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Universal properties of Yang-Lee edge singularity and QCD phase diagram

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Critical points are categorized based on the number of relevant variables. The standard critical point in systems like the Ising model involves two relevant variables, namely temperature and external magnetic field. In contrast, a tricritical point is characterized by four such variables. The protocritical point, widely known as the Yang-Lee edge singularity (YLE), is the simplest form of criticality and has just one relevant variable.

Unlike conventional critical points, the YLE singularity occurs at complex values of parameters. When two YLE singularities merge and pinch the real axis of the corresponding thermodynamic variable, a critical point with associated critical scaling emerges. In other words, the location of the YLE singularity is continuously connected to the location of the critical point.

In this talk, I explain why conventional methods fail to accurately locate YLE singularity and demonstrate the success of the Functional Renormalization Group approach in determining the universal location of these singularities. I also discuss how we can learn more about QCD phase diagram by combining our findings with lattice QCD results.

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1. Introduction

The phase diagram of Quantum Chromodynamics (QCD) is the only diagram in the Standard Model of elementary particle physics that can be studied in laboratory conditions. As a result, both theoretical and experimental efforts have been made to identify the most significant feature of any phase diagram - a critical point [1]. Presently, the location and existence of such a critical point is yet to be established (see Ref. [2] for a review).

Static theory of critical phenomena provides predictions for non-monotonic dependence of fluctuation observables, such as fluctuations of baryon number. They constitute the main direction in experimental quest to locate the critical point [2–5]. Due to a short-life time of the system created in relativistic heavy ion collisions, critical statics has to be amended by the dynamical evolution of these observables [6–8], which is not yet well studied. This and numerous non-critical contributions to fluctuation observables experimental identification of the QCD critical point.

To study QCD theoretically, the first principle simulations rely on Markov chain Monte-Carlo methods (functional methods have gained more prominence in recent years, see e.g. Ref. [9]). They have allowed us to probe the QCD phase diagram at zero baryon chemical potential (and at purely imaginary values), where the transition from hadronic matter to a quark gluon plasma is known to be a smooth "crossover" [10]. Monte-Carlo simulations require probabilistic interpretation for (or non-negativity of) the weights of configurations, which is lost at non-zero real values of the baryon chemical potential. This is the so-called sign problem which does not allow us to locate the critical point theoretically. To access the region of non-zero chemicla potential, several indirect methods have been explored. Most of the current knowledge about QCD phase diagram originate from methods based on carrying out Taylor expansions around zero baryon chemical potential [11, 12] or through analytic continuation from purely imaginary values of baryon chemical potential [13–16]. These methods heavily depend on the knowledge of the analytic structure of the QCD partition function.

For an illustration consider a function $f(x) = (\exp(x) + 1)^{-1}$ and its Taylor series expansion about zero argument, $f(x) \approx \frac{1}{2} - \frac{x}{4} + \frac{x^3}{48} + O\left(x^4\right)$. The function $f(x)$ is smooth for real values of x; despite this, the Taylor series expansion has a finite radius of convergence $|x| \le \pi$ (it can be found by analyzing the coefficients) due to the presence of a singularity at a purely imaginary value of $x_c = i\pi$. This demonstrates that (at least naive) application of Taylor series expansion to learn about the behaviour of the function for arguments $|x| > |x_c|$ is doomed to fail. Reversely, by finding the radius of convergence one can potentially determine the location of the singularity (see e.g. Ref. [17]) and thus deduce the structure of the phase diagram as we explain now.

Lee and Yang were the first to demonstrate the connection between the analytical structure of the equation of state and the phase structure [19, 20]. In the symmetric phase, Lee and Yang demonstrated that the equations of state of classic O(N)-symmetric ϕ^4 theories have a branch cut at purely imaginary values of the magnetic field h. The cut terminates at two branch points – the Yang-Lee edge (YLE) singularities [21, 22]. A second-order (first-order) phase transition at $t \propto T - T_c = 0$ occurs when the singularities pinch (cross) the real h-axis. This highlights the importance of the YLE as they are continuously connected to the corresponding critical point [23–31]. The edge singularities can be seen as critical points themselves. As illustrated in Fig. 1, variation of only one parameter h allows to tune the system to these critical points. The conventional Wilson-Fisher

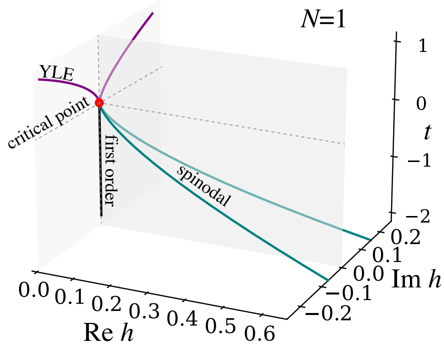

Figure 1: Analytic structure of the universal phase diagram for $N = 1$. Only branch points are displayed; the cuts are omitted for the clarity of the figure. See Ref. [18] for details.

critical point requires tuning two parameters t and h. Driving a system into a multi-critical point, e.g. a tricritical point, would require adjustment of four or more parameters. For that reason M. Fisher referred to YLE singularities as *protocritical* points. The number of relevant variables also determines the number of independent critical exponents in the leading order scaling relations. At the Wilson-Fisher critical point, the presence of two relevant perturbations leads to two independent critical exponents. At the YLE, we have only one independent critical exponent $\sigma_{\rm YLE} = 1/\delta_{\rm YLE}$. It determines the scaling of the magnetization, $M \sim M_{\rm c} + (h - h_c)^{\sigma_{\rm YLE}}$, where M_c and h_c are purely imaginary. The numerical value of the edge critical exponent in three dimensions (and for any N of the underlying universality class) has been determined by a variety of methods, see e.g. Refs. [32–35]. The most precise result is obtained using statistical bootstrap, $\sigma_{\rm YLE} = 0.085(1)$ [32].

