one valence quark flavor. We choose to this end the Wilson disretization of $\not{D}$, which in the TEK model reads [28]:

$$D_{\mathrm{W}}^{(\mathrm{TEK})}[U] = \frac{1}{2\kappa} - \frac{1}{2}\sum_{\mu}\left[(\mathbb{I}+\gamma_\mu)\otimes \mathcal{W}_\mu[U] + (\mathbb{I}-\gamma_\mu)\otimes \mathcal{W}_\mu^\dagger\right], \tag{2}$$

with κ the usual Wilson hopping parameter, and $\mathcal{W}_\mu[U] \equiv U_\mu \otimes \Gamma_\mu^*$, with Γ_μ representing the so-called *twist eaters*, i.e., SU(N) matrices satisfying $\Gamma_\mu\Gamma_\nu = z_{\nu\mu}^*\Gamma_\nu\Gamma_\mu$.

2.2 The Giusti–Lüscher method

The well-known Banks–Casher relation connects the formation of a non-zero quark condensate to an accumulation of near-zero eigenvalues in the Dirac spectrum in the chiral limit, meaning that the Dirac eigenvalue spectral density develops a non-zero value in $\lambda = 0$ when $m \to 0$, with m the quark mass:

$$\frac{\Sigma}{\pi} = \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty} \rho(\lambda, m) = \pi\rho(0). \tag{3}$$

From the lattice perspective, however, it is more convenient to consider the mode number of the Dirac operator [here $(\not{D}+m)u_\lambda = (i\lambda+m)u_\lambda$]:

$$\langle \nu(M)\rangle \equiv \langle \#\, |i\lambda+m| \leq M\rangle \tag{4}$$

$$= V\int_{-\Lambda}^{\Lambda}\rho(\lambda,m)d\lambda, \qquad \Lambda^2 \equiv M^2 - m^2, \tag{5}$$

which is of course related to the spectral density via an integral relation. Therefore, the Banks–Casher relation can be recast into a linear behavior of the mode number $\langle\nu(M)\rangle$ for M close to the quark mass threshold as follows:

$$\langle\nu(M)\rangle = \frac{2}{\pi}V\Sigma\Lambda + o(\Lambda^2), \tag{6}$$

where the $\sim O(\Lambda^2)$ corrections can be shown to be sub-leading in $1/N$ [45, 46].

In Ref. [47], L. Giusti and M. Lüscher pointed out that the chiral condensate Σ could be easily computed from numerical lattice simulations from the calculation of the slope of the Dirac mode number close to the threshold $M = m$:

$$\Sigma^{(\mathrm{eff})}(m) = \frac{\pi}{2V}\sqrt{1-\left(\frac{m}{\overline{M}}\right)^2}\left[\frac{\partial\langle\nu(M)\rangle}{\partial M}\right]\Bigg|_{M=\overline{M}} \longleftarrow \text{slope of } \langle\nu(M)\rangle \text{ in } \overline{M}, \tag{7}$$

$$\text{with } \Sigma = \lim_{m\to 0}\Sigma^{(\mathrm{eff})}(m). \tag{8}$$

For TEK models, although defined on a single site, the obtained results should actually be thought of as obtained on a lattice with an effective volume $V = \ell^4 = (aL)^4 = a^4N^2$, which is the value that was used to numerically evaluate Eq. (7).

3. Results

All results presented in this section have been obtained for $N = 289$ and $k(N) = 5$ (with k the parameter appearing in the twist factor earlier defined). Moreover, since we expect:

$$\Sigma(N) = N\bar{\Sigma}[1 + O(1/N)], \tag{9}$$

in the following we will always consider the quantity Σ/N, which is expected to have a finite large-N limit. Finally, given that the chiral condensate, after renormalization, is a scheme- and scale-dependent quantity, we will renormalize this quantity in the customary $\overline{\text{MS}}$ scheme at the usual renormalization point $\mu = 2$ GeV.

