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General $O(D)$-equivariant fuzzy hyperspheres via confining potentials and energy cutoffs

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We summarize our recent construction [1–3] of new fuzzy hyperspheres S^d_Λ of arbitrary dimension $d \in \mathbb{N}$ covariant under the *full* orthogonal group $O(D)$, $D = d+1$. We impose a suitable energy cutoff on a quantum particle in $\mathbb{R}^D$ subject to a confining potential well $V(r)$ with a very sharp minimum on the sphere of radius $r = 1$; the cutoff and the depth of the well diverge with $\Lambda \in \mathbb{N}$. Consequently, the commutators of the Cartesian coordinates $\overline{x}^i$ are proportional to the angular momentum components L_{ij}, as in Snyder's noncommutative spaces. The $\overline{x}^i$ generate the whole algebra of observables $\mathcal{A}_\Lambda$ and thus the whole Hilbert space $\mathcal{H}_\Lambda$ when applied to any state. $\mathcal{H}_\Lambda$ carries a reducible representation of $O(D)$ isomorphic to the space of harmonic homogeneous polynomials of degree Λ in the Cartesian coordinates of (commutative) $\mathbb{R}^{D+1}$; the latter carries an irreducible representation $\boldsymbol{\pi}_\Lambda$ of $O(D+1) \supset O(D)$. Moreover, $\mathcal{A}_\Lambda$ is isomorphic to $\boldsymbol{\pi}_\Lambda(Uso(D+1))$. We identify the subspace $C_\Lambda \subset \mathcal{A}_\Lambda$ spanned by fuzzy spherical harmonics. We interpret $\{\mathcal{H}_\Lambda\}_{\Lambda\in\mathbb{N}}$, $\{C_\Lambda\}_{\Lambda\in\mathbb{N}}$ as fuzzy deformations of the space $\mathcal{H}_s \equiv \mathcal{L}^2(S^d)$ of square integrable functions and the space $C(S^d)$ of continuous functions on S^d respectively, $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ as fuzzy deformation of the associated algebra $\mathcal{A}_s$ of observables, because they resp. go to $\mathcal{H}_s, C(S^d), \mathcal{A}_s$ as Λ diverges (with fixed $\hbar$). With suitable $\hbar = \hbar(\Lambda) \xrightarrow{\Lambda\to\infty} 0$, in the same limit $\mathcal{A}_\Lambda$ goes to the (algebra of functions on the) Poisson manifold T^*S^d; more formally, $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ yields a fuzzy quantization of a coadjoint orbit of $O(D+1)$ that goes to the classical phase space T^*S^d. These models might be useful in quantum field theory, quantum gravity or condensed matter physics.

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PoS(CORFU2022)344

1. Introduction and preliminaries

In the past decades noncommutative space(time) algebras have been introduced and studied as fundamental or effective arenas for regularizing ultraviolet (UV) divergences in quantum field theory (QFT) (see e.g. [4]), reconciling Quantum Mechanics and General Relativity in a satisfactory Quantum Gravity (QG) theory (see e.g. [5]), unifying fundamental interactions (see e.g. [6, 7]). Noncommutative Geometry (NCG) [8–11], i.e. differential geometry on noncommutative spaces, has become a sophisticated machinery. In particular, fuzzy (noncommutative) spaces have raised a big interest as a non-perturbative technique in QFT based on a finite discretization alternative to the lattice ones. A fuzzy space is a sequence $\{\mathcal{A}\}_{n\in\mathbb{N}}$ of *finite-dimensional* algebras such that $\mathcal{A}_n \overset{n\to\infty}{\longrightarrow} \mathcal{A}$ ≡algebra of regular functions on an ordinary manifold, with $\dim(\mathcal{A}_n) \overset{n\to\infty}{\longrightarrow} \infty$. Contrary to lattices, $\mathcal{A}_n$ can carry representations of Lie, beside discrete, groups. Fuzzy spaces can be used also to discretize internal (e.g. gauge) degrees of freedom (see e.g. [12]), or as a new tool in string and D-brane theories (see e.g. [13, 14]). In the seminal Madore-Hoppe Fuzzy Sphere (FS) of dimension $d = 2$ [15, 16] $\mathcal{A}_n \simeq M_n(\mathbb{C})$. $\mathcal{A}_n$ is generated by coordinates x^i ($i = 1, 2, 3$) fulfilling

