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Angular distributions of atmospheric leptons via two-dimensional matrix cascade equations

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The Matrix Cascade Equations (MCEQ) code is a numerical tool used to model the atmospheric lepton fluxes by solving a system of coupled differential equations for particle production, interaction, and decay at extremely low computational cost. Previous iterations of the MCEQ code relied on a longitudinal-only development of air showers, which was sufficient for modelling neutrino and muon fluxes at energies around 10 GeV and above. However, for precision calculations of atmospheric lepton angular distributions at energies below a few GeV, the lateral component of hadronic cascades becomes important. This study introduces a robust numerical technique for the combined longitudinal and angular evolution of air showers, which retains the low computational cost of the MCEQ code. We compare our numerical solutions to those obtained with the standard Monte Carlo code CORSIKA and show that our new "2D MCEQ" is sufficiently accurate. This approach enables fast two-dimensional evolution of hadronic cascades in arbitrary media and is an important bridge between the computationally efficient one-dimensional and the three-dimensional lepton flux calculations.

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1. Introduction

Neutrinos and muons are byproducts of hadronic cascades induced by interactions of cosmic rays in the Earth's atmosphere. The fluxes of few-GeV electron and muon neutrinos are of particular interest for atmospheric neutrino oscillation studies, to which they provide the main signal [1–8]. In this energy regime, the modelling of the angular evolution of air showers must be treated with care. Firstly, the trajectories of the charged particles are curved by the Earth's magnetic field; this affects both low-energy cosmic ray primaries and muons, which decay directly into neutrinos. Secondly, the angle of deflection of the secondary particles from the primary particle trajectories grows with decreasing energy and can vary from a few degrees to tens of degrees at GeV-scale energies. Both of these effects must be incorporated for a complete description of the low-energy lepton production in extensive air showers [9].

Monte Carlo simulations are a natural computational framework for modelling the full complexity of the air shower evolution via an event-by-event treatment of particle interactions and decays. The most widely used realizations of this method to date include the general-purpose codes such as Geant4 [10], FLUKA [11], MCNP [12] and PHITS [13] for particle propagation in matter, as well as CORSIKA [14] and AIRES [15] codes specialized in air shower evolution. While the Monte Carlo approach provides high level of detail as an inherent advantage, it is computationally expensive, fairly complex, and lacks sufficient flexibility for extraction of systematic uncertainties.

Another way to model particle production, interaction, and decay in the atmosphere is via a solution to the coupled transport equations (*cascade equations*). The MCEq software [16, 17] provides high-precision numerical solutions to the cascade equations, resulting in a significant speedup over the Monte Carlo approaches and the flexibility to study the impact of the systematic parameters. Since MCEq was originally written in the 1D approximation of the air shower development (longitudinal-only propagation), it did not allow for deflection of the secondary particles from the primary particle trajectories. The present study summarizes the recent efforts to extend this approach to the two-dimensional evolution of the air showers [18, 19]. This "2D MCEq" extension to the original MCEq software[1] has been benchmarked against the standard Monte Carlo code CORSIKA and made publicly available[2].

2. Overview of the Matrix Cascade Equations and the MCEq Code

The cascade equations are a form of integro-differential Boltzmann transport equations, which couple the evolution of fluxes of multiple particle species [20]. In the one-dimensional cascade theory, the evolving quantity for the particle h is the single-differential particle density n_h with respect to the kinetic energy E: $n_h(E) \equiv \frac{\mathrm{d}N_h}{\mathrm{d}E}$. Following particle interactions and decays in air, n_h changes as a function of slant depth X, i.e., the amount of material traversed. On a discrete kinetic

[1]https://github.com/mceq-project/MCEq
[2]https://github.com/kotania/MCEq/tree/2DShow/

energy grid with energy bins E_i, the cascade equation summarizes this evolution as follows:

$$\frac{\mathrm{d}n^h_{E_i}(X)}{\mathrm{d}X} = -\frac{n^h_{E_i}(X)}{\lambda^h_{\mathrm{int},E_i}} - \frac{n^h_{E_i}(X)}{\lambda^h_{\mathrm{dec},E_i}(X)} \tag{1a}$$

$$- \nabla_i[\mu^h_{E_i} n^h_{E_i}(X)] \tag{1b}$$

$$+ \sum_\ell \sum_{E^*_k \geq E^*_i} \frac{c_{\ell(E_k)\to h(E_i)}}{\lambda^\ell_{\mathrm{int},E_k}} n^\ell_{E_k}(X) + \sum_\ell \sum_{E^*_k \geq E^*_i} \frac{d_{\ell(E_k)\to h(E_i)}}{\lambda^\ell_{\mathrm{dec},E_k}(X)} n^\ell_{E_k}(X). \tag{1c}$$