The simplest way to introduce the YLE singularity and its universal location is by considering Landau's mean-field approximation. The Landau free energy neglecting the kinetic term gives

$$F = \int d^d x \left(\frac{1}{2}\tilde{t}\phi^2 + \frac{\lambda}{4}\phi^4 - \tilde{h}\phi \right) . \tag{1}$$

The uniform field that extremizes the free energy satisfy $\delta F/\delta\phi = 0$ or

$$\tilde{t}\phi^2 + \lambda\phi^3 = \tilde{h} . \tag{2}$$

It is convenient to rescale the external field $\tilde{h}$ and $\tilde{t}$ to absorb λ,

$$t\phi^2 + \phi^3 = h \tag{3}$$

or, rewriting in a more suggestive form,

$$\frac{t}{h^{2/3}} \left(\frac{\phi}{h^{1/3}} \right)^2 + \frac{\phi^3}{h} = 1 . \tag{4}$$

We conclude that the solution of this equation can be represented as $\phi(t,h) = h^{1/\delta_{\rm MF}} f_G(z)$ where f_G is defined by the so-called magnetic equation of state

$$f_G(z + f_G^2) = 1 . \tag{5}$$

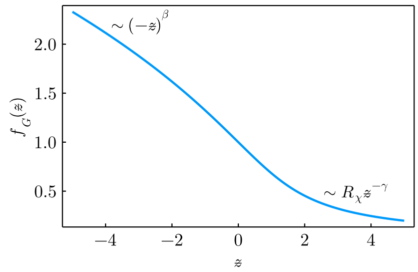

Figure 2: The magnetic equation of state in the mean-field approximation f_G as a function of the scaling variable z. In the mean-field approximation, the universal amplitude ratio $R_\chi = 1$ and the critical exponent $\gamma = 1$.

f_G is a function of a single "scaling" variable $z = t/h^{1/\beta_{\rm MV}\delta_{\rm MF}}$. Here $\beta_{\rm MF} = 1/2$ and $\delta_{\rm MF} = 3$ are mean-field critical exponents. Here (trough absorbing λ) and in general, one introduces the magnetic equation of state such as to satisfy the following normalization conditions $f_G(0) = 1$ and $f_G(z \to -\infty) \sim (-z)^{\beta_{\rm MF}}$. Function f_G is plotted in the Fig. 2

The solution of Eq. (5) has a singularity: $df_G/dz = \infty$ or $dz/df_G = 0$. Using the latter condition and differentiating Eq. (5) w.r.t. f_G leads to three solutions for z_c, two of which are complex conjugate. This complex conjugate pair are on the physical Riemann sheet (for $t > 0$):

$$z_c = |z_c| \exp\left(\frac{\pm i\pi}{2\beta_{\rm MF}\delta_{\rm MF}}\right), \quad |z_c| = \frac{3}{2^{2/3}}. \tag{6}$$

The above-found z_c are YLE singularities, see Fig. 3 . Going beyond mean-field approximation, the exact expression for the magnetic equation of state is not known for the physically interesting case of three spatial dimensions. However, in the vicinity of the critical point, it is universal; that is, the magnetic equation of state is only defined by global symmetries (in general, the number of dimensions) and is independent of the microscopic details of the system. The locations of the YLE singularities of the magnetic equation of state are also universal. For two-dimensional Ising model and three dimensional $O(N)$ universality classes, the location of YLE's was determined only recently (see Refs. [36–39] for two-dimensional Ising model and Refs. [18, 40–42] for three-dimensional classic universality classes) even though Lee and Yang formulated the problem more than seventy years ago. Why did this problem pose a challenge? The answer is in the failure of the commonly applied tools to describe critical statics:

- On the one hand, the widely used framework of the ε-expansion near four dimensions leads to non-perturbative contributions beyond the leading order, which significantly limits its predictive power for $d = 3$. This is related to the fact that the Wilson-Fisher fixed point has an upper critical dimension (the number of dimensions at which mean-field approximation becomes exact) of four, while the YLE fixed point at imaginary h has an upper critical dimension of six [22]. Specifically for $O(N)$ universality class, the ϵ-expansion of the

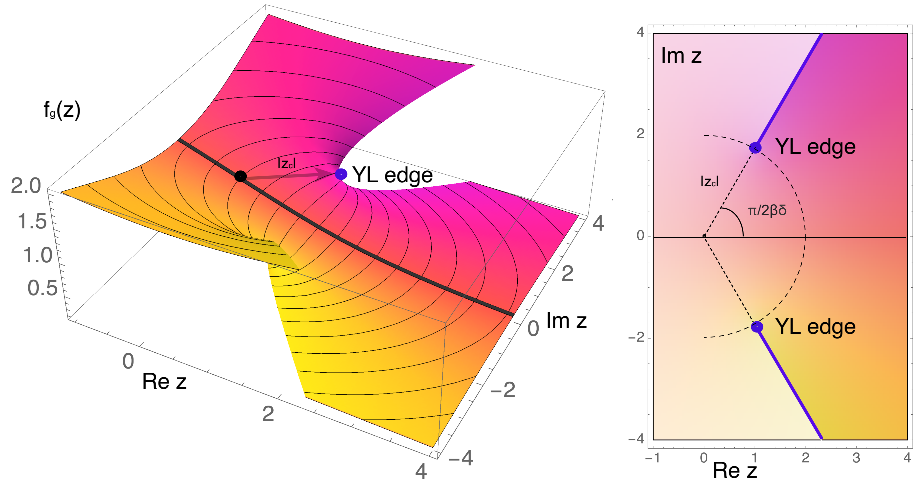

Figure 3: The real part of the magnetic equation of state f_G in the mean-field approximation as a function of complex z. The black line in the left panel coincides with f_G plotted in Fig. 2. The YLE singularities appear as branch points with cuts originating from them.

absolute value of z_c has the following form (see Ref. [18])

$$|z_c| \approx |z_c^{\mathrm{MF}}| \underbrace{\Big[\underbrace{1}_{\text{tree level}} + \frac{27\ln\left(\frac{3}{2}\right) - (N-1)\ln 2}{9(N+8)}\epsilon \Big]}_{\text{one loop}} + \underbrace{\epsilon^2 \log\epsilon \times (\cdots)}_{\text{all loops contribute}}, \qquad \epsilon = 4 - d, \qquad (7)$$

where the tree (mean-field approximation) and the one loop levels are computable only. The higher order contributions require a resummation of all loop orders which is not feasible, see Ref. [18] for more details.