3.1 Best fit of the mode number

Below we schematically resume how we practically implemented the Giusti–Lüscher method earlier outlined:

- The quantities introduces so far renormalize as follows: $\langle \nu_R \rangle = \langle \nu \rangle$, $\quad M_R = M/Z_P$, $\quad \lambda_R = \lambda/Z_P$, $\quad \Sigma_R = Z_P \Sigma$; in the following the quark mass will be defined through the PCAC mass, which renormalizes as $m_{PCAC} = Z_A m_R / Z_P$;
- The scale was set using the string tension, meaning that we built dimensionless quantities using the large-N results for $a\sqrt{\sigma}$ obtained in Refs. [22, 31] within the TEK model;
- The mode number is computed numerically solving $(\gamma_5 D_W^{(TEK)}[U])u_\lambda = \lambda u_\lambda$ for the first few hundred lowest-lying eigenvalues, and for an ensemble of a hundred well-decorrelated gauge configurations U;
- Using the TEK model large-N determinations of Z_A and m_{PCAC} reported in Ref. [31], we can express $\langle \nu_R \rangle$ as a function of M_R/m_R renormalizing the eigenvalues as follows: $\lambda_R/m_R = \lambda/(Z_A m_{PCAC})$;
- We perform a linear best fit of $\langle \nu_R(M_R) \rangle / N$ as a function of M_R/m_R to obtain the slope $\frac{1}{N}\left[\frac{\partial\langle \nu(M)\rangle}{\partial M}\right]\Big|_{M=\overline{M}}$, with $\overline{M}$ the middle point of the fit range; an example of such fit is reported in the top panel of Fig. 1;
- Using Eq. (7), and plugging in the slope and the volume $V\sigma^2 = (a\sqrt{\sigma})^4 N^2$, we calculate the Renormalization-Group-Invariant (RGI) quantity $\Sigma_R^{(eff)} m_R/(\sigma^2 N)$;
- Using again that $Z_A m_{PCAC} = Z_P m_R$, and assuming the conventional value $\sqrt{\sigma} = 440$ MeV, we are able to express the bare effective condensate $\Sigma^{(eff)}(m)/N = \Sigma_R^{(eff)}(m)/(N Z_P)$ in physical MeV^3 units;
- We explicitly checked that finite-N corrections where negligible within our typical statistical errors by comparing results obtained for $N = 289$ and $N = 361$, as shown in the bottom panel of Fig. 1, finding perfectly agreeing results for the mode number.

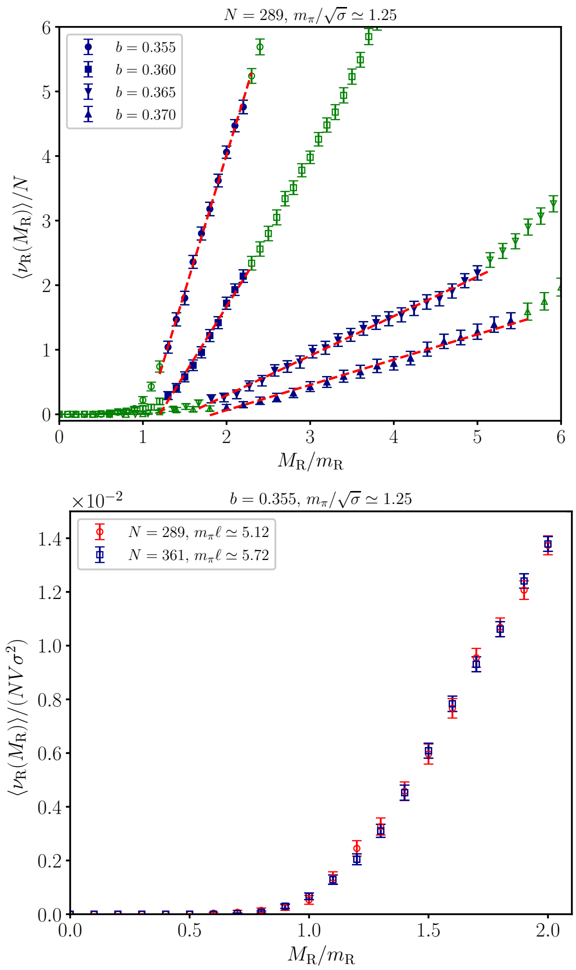

Figure 1: *Top panel: mode number $\langle \nu_{\rm R} \rangle / N$ as a function of $M_{\rm R}/m_{\rm R}$ for 4 different values of the 't Hooft coupling $1/b$ and of the hopping parameter κ, corresponding to 4 different lattice spacing and same pion mass in physical units $m_\pi/\sqrt{\sigma}$. Filled points have been fitted according to a linear behavior. Bottom panel: comparison between the mode number obtained for $N = 289$ and $N = 361$ for the same values of b and κ (in the latter case the twist parameter $k = 7$ was used). Figures taken from Ref. [15].*