$$[x^i, x^j] = \frac{2i}{\sqrt{n^2-1}}\varepsilon^{ijk}x^k, \qquad r^2 \equiv x^i x^i = 1, \qquad n \in \mathbb{N} \setminus \{1\}; \tag{1}$$

these are related via $x^i = 2L_i/\sqrt{n^2-1}$ to the standard basis $\{L_i\}_{i=1}^3$ of $so(3)$ in the unitary irreducible representation (irrep) (π^l, V^l) of dimension $n = 2l+1$ [i.e. V^l is the eigenspace of the Casimir $\boldsymbol{L}^2 = L_i L_i$ with eigenvalue $l(l+1)$]. Fuzzy spheres S^d of dimension $d = 4$ and any $d \geq 3$ were introduced resp. in [17], [18]; other versions of $d = 3, 4$ or $d \geq 3$ in [19–22]. Unfortunately, while for the $d = 2$ FS [15, 16] $\mathcal{A}_n$ admits a basis of spherical harmonics, for the $d > 2$ fuzzy S^d a product of spherical harmonics is not a combination thereof, but an element in a larger algebra $\mathcal{A}_n$.

The Hilbert space of a (zero-spin) quantum particle on configuration space S^d and the space of continuous functions on S^d carry a (same) *reducible* representation of $O(D)$, $D \equiv d+1$; they decompose into carrier spaces of irreducible representations (irreps) as follows

$$\mathcal{L}^2(S^d) \simeq \bigoplus_{l=0}^{\infty} V_D^l \simeq C(S^d), \tag{2}$$

where V_D^l is an eigenspace of the quadratic Casimir $\boldsymbol{L}^2$ with eigenvalue

$$E_l \equiv l(l+D-2) \tag{3}$$

($V_3^l \equiv V^l$); $C(S^d)$ acts an algebra of bounded operators on $\mathcal{L}^2(S^d)$. On the contrary, each of the mentioned fuzzy hyperspheres is based on a sequence parametrized by n either of irreps of $Spin(D)$ (so that $r^2 \propto \boldsymbol{L}^2$ is 1) [15–20], or of direct sums of small bunches of such irreps [21, 22]. In either case, even excluding the n's for which the associated representation of $O(D)$ is only *projective*, the carrier space does not go to (2) as $n \to \infty$; hence, interpreting these fuzzy spheres as fuzzy configuration spaces S^d (and the x^i as spatial coordinates) becomes questionable. Moreover, relations (1) for the Madore-Hoppe FS are equivariant under $SO(3)$, but not under the whole $O(3)$, e.g. not under parity $x^i \mapsto -x^i$. These difficulties are overcome by our recent fully $O(D)$-equivariant fuzzy quantizations [1, 3] S^d_Λ of spheres S^d of arbitrary dimension $d = D-1 \in \mathbb{N}$ (thought as configuration spaces) and of T^*S^d (thought as phase spaces), which we summarize here (the cases $d = 1, 2$ had been treated in [2, 23]); in particular, we recover (2) as $\Lambda \to \infty$.