- On the other hand, the lattice methods based on importance sampling prohibit direct computations at imaginary values of the magnetic field due to the sign problem; thus rendering the problem of extracting the location of the YLE singularity practically impossible.

Finally, the bootstrap approach (although very successful in extracting the critical exponents) is not equipped to determine the location of the YLE singularity.

Recently, the Functional Renormalization Group approach was successfully applied to locate the YLE singularity [18, 40, 41] in three dimensions. We will summarize the findings of these papers in the next section.

2. Functional renormalization group approach to locate of YLE singularity

The Functional Renormalization Group (FRG) (see Refs. [9, 43–47]) is a field-theoretical implementation of Wilson's idea of integrating over momentum shells which is achieved by the inclusion of fluctuations ordered in momentum. Practically this is done through modifying the path integral measure by adding a mass-like term $\Delta S_k[\varphi]$ suppressing contributions of momentum modes with $p \lesssim k$. $\Delta S_k[\varphi]$ is introduced such as the variation of the scale parameter k interpolates between the UV effective action at the scale Λ, $\Gamma_{k=\Lambda}[\phi] \approx S[\phi]$ and the full IR effective action at $k = 0$, $\Gamma_{k=0}[\phi] = \Gamma[\phi]$. Here, the expectation value/order parameter ϕ is given by $\phi(x) = \langle\varphi(x)\rangle$.

The functional $\Gamma_k[\phi]$ satisfies the Wetterich or flow equation [43],

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k \left(\frac{\delta^2\Gamma_k[\phi]}{\delta\phi_i\delta\phi_j} + R_k\right)^{-1}\right\}. \tag{8}$$

The FRG provides a versatile realization of the Wilsonian RG and is as such well-suited to study critical physics. Using FRG, both the scaling function and the critical exponents have been computed for $\mathrm{O}(N)$ theories, see, e.g., Refs. [48–54].

While the flow equation is exact, it is not practically solvable as it defines an infinite tower of coupled partial differential equations for the effective action and its functional derivatives. With few exceptions, truncation is necessary to tackle the equations in practice. There is often no obvious small parameter that can be used to set a systematic truncation scheme. Fortunately, this is not the case for critical physics, where the diverging correlation length facilitates a systematic expansion about vanishing momentum. Such a derivative expansion has been shown to have a finite radius of convergence with a well-defined scheme for systematic error estimate [54, 55]. To locate YLE singularity, the next-to-leading order of this expansion, i.e. first order in momentum-squared, was used in Refs. [18, 41].

At this order, the truncation of the effective action reads

$$\Gamma_k[\phi] = \int d^d x \left(U_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_i\phi)^2\right).$$

The flow equations can be reformulated as a set of equations for the effective potential U and the wave function renormalizations Z. For the average potential, one gets

$$\partial_t U_k(\rho) = \frac{1}{2}\int \bar{d}^d q\, \partial_t R_k\left(q^2\right)\left[G_k^{\parallel} + (N-1)G_k^{\perp}\right], \quad \rho = \frac{\phi^2}{2}$$

with

$$G_k^{\perp} = \frac{1}{Z_k^{\perp}(\rho)q^2 + U_k'(\rho) + R_k(q^2)}, \quad G_k^{\parallel} = \frac{1}{Z_k^{\parallel}(\rho)q^2 + U_k'(\rho) + 2\rho U_k''(\rho) + R_k(q^2)}.$$

Here we introduced the longitudinal and the transverse (Goldstone) modes.

The flow equation for the wave function renormalization for the longitudinal mode reads

$$\begin{aligned}
\partial_t Z_{\parallel}(\phi) = \int \bar{d}^d q \partial_t R_k(q^2)\Bigg\{ & G_{\parallel}^2\Big[\gamma_{\parallel}^2(G_{\parallel}' + 2G_{\parallel}''\frac{q^2}{d}) + 2\gamma_{\parallel}Z_{\parallel}'(\phi)(G_{\parallel} + 2G_{\parallel}'\frac{q^2}{d}) \\
& + (Z_{\parallel}'(\phi))^2 G_{\parallel}\frac{q^2}{d} - \frac{1}{2}Z_{\parallel}''(\phi)\Big] \\
& + (N-1)G_{\perp}^2\Big[\gamma_{\perp}^2(G_{\perp}' + 2G_{\perp}''\frac{q^2}{d}) + 4\gamma_{\perp}Z_{\perp}'(\phi)G_{\perp}'\frac{q^2}{d} + (Z_{\perp}'(\phi))^2 G_{\perp}\frac{q^2}{d} \\
& + 2\frac{Z_{\parallel}(\phi) - Z_{\perp}(\phi)}{\phi}\gamma_{\perp}G_{\perp} - \frac{1}{2}\left(\frac{1}{\phi}Z_{\parallel}'(\phi) - \frac{2}{\phi^2}(Z_{\parallel} - Z_{\perp})\right)\Big]\Bigg\}.
\end{aligned}$$