As reflected in Eq. (1a), the decrease in the density of particle h in the energy bin E_i can be caused by inelastic collisions with the atmospheric nuclei or decay into other species, with the corresponding probabilities defined by the interaction length $\lambda^h_{\mathrm{int},E_i}$ and the decay length $\lambda^h_{\mathrm{dec},E_i}$. If the particle is charged, it can also lose energy via ionization as described by Eq. (1b), where $\mu_{E_i} = -\langle \frac{\mathrm{d}E}{\mathrm{d}X} \rangle \big|_{E=E_i}$ is the average stopping power per unit length and ∇_i is the energy derivative. The same particle can also be produced by other cascade species l with total energies $E^*_k \geq E^*_i$ as per Eq. (1c). The probabilities of producing the secondary h are represented as the yield coefficients $c_{l(E_k)\to h(E_i)}$ and $d_{l(E_k)\to h(E_i)}$ for interactions and decays, respectively. The MCEq cascade equation solver relies on the yield coefficients extracted from event generators (e.g. UrQMD [21], DPMJet [22, 23], Sibyll [17, 24], or EPOS-LHC [25] for hadron-nucleus collisions, and Pythia [26] for decays) and stored as matrices. In the 1D shower evolution approximation, these matrices contain the angle-integrated secondary particle yields, which are histogrammed on the combined (primary energy, secondary energy) grid. MCEq then performs the forward integration of Eq. (1) in the matrix form until the Earth's surface is reached (see [16, 17] for details).

3. Two-dimensional cascade equations and 2D MCEq

3.1 Incorporating the second (angular) dimension

The relevant quantity to evolve in two dimensions is the double-differential particle density η_h with respect to the energy E and the polar angle θ: $\eta_h(E,\theta) \equiv \frac{1}{\theta}\frac{\mathrm{d}^2 N_h(\theta)}{\mathrm{d}\theta \mathrm{d}E}$. As recently shown in [18, 19], the evolution of η_h can be modelled as a sequence of convolutions of the angular densities of the primary particles with the *convolution kernels* describing the angular deflections of the secondaries from the primary particle directions. For the secondaries of species h obtained via interactions of species l, we denote this kernel as $\varsigma_{l\to h}$ and write

$$\eta_h(E,\theta) = \eta_\ell(E,\theta_\ell) ** \varsigma_{\ell\to h}(E_\ell, E, \theta_{\ell\to h}), \tag{2a}$$

$$= \int_0^{\theta_{\max}} \eta_\ell(\theta_\ell)\, \varsigma_{\ell(E_\ell,\theta_\ell)\to h(E,\theta)} \theta_\ell \mathrm{d}\theta_\ell, \tag{2b}$$

where the "$**$" operator represents two-dimensional convolution, and $\varsigma_{l\to h}$ is to be replaced with $\delta_{l\to h}$ for the case of decays. The maximum considered angle of deflection is represented by $\theta_{\max}$, which we set to $\pi/2$ to consider only forward-going particles.

The integrals of the type (2b) could be readily included in Eq. (1) by replacing the single-differential densities/kernels with the double-differential densities/kernels. However, depending on the energy scales of hadronic interactions and decays, the widths of convolution kernels can vary by orders of magnitude (from hundredths of a degree to tens of degrees). As a result, the explicit evaluation of Eq. (2b) would require a "universal" θ grid which could accommodate both large and small angular deflections. This would be either prohibitively computationally expensive on a finely discretized linear angular grid or numerically unstable on a logarithmic grid, where the grids pre- and post-convolution would not be aligned. To avoid the complications of the 2D convolutions in the "real" (θ) space , we operate in the spectral ("frequency") domain instead. This is motivated by the existence of the *convolution theorem*, which transforms the convolutions in the real space into multiplications in the frequency space. For the 2D convolution of the azimuthally symmetric functions $\varsigma_{\ell\to h}, \delta_{\ell\to h}$, and $\eta(X, \theta_\ell)$, the correct transform enabling the use of the convolution theorem is the zeroth-order Hankel transform $\mathcal{H}$ [27]:

$$\mathcal{H}[f(\theta)](\kappa) = \int_0^\infty f(\theta) J_0(\kappa\theta)\,\theta\,\mathrm{d}\theta, \tag{3}$$

where $f(\theta)$ is a function of the continuous variable θ, κ is the spectral frequency mode ($\kappa \geq 0$), and J_0 is the zeroth-order Bessel function of the first kind[3].