Our fuzzy quantization uses: 1. the *projection* of a quantum theory $\mathcal{T}$ on $\mathbb{R}^D$ below an *energy cutoff*; 2. a *dimensional reduction* induced by a *confining potential* on $S^d \subset \mathbb{R}^D$. One can apply it to quantize also other submanifolds $M \subset \mathbb{R}^D$. Given a generic quantum theory $\mathcal{T}$ with Hilbert space $\mathcal{H}$, algebra of observables on $\mathcal{H}$ (or with a domain dense in $\mathcal{H}$) $\mathcal{A} \equiv \mathrm{Lin}(\mathcal{H})$, Hamiltonian $H \in \mathcal{A}$, for any subspace $\overline{\mathcal{H}} \subset \mathcal{H}$ preserved by H let $\overline{P} : \mathcal{H} \mapsto \overline{\mathcal{H}}$ be the associated projector and

$$\overline{\mathcal{A}} \equiv \mathrm{Lin}(\overline{\mathcal{H}}) = \{\overline{A} \equiv \overline{P}A\overline{P} \mid A \in \mathcal{A}\}.$$

By construction $\overline{H} = \overline{P}H = H\overline{P}$. The projected Hilbert space $\overline{\mathcal{H}}$, algebra of observables $\overline{\mathcal{A}}$ and Hamiltonian $\overline{H}$ provide a new quantum theory $\overline{\mathcal{T}}$ [24]; we will ascribe the observable $\overline{A}$ the same physical meaning of A in $\mathcal{T}$. If $\overline{\mathcal{H}}, H$ are invariant under some group G, then $\overline{P}, \overline{\mathcal{A}}, \overline{H}, \overline{\mathcal{T}}$ will be as well. The relations among the generators of $\overline{\mathcal{A}}$ differ from those among the generators of $\mathcal{A}$. In particular, if $\mathcal{T}$ is based on commuting coordinates x^i (commutative space) this will be in general no longer true for $\overline{\mathcal{T}}$: $[\overline{x}^i, \overline{x}^j] \neq 0$, and we have generated a quantum theory on a NC space. In particular, if $\overline{\mathcal{H}} \subset \mathcal{H}$ is characterized by energies $E \leq \overline{E}$ below a certain cutoff $\overline{E}$, then $\overline{\mathcal{T}}$ is a low-energy approximation of $\mathcal{T}$ preserved by the dynamical evolution ruled by H. $\overline{\mathcal{T}}$ may be used as an effective theory for $E \leq \overline{E}$, or may even help to figure out a new theory $\mathcal{T}'$ valid for all E if at $E > \overline{E}$ physics is not accounted for by $\mathcal{T}$. If $\overline{\mathcal{T}}$ describes an ordinary (for simplicity, zero-spin) quantum particle in the Euclidean (configuration) space $\mathbb{R}^D$, then $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^d)$. If $H = T+V$, with kinetic energy T and a confining potential $V(x)$, then the classical region $\mathcal{B}_{\overline{E}}$ in phase space fulfilling $H(x,p) \leq \overline{E}$ and the one $v_{\overline{E}} \subset \mathbb{R}^D$ in configuration space fulfilling $V \leq \overline{E}$ are bounded at least for sufficiently small $\overline{E}$, and the dimension $\dim(\overline{\mathcal{H}}) \approx \mathrm{Vol}(\mathcal{B}_{\overline{E}})/h^D$ of $\overline{\mathcal{H}}$ is finite. In the sequel we rescale x, p, H, V so that they are dimensionless and, denoting by Δ the Laplacian in $\mathbb{R}^D$,

$$H = -\Delta + V. \tag{4}$$

We choose a sequence of pairs $(V, \overline{E})$ satisfying the following requirements. $V = V(r)$ has a very sharp minimum, parametrized by a very large $k \equiv V''(1)/4$, on the sphere $S^d \subset \mathbb{R}^D$ of radius $r = 1$; we fix $V_0 \equiv V(1)$ so that the ground state ψ_0 has zero energy, $E_0 = 0$. We choose $\overline{E}$ fulfilling first of all the condition $V(r) \simeq V_0 + 2k(r-1)^2$ in $v_{\overline{E}}$, so that we can approximate $v_{\overline{E}}$ by the spherical shell $|r-1| \leq \sqrt{\frac{\overline{E}-V_0}{2k}}$ and the potential by a harmonic one. If $\overline{E}-V_0$ and k diverge, while their ratio goes to zero, then in this limit $v_{\overline{E}} \to S^d$, $\dim(\overline{\mathcal{H}}) \to \infty$, and we recover quantum mechanics on S^d.