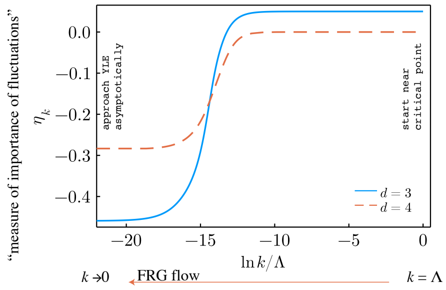

Figure 4: The anomalous dimension as a function of the cut-off momentum k in FRG evolution in $d = 3$ and $d = 4$. The initial conditions for the FRG flow are formulated in the vicinity of the Wilson-Fisher critical point; the flow equation is constructed in such a way as to track the position of the YLE singularity $m_R = 0+$. In the IR ($k \to 0$), the anomalous dimension reaches its physical value at the YLE singularity. The magnitude of η can be considered as a measure of the importance of fluctuations. In this context, it is interesting to consider $d = 4$. Near Wilson-Fisher point $\eta = 0$, that is fluctuations can be neglected and the men-filed approximation becomes exact. As the system is driven towards the YLE fixed point, the magnitude of the anomalous dimension increases drastically; this illustrates the fact that the upper critical dimension of the YLE fixed point (6) is larger than the one of the Wilson-Fisher fixed point (4).

A similar equation for the transverse mode can be found in Ref. [18]. In the above set of equations we used the shorthand notations for

$$\gamma_{\parallel} = q^2 Z'_{\parallel}(\phi) + U^{(3)}(\phi), \quad \gamma_{\perp} = q^2 Z'_{\perp}(\phi) + \frac{\partial}{\partial \phi}\left(\frac{1}{\phi}U'(\phi)\right), \quad G' = \frac{\partial G}{\partial q^2}.$$

In summary, the flow equation leads to three coupled partial-differential equations on U, $Z_{\parallel}$ and $Z_{\perp}$. The equations are stiff and solving them is rather challenging. Instead, one considers Taylor series expansion near k-dependent expansion point ϕ_k and corresponding equations for the expansion coefficients and ϕ_k. Traditionally the expansion point is selected to coincide with the minimum of the effective potential $U'_k[\phi_k] = h = \text{const}$. To locate the YLE, this is however is not the best choice – it is better to expand near ϕ_k defined by $U''_k[\phi_k] = m^2 \to 0$. Then, the FRG equation tracks the critical manifold in the parameter space and thus interpolates between the

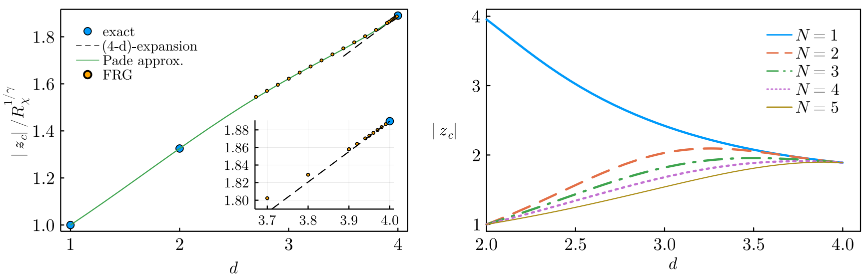

Figure 5: Left panel: the universal location of the YLE singularity as a function of the number of dimensions d for N=1. The result for the location of the YLE singularity in two-dimensional Ising model is taken from Ref. [37]. See Ref. [41] for details. Right panel: four-parameter Padé approximation for the dependence of the YLE location on the number of spatial dimension for different number of filed components, N; see Ref. [18] for details.

N	1	2	3	4	5
$\lvert z_c\rvert/R_\chi^{1/\gamma}$	1.621(4)(1)	1.612(9)(0)	1.604(7)(0)	1.597(3)(0)	1.5925(2)(1)
$\lvert z_c\rvert$	2.43(4)	2.04(8)	1.83(6)	1.69(3)	1.55(4)

Table 1: The location of the YLE singularity, $|z_c|/R_\chi^{1/\gamma}$ and $|z_c|$ for a representative number of components N. For $|z_c|/R_\chi^{1/\gamma}$, the numbers in the parentheses show the truncation error and the error due to residual regulator dependence. For $|z_c|$, the number in the parentheses represents a combined uncertainty, including that originating from the universal amplitude ratio R_χ and the critical exponent γ. See Ref. [18] for more details.

Wilson-Fisher fixed point and the YLE singularity as illustrated in Fig. 4. Note that in this case the value of the external field changes as a function of k according to $U'_k[\phi_k] = h_k \neq \text{const}$.

The results of these calculations are presented in Table 1. Note that our approach gives direct access to the universal ratio $|z_c|/R_\chi^{1/\gamma}$ not $|z_c|$. To find $|z_c|$, we used known results for the universal amplitude ratio R_χ and the critical exponent γ. Remarkably, FRG calculations can be performed for a non-integer number of dimensions, see Fig. 5. The left panel of the figure demonstrates that $|z_c|/R_\chi^{1/\gamma}$ is consistent with results obtained at the one loop order of ϵ expansion, Eq. (7) and with two- and one-dimensional Ising models. The right panel of Fig. 5 shows z_c dependence on the number of dimensions and the number of the field components N.

In the context of lattice QCD, the results shown in Table 1 were used to a) to determine the metric factor near Roberge-Weiss phase transition, see Ref. [15, 56]; b) to compare the results extracted from finite volume analysis of Lee-Yang zeroes, see Ref. [15]; c) to predict the location of the critical point, see Ref. [57].