The convolution theorem states that, for the azimuthally symmetric functions $f(\theta)$ and $g(\theta)$,

$$\mathcal{H}[f(\theta) ** g(\theta)] = \mathcal{H}[f(\theta)](\kappa) \cdot \mathcal{H}[g(\theta)](\kappa), \tag{4}$$

i.e., the Hankel transform of the convolution result is a product of the Hankel transforms of the input functions in the frequency space [27]. We therefore bring the convolution kernels and the angular densities of the cascade particles to the Hankel frequency space by defining their zeroth-order Hankel transforms as follows:

$$\tilde{\eta}^h_{E_i}(X, \kappa) \equiv \mathcal{H}[\eta^h_{E_i}(X, \theta)](\kappa); \tag{5a}$$

$$\tilde{\varsigma}_{\ell(E_k)\to h(E_i)}(\kappa) \equiv \mathcal{H}[\varsigma_{\ell(E_k)\to h(E_i)}(\theta)](\kappa); \tag{5b}$$

$$\tilde{\delta}_{\ell(E_k)\to h(E_i)}(\kappa) \equiv \mathcal{H}[\delta_{\ell(E_k)\to h(E_i)}(\theta)](\kappa). \tag{5c}$$

Then, we can write down the *two-dimensional cascade equation in the frequency domain*:

$$\frac{\mathrm{d}\tilde{\eta}^h_{E_i}(X,\kappa)}{\mathrm{d}X} = -\frac{\tilde{\eta}^h_{E_i}(X,\kappa)}{\lambda^h_{\mathrm{int},E_i}} - \frac{\tilde{\eta}^h_{E_i}(X,\kappa)}{\lambda^h_{\mathrm{dec},E_i}(X)} \tag{6a}$$

$$- \nabla_i[\mu^h_{E_i}\tilde{\eta}^h_{E_i}(X,\kappa)] \tag{6b}$$

$$+ \sum_{E^*_k \geq E^*_i} \sum_\ell \frac{[\tilde{\varsigma}_{\ell(E_k)\to h(E_i)} \cdot \tilde{\eta}^l_{E_k}](\kappa)}{\lambda^\ell_{\mathrm{int},E_k}} \tag{6c}$$

$$+ \sum_{E^*_k \geq E^*_i} \sum_\ell \frac{[\tilde{\delta}_{\ell(E_k)\to h(E_i)} \cdot \tilde{\eta}^\ell_{E_k}](\kappa)}{\lambda^\ell_{\mathrm{dec},E_k}(X)}, \tag{6d}$$

which is the main equation solved in "2D MCEQ".

[3]In the formal definition of $\mathcal{H}$, the upper limit of the θ integral in Eq. (3) is ∞, however we only consider the forward-going particles with $\theta \leq \pi/2$.

3.2 Practical implementation of 2D MCEq

For practical applications, the Hankel frequency grid κ is made discrete and integer-valued. Our specific implementation involves 24 logarithmically spaced integer modes between 0 and 2000, which suffices to accurately represent the angular distributions of GeV-scale atmospheric leptons [19]. During the integration of Eq. (6), the amplitude of the primary angular distribution corresponding to the mode κ is multiplied by the amplitude of the interaction/decay kernel corresponding to exactly the same mode, i.e., the modes κ_1 and κ_2 are not coupled if $\kappa_1 \neq \kappa_2$. This allows us to treat each of the $N_\kappa = 24$ equations of the 2D MCEq completely independently and solve them using the strategy analogous to that of the 1D MCEq [16, 17]. This method is illustrated in Fig. 1.