3.2 Chiral limit at fixed lattice spacing

The physical value of the chiral condensate sits at vanishing quark mass, meaning that the effective chiral condensate needs to be chirally extrapolated. In all explored cases, we found that it was sufficient to rely on Leading Order (LO) Chiral Perturbation Theory (ChPT) to take the chiral limit:

$$[\Sigma^{(\rm eff)}(m)/N]^{1/3} = (\Sigma/N)^{1/3} + k\, m + o(m) \tag{10}$$

$$= (\Sigma/N)^{1/3} + k'\, m_\pi^2 + o(m_\pi^2), \tag{11}$$

As it can be observed from Fig. 2, indeed, our data are perfectly described by a linear behavior in the squared pion mass.

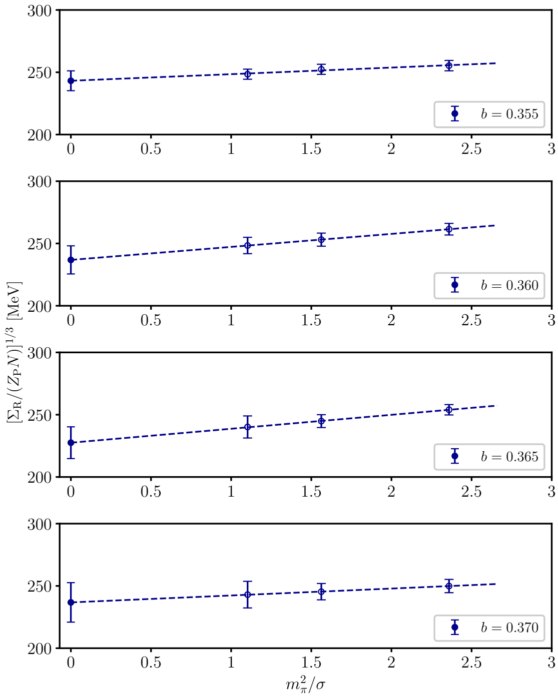

Figure 2: *Chiral limit extrapolation of the third root of the bare effective chiral condensate* $[\Sigma^{(\text{eff})}/N]^{1/3} = [\Sigma_R^{(\text{eff})}/(NZ_P)]^{1/3}$ *expressed in* MeV *units for all explored values of the lattice spacing. Figure taken from Ref. [15].*

3.3 Continuum limit

In order to take the continuum limit, we renormalized the chiral condensate using the non-perturbative large-N determinations of Z_P reported in Ref. [48], obtained using the so-called Rome-Southampton method.

We took the continuum limit assuming leading $O(a^2)$ finite-lattice-spacing corrections, and computed the final continuum result from a best fit of our data according to:

$$[\Sigma_R(a)/N]^{1/3} = (\Sigma_R/N)^{1/3} + c\,a^2 + o(a^2). \tag{12}$$

We also obtained an independent determination of the chiral condensate from the Gell-Man–Oakes–Renner (GMOR) relation, using the large-N determinations of m_π and $F_\pi/\sqrt{N}$ obtained in Ref. [31] from the TEK model:

$$m_\pi^2 = 2\frac{\Sigma}{F_\pi^2}m = 2\frac{\Sigma_R}{F_\pi^2}m_R = 2\frac{\Sigma_R}{N}\frac{N}{F_\pi^2}m_R. \tag{13}$$