3. Fourier coefficients

YLE singularity affects many thermodynamic properties. One striking example, that we consider in this section, is Fourier coefficients of the baryon chemical potential. The QCD partition function is periodic in baryon chemical potential $\hat{\mu} = \mu/T$: $Z(\hat{\mu} + 2\pi i) = Z(\hat{\mu})$. Due to this periodicity, analysis of the data obtained for purely imaginary values of baryon chemical potential is natural in terms of the corresponding Fourier coefficients [58–63]. Usually one computes Fourier

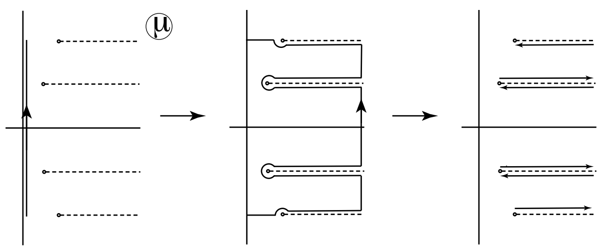

Figure 6: Complex chemical potential plane with the chiral and RW YLE. The integration path along the imaginary chemical potential axis can be deformed to the integration around the branch point singularities and the cuts.

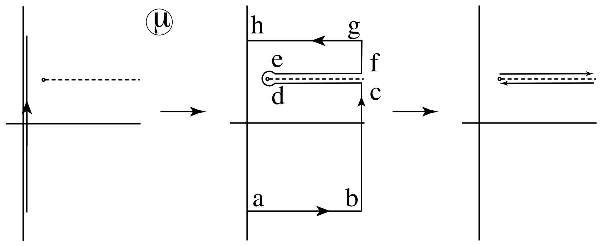

Figure 7: Computation of the Fourier coefficient for the function with one singularity in the right half-plane of complex chemical potential.

transformation of the baryon number density $n_B = V^{-1}\partial_{\hat{\mu}} \ln Z(\hat{\mu})$:

$$b_k = \frac{1}{i\pi}\int_{-\pi}^{\pi} d\theta \, \hat{n}_B(\hat{\mu} = i\theta) \sin(k\theta) \, , \tag{9}$$

where we explicitly took into account the symmetry property of the baryon density $n_B(\hat{\mu}) = -n_B(-\hat{\mu})$ and introduced $\hat{n}_B = n_B/T^3$.

The Fourier coefficients are sensitive to the structure of the phase diagram through the following line of arguments, see Ref. [64]. Consider the integral (9) in the complex plane, as illustrated in Fig. 6. In this figure, we accounted for the presence of the YLE singularities associated with the chiral and Roberge-Weiss phase transitions. Now since the continuous deformation of the contour does not change the value of the integral as long as it does not cross the singularities, one can modify the integration to that over both sides of the cuts, as shown in Fig. 6. To proceed with an actual realization of this program it is useful first consider just one singularity and the corresponding cut in isolation.

Consider an odd function $n_B(\hat{\mu})$ periodic in an imaginary argument having brunch points in the complex plane located at $\pm\hat{\mu}^{\rm br}$, where $\hat{\mu}^{\rm br} = \hat{\mu}^{\rm br}_r + i\,\hat{\mu}^{\rm br}_i$. Here, we expand n_B near the branch point $\hat{\mu} \to +\hat{\mu}^{\rm br}$:

$$\hat{n}_B(\hat{\mu}) = A(\hat{\mu} - \hat{\mu}^{\rm br})^{\sigma}(1 + B(\hat{\mu} - \hat{\mu}^{\rm br})^{\theta_c} + \ldots) + \sum_{n=0}^{\infty} a_n(\hat{\mu} - \hat{\mu}^{\rm br})^n \tag{10}$$

with $\sigma > -1$ and $\theta_c > 0$. In the context of the YLE singularity, θ_c is the confluent critical exponent (not to be confused with integration variable θ). The regular part of $\hat{n}_B$ on the cuts is encoded by the coefficients a_n.

From the definition of the Fourier coefficients, we have

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \, \hat{n}_B(\hat{\mu} = i\theta) e^{-ik\theta} \, . \tag{11}$$

To compute the integral, we will deform the contour as shown in Fig. 7. In the figure, we assume that the right-most points are extended to the infinity, i.e. ${\rm Re}\,\mu \to \infty$. The contribution of the segments (ab) and (gh) cancel each other due to the periodicity of the integrand and the opposite direction of the segments. The contributions from (bc) and (fg) is zero due to the exponential decay of $\exp(-ik\theta) = \exp(-k\hat{\mu})$ for any $k > 0$. The integral around the branch point (de) is vanishing owing to $\sigma > -1$. Thus, only the segments on both sides of the cut (cd) and (ef) give a non-trivial contribution to the integral as illustrated in Fig. 7. To evaluate the contribution of these segments, we consider the parametrization $i\theta = s + \hat{\mu}^{\rm br}$

$$\frac{1}{\pi}\int_{(ef)} d\theta\, \hat{n}_B(\hat{\mu} = i\theta)e^{-ik\theta} = \frac{1}{i\pi}e^{-\mu^{\rm br}k}\int_0^\infty ds\, \hat{n}_B(\hat{\mu} = s + \hat{\mu}^{\rm br})e^{-ks}\,. \tag{12}$$

Evaluating the ensuing integrals, we get

$$\frac{1}{\pi}\int_{(ef)} d\theta\, \hat{n}_B(\hat{\mu} = i\theta)e^{-ik\theta} = \frac{e^{-\mu^{\rm br}k}}{i\pi}\left(A\frac{\Gamma(1+\sigma)}{k^{1+\sigma}}\left[1 + \frac{B}{k^{\theta_c}}\frac{\Gamma(1+\sigma+\theta_c)}{\Gamma(1+\sigma)} + \dots\right]\right.$$
$$\left. + \sum_{n=0}^{\infty} a_n \frac{\Gamma(1+n)}{k^{1+n}}\right)\,. \tag{13}$$

In the second line, the contribution is attributed to the analytic part in Eq. (10).