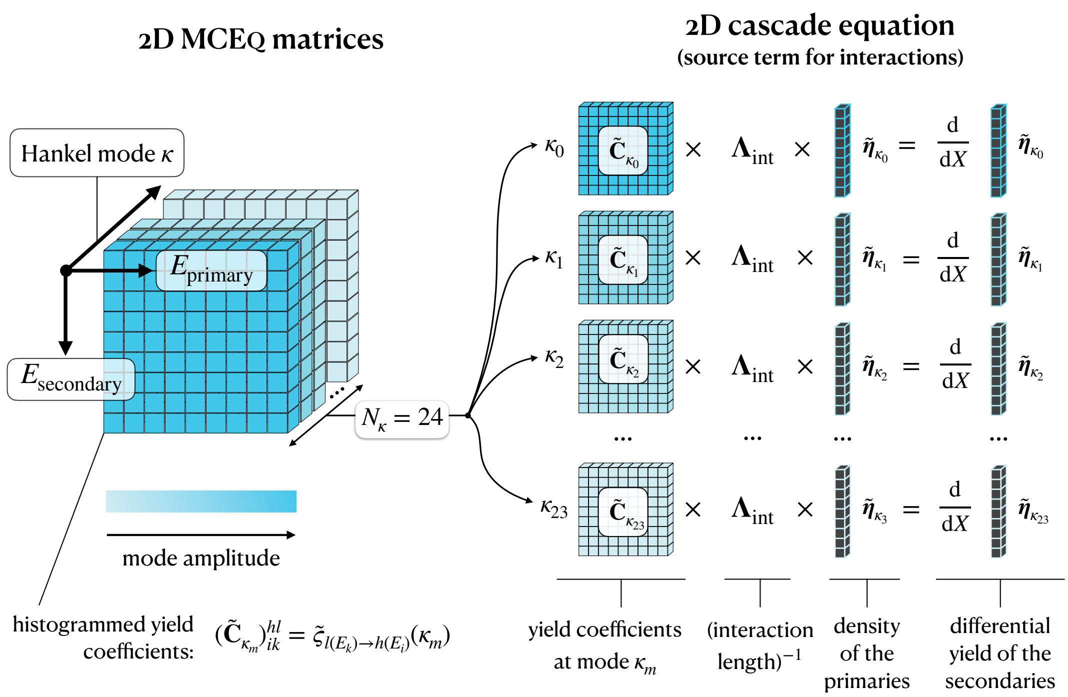

Figure 1: Schematic representation of the 2D MCEq cascade equations solver, which illustrates the computation of the source term (6c) for the Hankel-transformed angular densities $\tilde{\boldsymbol{\eta}}(\kappa)$ of all secondary particles h across all energy bins E_i. The inverse interaction lengths $(\lambda^{\ell}_{\text{int},E_k})^{-1}$ are arranged on the diagonal of $\mathbf{\Lambda}_{\text{int}}$.

4. Benchmarking against CORSIKA

To validate the two-dimensional cascade equation approach described in Section 3, we compare angular densities and energy spectra of atmospheric leptons from 2D MCEq to the equivalent distributions obtained with the CORSIKA Monte Carlo code. In Fig. 2, we present a benchmark comparison for the distributions of $\nu_\mu + \bar{\nu}_\mu$, $\nu_e + \bar{\nu}_e$, and $\mu^+ + \mu^-$ obtained at the level of the Earth's surface from a cosmic ray shower induced by a single 100 GeV proton at 30° inclination. We refer the reader to [19] for a more comprehensive overview of the cross-checks performed.