Let $x \equiv (x^1, \ldots x^D)$ be Cartesian coordinates of $\mathbb{R}^D$, $r^2 = x^i x^i$, $\partial_i \equiv \partial/\partial x^i$; $\Delta = \partial_i \partial_i$ decomposes as

$$\Delta = \partial_r^2 \; + \; (D-1)\, r^{-1} \partial_r \; - \; r^{-2} \boldsymbol{L}^2, \tag{5}$$

where $\partial_r \equiv \partial/\partial r$ and $\boldsymbol{L}^2 \equiv L_{ij} L_{ij}/2$ is the square angular momentum (in normalized units), i.e. the quadratic Casimir of $Uso(D)$ and the Laplacian on the sphere S^d, the angular momentum components $L_{ij} \equiv i(x^j \partial_i - x^i \partial_j)$ are vector fields tangent to all spheres $r = \text{const}$ satisfying

$$[L_{ij}, L_{hk}] = i\left(L_{jk}\delta_{hi} + L_{ih}\delta_{kj} - L_{jh}\delta_{ki} - L_{ik}\delta_{hj}\right), \qquad [L_{ij}, S] = 0, \tag{6}$$

$$[iL_{ij}, v^h] = v^i \delta^h_j - v^j \delta^h_i, \qquad \varepsilon^{i_1 i_2 i_3 \ldots i_D} x^{i_1} L_{i_2 i_3} = 0, \tag{7}$$

where S is any scalar and v^h are the components of any vector depending on x^h, ∂_h, in particular $v^h = x^h, \partial_h$. The Ansatz $\boldsymbol{\psi} = f(r) Y_l(\boldsymbol{\theta})$, with $f(r) = r^{-d/2} g(r)$ and $Y_l \in V_D^l$ an E_l-eigenfunction of $\boldsymbol{L}^2$, transforms the Schrödinger PDE $H\boldsymbol{\psi} = E\boldsymbol{\psi}$ into the Fuchsian ODE in the unknown $g(r)$

$$-g''(r) + \left[V(r) + \frac{D^2-4D+3+4l(l+D-2)}{4}\, r^{-2}\right] g(r) = Eg(r) \tag{8}$$

(by similar product Ansätze one can reduce numerous different PDEs to ODEs, see e.g. [25]). Requiring $\lim_{r\to 0^+} r^2V(r) > 0$, $f(0) = 0$, we make H self-adjoint. As $V(r)$ is very large outside $v_{\overline{E}}$, there $g, f, \boldsymbol{\psi}$ are negligibly small, and the lowest eigenvalues E are at leading order those of the 1-dimensional harmonic oscillator approximation [3] of (8)

$$-g''(r) + g(r)k_l\,(r-\widetilde{r}_l)^2 = \widetilde{E}_l g(r), \tag{9}$$

obtained neglecting terms $O\big((r-1)^3\big)$ in the Taylor expansions of $1/r^2, V(r)$ about $r=1$. Here

$$\widetilde{r}_l \equiv 1 + \frac{b(l,D)}{3b(l,D)+2k}, \qquad \widetilde{E}_l \equiv E - V_0\frac{2b(l,D)[k+b(l,D)]}{3b(l,D)+2k},$$
$$k_l \equiv 2k + 3b(l,D), \qquad b(l,D) \equiv \frac{D^2-4D+3+4l(l+D-2)}{4}.$$