The integral over the segment (cd) is identical to the expression above except for the 2π rotation around the branch point and an extra minus sign due to the direction of the segment. Adding both integrals together cancels the analytic part to yield

$$b_k = \frac{e^{-\mu^{\rm br}k}}{i\pi}A\frac{\Gamma(1+\sigma)}{k^{1+\sigma}}\left(1 - e^{i2\pi\sigma} + \frac{B}{k^{\theta_c}}\left[1 - e^{i2\pi(\sigma+\theta_c)}\right]\frac{\Gamma(1+\sigma+\theta_c)}{\Gamma(1+\sigma)} + \dots\right)\,. \tag{14}$$

This can be rewritten by Absorbing k independent factors into constants A and B

$$b_k = \tilde{A}\frac{e^{-\mu^{\rm br}k}}{k^{1+\sigma}}\left(1 + \frac{\tilde{B}}{k^{\theta_c}} + \dots\right)\,. \tag{15}$$

Now we are ready to generalize this result to the case when both YLE and RW singularities are present, we obtain

$$b_k = \tilde{A}_{\rm YLE}\frac{e^{-\hat{\mu}^{\rm YLE}k}}{k^{1+\sigma}}\left(1 + \frac{\tilde{B}_{\rm YLE}}{k^{\theta_c}} + \dots\right) + \tilde{A}_{\rm RW}\frac{e^{-\hat{\mu}^{\rm RW}k}}{k^{1+\sigma}}\left(1 + \frac{\tilde{B}_{\rm RW}}{k^{\theta_c}} + \dots\right) + \text{c.c.}\,. \tag{16}$$

Here, the coefficients $\tilde{A}_{\rm YLE,RW}$ and $\tilde{B}_{\rm YLE,RW}$ are generally complex numbers. Taking into account that ${\rm Im}\,\hat{\mu}^{\rm RW} = \pi$, we have

$$b_k = |\tilde{A}_{\rm YLE}|\frac{e^{-\hat{\mu}_r^{\rm YLE}k}}{k^{1+\sigma}}\left(\cos(\hat{\mu}_i^{\rm YLE}k + \phi_a^{\rm YLE}) + \frac{|\tilde{B}_{\rm YLE}|}{k^{\theta_c}}\cos(\hat{\mu}_i^{\rm YLE}k + \phi_b^{\rm YLE}) + \dots\right)$$
$$+ |\hat{A}_{\rm RW}|(-1)^k\frac{e^{-\hat{\mu}_r^{\rm RW}k}}{k^{1+\sigma}}\left(1 + \frac{|\hat{B}_{\rm RW}|}{k^{\theta_c}} + \dots\right)\,, \tag{17}$$

where ϕ_a and ϕ_b are phases due to non trivial phases of $\tilde{A}^{\rm YLE}$ and $\tilde{B}^{\rm YLE}$ and trivial real factors were absorbed into $|\hat{A}_{\rm RW}|$ and $|\hat{B}_{\rm RW}|$.

A few comments about the obtained results:

- The coefficients, b_k, are exponentially sensitive to the imaginary values of the positions of the YLE singularities and also is sensitive to the edge critical exponent σ.
- The confluent critical exponent $\theta_c = \nu_c\omega = \frac{\sigma+1}{3}\omega$ is about 0.6 (ω can be found in Ref. [35]) and thus leads to an appreciable suppression of the corresponding terms. For a large enough order of k, it is safe to drop these corrections.

4. Conclusions

In these proceedings, we reviewed the importance and the universal properties of the Yang-Lee edge singularity. We specifically concentrated on the universal location of the YLE, as it has been determined recently. From the perspective of establishing the phase diagram of QCD, tracking the position of the YLE singularity may reveal the existence and the position of critical point.

One way to determine the position of the YLE in QCD is by analyzing the Fourier coefficients on the baryon chemical potential. Here we reviewed the asymptotic form of these coefficients and showed that they are exponentially sensitive to the location of the singularity. Currently, an application of our analysis to lattice QCD results for Fourier coefficients is not possible as a) Fourier coefficient has to be considered in the infinite volume limit and b) the statistical error in the lattice data would require a substantial reduction. For more details, see Ref. [64].

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References

[1] M.A. Stephanov, K. Rajagopal and E.V. Shuryak, *Signatures of the tricritical point in QCD*, *Phys. Rev. Lett.* **81** (1998) 4816 [hep-ph/9806219].

[2] A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, *Mapping the Phases of Quantum Chromodynamics with Beam Energy Scan*, *Phys. Rept.* **853** (2020) 1 [1906.00936].

[3] M. Bluhm et al., *Dynamics of critical fluctuations: Theory – phenomenology – heavy-ion collisions*, *Nucl. Phys. A* **1003** (2020) 122016 [2001.08831].

[4] X. An et al., *The BEST framework for the search for the QCD critical point and the chiral magnetic effect*, *Nucl. Phys. A* **1017** (2022) 122343 [2108.13867].

[5] STAR collaboration, *Nonmonotonic Energy Dependence of Net-Proton Number Fluctuations*, *Phys. Rev. Lett.* **126** (2021) 092301 [2001.02852].

[6] K. Rajagopal and F. Wilczek, *Static and dynamic critical phenomena at a second order QCD phase transition*, *Nucl. Phys. B* **399** (1993) 395 [hep-ph/9210253].

[7] B. Berdnikov and K. Rajagopal, *Slowing out-of-equilibrium near the QCD critical point*, *Phys. Rev. D* **61** (2000) 105017 [hep-ph/9912274].

[8] D.T. Son and M.A. Stephanov, *Dynamic universality class of the QCD critical point*, *Phys. Rev. D* **70** (2004) 056001 [hep-ph/0401052].

[9] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J.M. Pawlowski, M. Tissier et al., *The nonperturbative functional renormalization group and its applications*, *Phys. Rept.* **910** (2021) 1 [2006.04853].