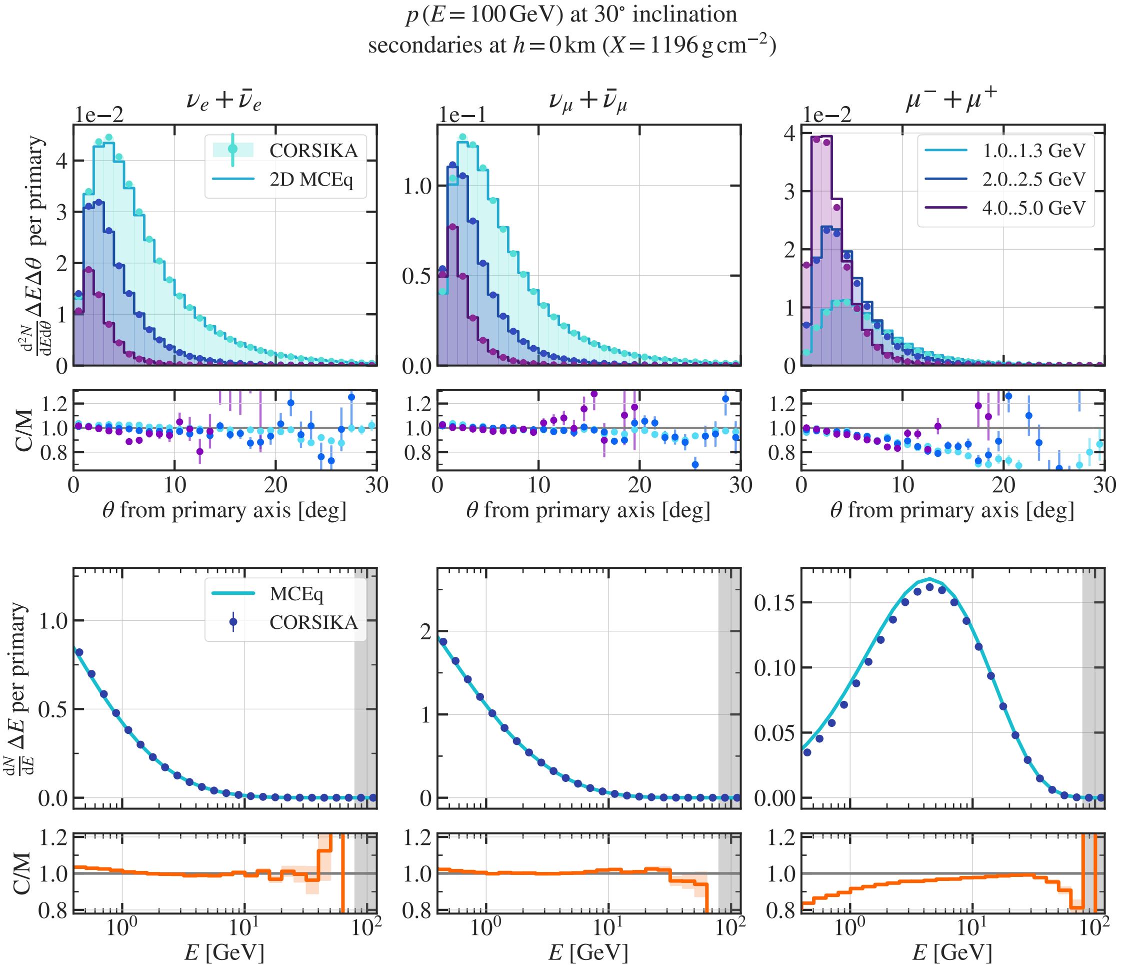

Figure 2: Comparison of the angular distributions (top) and the energy spectra (bottom) between 2D MCEq and CORSIKA. The energy spectra were extracted from the $\kappa = 0$ mode of Eq. (6). In both codes, the UrQMD hadronic model was used at $E \leq 150\,\mathrm{GeV}$, and EPOS-LHC was used at higher energies. CORSIKA uses the older UrQMD-1.3, while the 2D MCEq matrices were produced with the UrQMD-3.4 model accessed via the CHROMO tool [28][4]. The gray band indicates the region where the MCEq solution exhibits a numerical artefact due to the discretization of the delta function-like initial condition on the MCEq energy grid [19].

We find that the angular distributions of the few-GeV leptons with respect to the primary proton axis are in a very good agreement between 2D MCEq and CORSIKA. For neutrinos, the differences between the two codes are mainly statistical and reach at most 10% in the tails of the distributions. This level of agreement holds across all altitudes and energy bins considered. For muons, a characteristic tilt of the CORSIKA-to-MCEq angular distribution ratio is observed at all altitudes, reaching ∼20% in the distribution tails. The energy spectra from MCEq and CORSIKA agree within a few % in the 1–10 GeV region, which is the main energy range of interest in this study. Above 10 GeV, the difference between the two codes grows as a function of energy, which could imply difference in the treatment of hadronic interactions, e.g. the hadron yields between the different

[4] https://github.com/impy-project/chromo

interaction models (see the caption of Fig. 2).

5. Summary and outlook

In this work, we extended of MCEQ software for atmospheric lepton flux calculations to two dimensions. The new 2D MCEQ code provides an efficient numerical approach to angular evolution of hadronic cascades with broad particle physics applications. This tool considers all crucial aspects of hadronic and leptonic physics, such as inelastic interactions of hadrons with atmospheric nuclei, decays of unstable particles, energy losses, muon polarization, and muon multiple scattering.

Validation of 2D MCEQ was performed against the standard Monte Carlo code, CORSIKA. Given the very high level of agreement and a significant computational superiority over the Monte Carlo approach, 2D MCEQ provides a very appealing option for atmospheric lepton flux calculations. Our tool opens the pathway to fast exploration of the systematic uncertainties on the angular distributions of atmospheric leptons, including those associated with the hadronic interaction models and the cosmic ray primary flux. The 2D MCEQ code can further be utilized within hybrid air-shower calculation frameworks.

Future enhancements will involve the integration of three-dimensional calculations, accounting for factors such as the Earth's spherical geometry, the initial angular distribution of cosmic ray primaries, the geomagnetic cutoff for these primaries, and the deflection of cascade secondaries within the geomagnetic field.

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