The square-integrable solutions of (9) $g_{n,l}(r)$ lead to

$$f_{n,l}(r) = M_{n,l}\ r^{-d/2}\, e^{-\sqrt{k_l}(r-\widetilde{r}_l)^2/2} \cdot H_n\left((r-\widetilde{r}_l)\sqrt[4]{k_l}\right) \qquad \text{with } n \in \mathbb{N}_0; \tag{10}$$

here $M_{n,l}$ are normalization constants and H_n are the Hermite polynomials. The corresponding 'eigenvalues' in (9) $\widetilde{E}_{n,l} = (2n+1)\sqrt{k_l}$ lead to energies $E_{n,l} = (2n+1)\sqrt{k_l} + V_0 + \frac{2b(l,D)[k+b(l,D)]}{3b(l,D)+2k}$. As said, we fix V_0 requiring that the lowest one $E_{0,0}$ be zero; this implies $V_0 = -\sqrt{2k} - b(0,D) - \frac{3b(0,D)}{2\sqrt{2k}} + O\big(k^{-1/2}\big)$, and the expansions of $E_{n,l}$ and $\widetilde{r}_l$ at leading order in k become

$$E_{n,l} = l(l+D-2) + 2n\sqrt{2k} + O\big(k^{-2}\big), \qquad \widetilde{r}_l = 1 + b(l,D)/2k + O\big(k^{-2}\big). \tag{11}$$

$E_{0,l}$ coincide at lowest order with the desired eigenvalues E_l (coloured blue) of $\boldsymbol{L}^2$, while if $n>0$ $E_{n,l}$ diverge as $k\to\infty$ (due to the red part); to exclude all states with $n>0$ (i.e., to 'freeze' radial oscillations, so that all corresponding classical trajectories are circles; this can be considered as a *quantum* version of the *constraint* $r=1$) we impose the energy cutoff

$$E_{n,l} \le \overline{E}(\Lambda) \equiv \Lambda(\Lambda+D-2) < 2\sqrt{2k}, \qquad \Lambda\in\mathbb{N}. \tag{12}$$

The right inequality is satisfied prescribing a suitable dependence $k(\Lambda)$, e.g. $k(\Lambda) \equiv [\Lambda(\Lambda+D-2)]^2$; the left one is satisfied setting $n=0$ and $l\le\Lambda$. We rename $\overline{H}, \overline{\mathcal{H}}, \overline{P}, \overline{\mathcal{A}}, \overline{\mathcal{T}}$ as $H_\Lambda, \mathcal{H}_\Lambda, P_\Lambda, \mathcal{A}_\Lambda, \mathcal{T}_\Lambda$. $\mathcal{T}_\Lambda$ is $O(D)$-equivariant. We end up with eigenfunctions and eigenvalues (at leading order in $1/\Lambda$)

$$\boldsymbol{\psi}_l(r,\boldsymbol{\theta}) = f_l(r)\,Y_l(\boldsymbol{\theta}), \qquad H_\Lambda\boldsymbol{\psi}_l = E_l\,\boldsymbol{\psi}_l, \qquad l=0,1,\ldots,\Lambda, \tag{13}$$

abbreviating $f_l \equiv f_{0,l}$. Hence $\mathcal{H}_\Lambda$ decomposes into irreps of $O(D)$ (and eigenspaces of $\boldsymbol{L}^2, H_\Lambda$) as

$$\mathcal{H}_\Lambda = \bigoplus_{l=0}^{\Lambda}\mathcal{H}_\Lambda^l, \qquad \mathcal{H}_\Lambda^l \equiv f_l(r)\,V_D^l. \tag{14}$$

As $\Lambda\to\infty$ the spectrum $\{E_l\}_{l=0}^{\Lambda}$ of H_Λ goes to the whole spectrum $\{E_l\}_{l\in\mathbb{N}_0}$ of $\boldsymbol{L}^2$, and we recover (2). We can express the projectors $P_\Lambda^l : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda^l$ as the following polynomials in $\overline{\boldsymbol{L}}^2$:

$$P_\Lambda^l = \prod_{n=0,n\neq l}^{\Lambda}\frac{\overline{\boldsymbol{L}}^2-E_n}{E_l-E_n}. \tag{15}$$