[10] Y. Aoki, G. Endrodi, Z. Fodor, S.D. Katz and K.K. Szabo, *The Order of the quantum chromodynamics transition predicted by the standard model of particle physics*, *Nature* **443** (2006) 675 [hep-lat/0611014].

[11] A. Bazavov et al., *The QCD Equation of State to $O(\mu_B^6)$ from Lattice QCD*, *Phys. Rev. D* **95** (2017) 054504 [1701.04325].

[12] S. Borsanyi, Z. Fodor, J.N. Guenther, S.K. Katz, K.K. Szabo, A. Pasztor et al., *Higher order fluctuations and correlations of conserved charges from lattice QCD*, *JHEP* **10** (2018) 205 [1805.04445].

[13] P. de Forcrand and O. Philipsen, *The QCD phase diagram for small densities from imaginary chemical potential*, *Nucl. Phys. B* **642** (2002) 290 [hep-lat/0205016].

[14] M. D'Elia and M.-P. Lombardo, *Finite density QCD via imaginary chemical potential*, *Phys. Rev. D* **67** (2003) 014505 [hep-lat/0209146].

[15] P. Dimopoulos, L. Dini, F. Di Renzo, J. Goswami, G. Nicotra, C. Schmidt et al., *Contribution to understanding the phase structure of strong interaction matter: Lee-Yang edge singularities from lattice QCD*, *Phys. Rev. D* **105** (2022) 034513 [2110.15933].

[16] S. Borsányi, Z. Fodor, J.N. Guenther, R. Kara, S.D. Katz, P. Parotto et al., *Lattice QCD equation of state at finite chemical potential from an alternative expansion scheme*, *Phys. Rev. Lett.* **126** (2021) 232001 [2102.06660].

[17] C. Hunter and B. Guerrieri, *Deducing the properties of singularities of functions from their taylor series coefficients*, *SIAM Journal on Applied Mathematics* **39** (1980) 248 [https://doi.org/10.1137/0139022].

[18] G. Johnson, F. Rennecke and V.V. Skokov, *Universal location of Yang-Lee edge singularity in classic O(N) universality classes*, *Phys. Rev. D* **107** (2023) 116013 [2211.00710].

[19] C.N. Yang and T.D. Lee, *Statistical theory of equations of state and phase transitions. i. theory of condensation*, *Phys. Rev.* **87** (1952) 404.

[20] T.D. Lee and C.N. Yang, *Statistical theory of equations of state and phase transitions. ii. lattice gas and ising model*, *Phys. Rev.* **87** (1952) 410.

[21] P.J. Kortman and R.B. Griffiths, *Density of Zeros on the Lee-Yang Circle for Two Ising Ferromagnets*, *Phys. Rev. Lett.* **27** (1971) 1439.

[22] M.E. Fisher, *Yang-Lee Edge Singularity and phi**3 Field Theory*, *Phys. Rev. Lett.* **40** (1978) 1610.

[23] T.D. Lee and C.-N. Yang, *Statistical theory of equations of state and phase transitions. 2. Lattice gas and Ising model*, *Phys. Rev.* **87** (1952) 410.

[24] C.-N. Yang and T.D. Lee, *Statistical theory of equations of state and phase transitions. 1. Theory of condensation*, *Phys. Rev.* **87** (1952) 404.

[25] C. Itzykson, R.B. Pearson and J.B. Zuber, *Distribution of Zeros in Ising and Gauge Models*, *Nucl. Phys. B* **220** (1983) 415.

[26] M.A. Stephanov, *QCD critical point and complex chemical potential singularities*, *Phys. Rev. D* **73** (2006) 094508 [hep-lat/0603014].

[27] S. Mukherjee and V. Skokov, *Universality driven analytic structure of the QCD crossover: radius of convergence in the baryon chemical potential*, *Phys. Rev. D* **103** (2021) L071501 [1909.04639].

[28] S. Mukherjee, F. Rennecke and V.V. Skokov, *Analytical structure of the equation of state at finite density: Resummation versus expansion in a low energy model*, *Phys. Rev. D* **105** (2022) 014026 [2110.02241].

[29] G. Basar, *Universality, Lee-Yang Singularities, and Series Expansions*, *Phys. Rev. Lett.* **127** (2021) 171603 [2105.08080].

[30] G. Basar, *On the QCD critical point, Lee-Yang edge singularities and Pade resummations*, 2312.06952.

[31] Z.-Y. Wan, Y. Lu, F. Gao and Y.-X. Liu, *Lee-Yang edge singularities in QCD via the Dyson-Schwinger Equations*, 2401.04957.

[32] F. Gliozzi and A. Rago, *Critical exponents of the 3d Ising and related models from Conformal Bootstrap*, *JHEP* **10** (2014) 042 [1403.6003].

[33] X. An, D. Mesterházy and M.A. Stephanov, *Functional renormalization group approach to the Yang-Lee edge singularity*, *JHEP* **07** (2016) 041 [1605.06039].

[34] L. Zambelli and O. Zanusso, *Lee-Yang model from the functional renormalization group*, *Phys. Rev. D* **95** (2017) 085001 [1612.08739].

[35] M. Borinsky, J.A. Gracey, M.V. Kompaniets and O. Schnetz, *Five-loop renormalization of $\phi 3$ theory with applications to the Lee-Yang edge singularity and percolation theory*, *Phys. Rev. D* **103** (2021) 116024 [2103.16224].

[36] P. Fonseca and A. Zamolodchikov, *Ising field theory in a magnetic field: Analytic properties of the free energy*, hep-th/0112167.

[37] H.-L. Xu and A. Zamolodchikov, *2D Ising Field Theory in a magnetic field: the Yang-Lee singularity*, *JHEP* **08** (2022) 057 [2203.11262].

[38] H.-L. Xu and A. Zamolodchikov, *Ising field theory in a magnetic field: φ^3 coupling at $T > T_c$*, *JHEP* **08** (2023) 161 [2304.07886].