In the top panel of Fig. 3 we compare these two determinations. As it can be appreciated, they perfectly match among themselves both at finite lattice spacing and in the continuum limit. For this reason, we take as our final result for the condensate the continuum limit obtained from a joint best fit of the two data sets (cf. Fig. 3, bottom panel), yielding:

$$(\Sigma_R/N)^{1/3} = 184(13)\text{ MeV}. \tag{14}$$

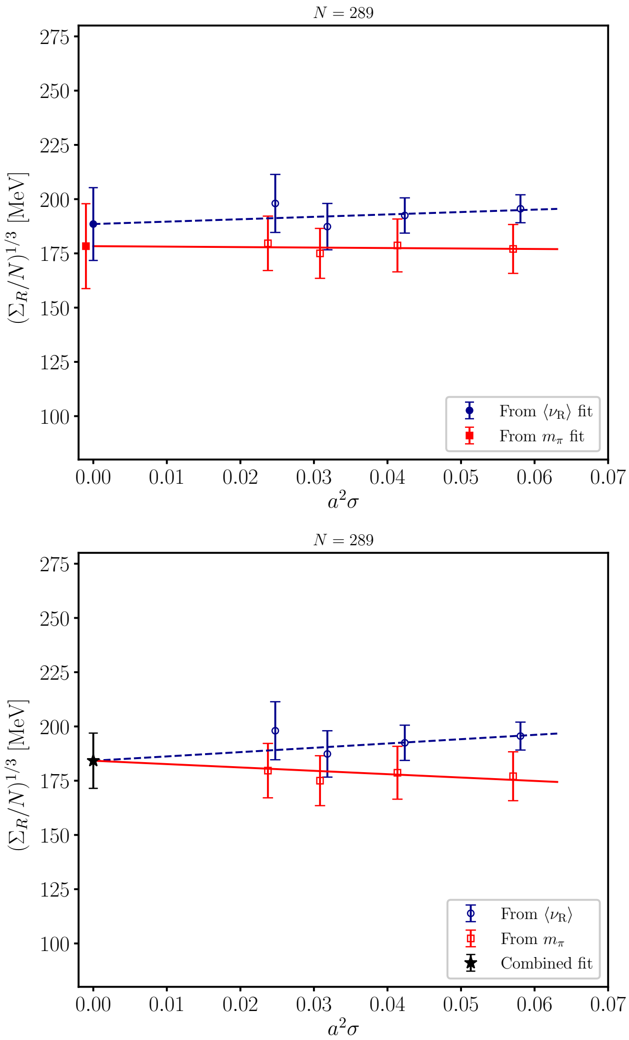

Figure 3: *Top panel: individual continuum limits of the determinations of the chiral condensate obtained from the Giusti–Lüscher method (circles) and from the GMOR relation in Eq.* (13) *(squares). Bottom panel: joint continuum limit of the two different data sets. Figures taken from Ref. [15].*

4. Conclusions

This proceeding reports on the main results of Ref. [15]. In that paper we performed a solid and reliable calculation of the chiral condensate of large-N QCD using twisted volume-reduced models. In particular, we performed our calculation for $N = 289$ (after checking that $N = 361$ gave perfectly identical results), and performed controlled continuum and chiral limits using 4 different values of the lattice spacing, and considering 3 different values of the pion mass each.

The spectral determination of the condensate was shown to be in perfect agreement with the one obtained from the quark-mass dependence of the pion mass using the well-known GMOR formula. Our final result for the chiral condensate of large-N QCD reads:

$$\lim_{N\to\infty} \frac{\Sigma_{\mathrm{R}}(N)}{N} = [184(13)\ \mathrm{MeV}]^3, \qquad \left(\overline{\mathrm{MS}},\ \mu = 2\ \mathrm{GeV},\ \sqrt{\sigma} = 440\ \mathrm{MeV}\right). \tag{15}$$

Remarkably, our large-N determination perfectly matches, within errors, the FLAG21 [49] averaged results for the 2-flavor $N = 3$ QCD condensate $\Sigma_R(N = 3)/3 = [184(7)\ \mathrm{MeV}]^3$ ($\overline{\mathrm{MS}}$, $\mu = 2$ GeV). Our calculation thus points out that higher-order corrections in $1/N$ are pretty small, meaning that the $N = \infty$ theory is actually a very good approximation of the physical $N = 3$ one. Such conclusion perfectly matches with several other large-N lattice results obtained in recent years pointing towards the same scenario [13, 34–37, 39–44, 50].

In the next future, it would be interesting to apply the same numerical techniques presented here to calculate the gluino condensate of large-N SUSY Yang–Mills theory, being it a quantity of great theoretical interest.

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