PoS(CORFU2022)344

The space V_D^l consists of harmonic homogeneous polynomials of degree l in the x^i restricted to the sphere S^d. In section 2 we show: i) how to explicitly determine V_D^l, as well as the action of L_{hk} and $t^h \equiv x^h/r$ on V_D^l, applying the trace-free completely symmetric projector $\mathcal{P}^l$ of $\left(\mathbb{R}^D\right)^{\otimes l}$ to the homogeneous polynomials of degree l in x^i; ii) that not only $\mathcal{H}_\Lambda$, but also V_{D+1}^Λ decomposes into irreps of $O(D)$ as follows $V_{D+1}^\Lambda \simeq \bigoplus_{l=0}^\Lambda V_D^l$. In section 3 we write down the relations fulfilled by $\overline{x}^i, \overline{L}_{hk}$ and point out that: the $*$-algebra $\mathcal{A}_\Lambda$ generated by the latter is also generated by the $\overline{x}^i$ alone; ii) the unitary irrep of $\mathcal{A}_\Lambda$ on $\mathcal{H}_\Lambda$ is isomorphic to the irrep $\boldsymbol{\pi}_\Lambda$ of $Uso(D+1)$ on V_{D+1}^Λ. In section 4 we show in which sense $\mathcal{H}_\Lambda, \mathcal{A}_\Lambda$ go to $\mathcal{H}, \mathcal{A}$ as $\Lambda \to \infty$, in particular how one can recover the multiplication operator $f\cdot \in C(S^d) \subset \mathcal{A}$ of wavefunctions in $\mathcal{L}^2(S^d)$ by a continuous function f as the strong limit of a suitable sequence $f_\Lambda \in \mathcal{A}_\Lambda$. In section 5 we discuss our results and possible developments in comparison with the literature; in particular, we point out that with a suitable $\hbar(\Lambda)$ our pair $(\mathcal{H}_\Lambda, \mathcal{A}_\Lambda)$ can be seen as a fuzzy quantization of a coadjoint orbit of $O(D)$ that can be identified with the cotangent space T^*S^d, the classical phase space over the d-dimensional sphere.

2. Representations of $O(D)$ via polynomials in x^i, $t^i \equiv x^i/r$

Let $\mathbb{C}[x^1, ..., x^D] = \bigoplus_{l=0}^\infty W_D^l$ be the decomposition of the space of complex polynomial functions on $\mathbb{R}^D$ into subspaces W_D^l of homogeneous ones of degree l. If $l \geq 2$ then W_D^l carries a reducible representation of $O(D)$, as well as $Uso(D)$, because by (6b) the subspace $r^2 W_D^{l-2} \subset W_D^l$ carries a smaller one. The 'trace-free' component $\check{V}_D^l$ in the decomposition $W_D^l = r^2 W_D^{l-2} \oplus \check{V}_D^l$ carries the irrep $\boldsymbol{\pi}_D^l$ of $Uso(D)$ and $O(D)$ characterized by the highest eigenvalue of $\boldsymbol{L}^2$ within W_D^l, namely E_l. In fact, for all $h,k \in \{1, ..., D\}$ $X_{l,\pm}^{hk} \equiv (x^h \pm i x^k)^l \in W_D^l$ are eigenvectors of $\boldsymbol{L}^2$ with eigenvalue E_l, of L_{hk} with eigenvalue $\pm l$, and of Δ with eigenvalue 0. Hence $X_{l,+}^{hk}, X_{l,-}^{hk}$ can be used as the highest and lowest weight vectors of $(\boldsymbol{\pi}_D^l, \check{V}_D^l)$ [1]. Since all the L_{ij} commute with Δ, $\check{V}_D^l$ can be characterized also as the subspace of W_D^l that is annihilated by Δ. A complete set in $\check{V}_D^l$ consists of trace-free homogeneous polynomials $X_l^{i_1 i_2 \dots i_l}$, which we obtain below applying the completely symmetric trace-free projector $\mathcal{P}^l$ to the monomials $x^{i_1} x^{i_2} \dots x^{i_l}$. We slightly enlarge $\mathbb{C}[x^1, ... x^D]$ by new scalar generators r, r^{-1} fulfilling the relations $r^2 = x^i x^i$, $r r^{-1} = 1$. Its elements