[39] V.V. Mangazeev, B. Hagan and V.V. Bazhanov, *Corner transfer matrix approach to the Yang-Lee singularity in the two-dimensional Ising model in a magnetic field*, *Phys. Rev. E* **108** (2023) 064136 [2308.15113].

[40] A. Connelly, G. Johnson, F. Rennecke and V. Skokov, *Universal Location of the Yang-Lee Edge Singularity in $O(N)$ Theories*, *Phys. Rev. Lett.* **125** (2020) 191602 [2006.12541].

[41] F. Rennecke and V.V. Skokov, *Universal location of Yang–Lee edge singularity for a one-component field theory in 1≤d≤4*, *Annals Phys.* **444** (2022) 169010 [2203.16651].

[42] F. Karsch, C. Schmidt and S. Singh, *Lee-Yang and Langer edge singularities from analytic continuation of scaling functions*, *Phys. Rev. D* **109** (2024) 014508 [2311.13530].

[43] C. Wetterich, *Exact evolution equation for the effective potential*, *Phys. Lett. B* **301** (1993) 90 [1710.05815].

[44] J. Berges, N. Tetradis and C. Wetterich, *Nonperturbative renormalization flow in quantum field theory and statistical physics*, *Phys. Rept.* **363** (2002) 223 [hep-ph/0005122].

[45] B. Delamotte, D. Mouhanna and M. Tissier, *Nonperturbative renormalization group approach to frustrated magnets*, *Phys. Rev. B* **69** (2004) 134413 [cond-mat/0309101].

[46] J. Braun, *Fermion Interactions and Universal Behavior in Strongly Interacting Theories*, *J. Phys. G* **39** (2012) 033001 [1108.4449].

[47] P. Kopietz, L. Bartosch and F. Schütz, *Introduction to the functional renormalization group*, vol. 798 (2010), 10.1007/978-3-642-05094-7.

[48] J. Berges, N. Tetradis and C. Wetterich, *Critical equation of state from the average action*, *Phys. Rev. Lett.* **77** (1996) 873 [hep-th/9507159].

[49] N. Defenu, A. Trombettoni and A. Codello, *Fixed-point structure and effective fractional dimensionality for $O(N)$ models with long-range interactions*, *Phys. Rev. E* **92** (2015) 052113 [1409.8322].

[50] A. Eichhorn, L. Janssen and M.M. Scherer, *Critical O(N) models above four dimensions: Small-N solutions and stability*, *Phys. Rev. D* **93** (2016) 125021 [1604.03561].

[51] D.F. Litim and E. Marchais, *Critical $O(N)$ models in the complex field plane*, *Phys. Rev. D* **95** (2017) 025026 [1607.02030].

[52] D. Roscher and I.F. Herbut, *Critical $O(2)$ field theory near six dimensions beyond one loop*, *Phys. Rev. D* **97** (2018) 116019 [1805.01480].

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[53] S. Yabunaka and B. Delamotte, *Why Might the Standard Large N Analysis Fail in the O(N) Model: The Role of Cusps in the Fixed Point Potentials*, Phys. Rev. Lett. **121** (2018) 231601 [1807.04681].

[54] G. De Polsi, I. Balog, M. Tissier and N. Wschebor, *Precision calculation of critical exponents in the $O(N)$ universality classes with the nonperturbative renormalization group*, Phys. Rev. E **101** (2020) 042113 [2001.07525].

[55] I. Balog, H. Chaté, B. Delamotte, M. Marohnic and N. Wschebor, *Convergence of Nonperturbative Approximations to the Renormalization Group*, Phys. Rev. Lett. **123** (2019) 240604 [1907.01829].

[56] G. Nicotra, P. Dimopoulos, L. Dini, F. Di Renzo, J. Goswami, C. Schmidt et al., *Lee-Yang edge singularities in 2+1 flavor QCD with imaginary chemical potential.*, PoS **LATTICE2021** (2022) 260 [2111.05630].

[57] K. Zambello, D.A. Clarke, P. Dimopoulos, F. Di Renzo, J. Goswami, G. Nicotra et al., *Determination of Lee-Yang edge singularities in QCD by rational approximations*, PoS **LATTICE2022** (2023) 164 [2301.03952].

[58] A. Roberge and N. Weiss, *Gauge Theories With Imaginary Chemical Potential and the Phases of QCD*, Nucl. Phys. B **275** (1986) 734.

[59] V. Vovchenko, A. Pasztor, Z. Fodor, S.D. Katz and H. Stoecker, *Repulsive baryonic interactions and lattice QCD observables at imaginary chemical potential*, Phys. Lett. B **775** (2017) 71 [1708.02852].

[60] G.A. Almasi, B. Friman, K. Morita, P.M. Lo and K. Redlich, *Fourier coefficients of the net-baryon number density and chiral criticality*, Phys. Rev. D **100** (2019) 016016 [1805.04441].

[61] V.G. Bornyakov, D.L. Boyda, V.A. Goy, A.V. Molochkov, A. Nakamura, A.A. Nikolaev et al., *New approach to canonical partition functions computation in $N_f = 2$ lattice QCD at finite baryon density*, Phys. Rev. D **95** (2017) 094506 [1611.04229].

[62] V.G. Bornyakov, N.V. Gerasimeniuk, V.A. Goy, A.A. Korneev, A.V. Molochkov, A. Nakamura et al., *Numerical study of the Roberge-Weiss transition*, Phys. Rev. D **107** (2023) 014508 [2203.06159].

[63] C. Schmidt, *Fourier coefficients of the net-baryon number density*, PoS **LATTICE2022** (2023) 159 [2301.04978].

[64] M. Bryant, C. Schmidt and V.V. Skokov, *Asymptotic behavior of the Fourier coefficients and the analytic structure of the QCD equation of state*, 2401.06489.