$$t^i \equiv r^{-1} x^i, \qquad\qquad T_{l,\pm}^{hk} \equiv (t^h \pm i t^k)^l = r^{-l} X_{l,\pm}^{hk} \tag{16}$$

fulfill the following relations: i) $t^i t^i = 1$, which characterizes the coordinates of points of S^d; hence $V_D^l \equiv r^{-l} \check{V}_D^l$ can be seen as the restriction of $\check{V}_D^l$ to S^d. ii) $T_{l,\pm}^{hk} \in V_D^l$ are eigenvectors of $\boldsymbol{L}^2$ with eigenvalue E_l and of L_{hk} with eigenvalue $\pm l$; hence $T_{l,+}^{hk}, T_{l,-}^{hk}$ can be used as the highest and lowest weight vectors of $(\boldsymbol{\pi}_D^l, V_D^l)$. We denote by Pol_D the algebra of complex polynomials in the t^i, by Pol_D^Λ the subspace of polynomials of degree Λ, by $P^\Lambda : Pol_D \to Pol_D^\Lambda$ the corresponding projector. Pol_D endowed with the scalar product $\langle T, T' \rangle \equiv \int_{S^d} d\alpha\, T^* T'$ is a pre-Hilbert space, whose completion is $\mathcal{L}^2(S^d)$; here $d\alpha = \varepsilon^{i_1 \dots i_D} x^{i_1} dx^{i_2} \dots dx^{i_D}$ is the $O(D)$-invariant measure on S^d. We extend P^Λ to all of $\mathcal{L}^2(S^d)$ by continuity in the norm of the latter. Also Pol_D^Λ, V_D^l are Hilbert subspaces of $\mathcal{L}^2(S^d)$. $Pol_D^\Lambda = W_D^\Lambda r^{-\Lambda} \oplus W_D^{\Lambda-1} r^{1-\Lambda}$ carries a reducible representation of $O(D)$ [and $Uso(D)$] that splits into irreps as $Pol_D^\Lambda = \bigoplus_{l=0}^\Lambda V_D^l$. One finds $\mathcal{H}_\Lambda \simeq Pol_D^\Lambda \simeq V_{D+1}^\Lambda$ as $Uso(D)$ representations. The first isomorphism follows from (14), the second from section 2.2.

PoS(CORFU2022)344

2.1 $O(D)$-irreps via trace-free completely symmetric projectors

Let $(\pi, \mathcal{E})$ be the fundamental (D-dimensional irreducible unitary) representation of $Uso(D)$ and $O(D)$; the carrier space $\mathcal{E}$ is isomorphic to V_D^1. As a vector space $\mathcal{E} \simeq \mathbb{R}^D$; the set of Cartesian coordinates $x \equiv (x^1, ...x^D) \in \mathbb{R}^D$ can be seen as the set of components of an element of $\mathcal{E}$ with respect to (w.r.t.) an orthonormal basis. The permutator on $\mathcal{E}^{\otimes^2} \equiv \mathcal{E} \otimes \mathcal{E}$ is defined via $\mathsf{P}(u \otimes v) = v \otimes u$ and linearly extended. In all bases it is represented by the $D^2 \times D^2$ matrix $\mathsf{P}^{hi}_{jk} = \delta^h_k \delta^i_j$. The symmetric and antisymmetric projectors $\mathcal{P}^+, \mathcal{P}^-$ on $\mathcal{E}^{\otimes^2}$ are obtained as

$$\mathcal{P}^{\pm} = \frac{1}{2}\left(\mathbf{1}_{D^2} \pm \mathsf{P}\right). \tag{17}$$

Here and below we denote by $\mathbf{1}_{D^l}$ the identity operator on $\mathcal{E}^{\otimes^l}$; in all bases it is represented by the $D^l \times D^l$ matrix $\mathbf{1}_{D^l}{}^{h_1...h_l}_{i_1...i_l} \equiv \delta^{h_1}_{i_1}...\delta^{h_l}_{i_l}$. $\mathcal{P}^-\mathcal{E}^{\otimes^2}$ carries an irrep under $O(D)$, while $\mathcal{P}^+\mathcal{E}^{\otimes^2}$ is the direct sum of two irreps: the 1-dimensional *trace* and the $\frac{1}{2}(D-1)(D+2)$-dimensional *trace-free symmetric* ones. The associated projectors $\mathcal{P}^t, \mathcal{P}^s$ from $\mathcal{E}^{\otimes^2}$ are given by

$$\mathcal{P}^t{}^{ij}_{kl} = \frac{1}{D}\delta^{ij}\delta_{kl}, \qquad \mathcal{P}^s = \mathcal{P}^+ - \mathcal{P}^t = \frac{1}{2}\left(\mathbf{1}_{D^2} + \mathsf{P}\right) - \mathcal{P}^t\,; \tag{18}$$

here and below we adopt an orthonormal basis of $\mathcal{E}$ for the matrix representation of $\mathcal{P}^t$. Hence $\mathcal{P}^t{}^{ij}_{kl}x^ix^j = \delta^{ij}r^2/D$. These projectors satisfy the equations $\mathcal{P}^\alpha\mathcal{P}^\beta = \mathcal{P}^\alpha\delta^{\alpha\beta}$, $\sum_\alpha \mathcal{P}^\alpha = \mathbf{1}_{D^2}$, where $\alpha, \beta = -, s, t$. $\mathsf{P}, \mathcal{P}^t$ are symmetric matrices, i.e. invariant under transposition ^{T}, and therefore also the other projectors are, $\mathsf{P}^T = \mathsf{P}$, $\mathcal{P}^{\alpha T} = \mathcal{P}^\alpha$. In the sequel we abbreviate $\mathcal{P} \equiv \mathcal{P}^s$. Given a (linear) operator M on $\mathcal{E}^{\otimes^n}$, for all integers l, h with $l > n$, and $1 \le h \le l+1-n$ we denote by $M_{h(h+1)...(h+n-1)}$ the operator on $\mathcal{E}^{\otimes^l}$ acting as the identity on the first $h-1$ and the last $l+1-n-h$ tensor factors, and as M in the remaining central ones. For instance, if $M = \mathsf{P}$ and $l = 3$ we have $\mathsf{P}_{12} = \mathsf{P} \otimes \mathbf{1}_D$, $\mathsf{P}_{23} = \mathbf{1}_D \otimes \mathsf{P}$. All the projectors $A = \mathcal{P}^+, \mathcal{P}^-, \mathcal{P}, \mathcal{P}^t$ fulfill the relations

$$A_{12}\,\mathsf{P}_{23}\,\mathsf{P}_{12} = \mathsf{P}_{23}\,\mathsf{P}_{12}\,A_{23}, \tag{19}$$

$$D\,\mathcal{P}^t_{23}\mathcal{P}^t_{12} = \mathsf{P}_{12}\mathsf{P}_{23}\mathcal{P}^t_{12}, \qquad D\mathsf{P}_{12}\mathcal{P}^t_{23}\mathcal{P}^t_{12} = \mathsf{P}_{23}\mathcal{P}^t_{12}, \tag{20}$$

$$D\,\mathcal{P}^t_{12}\mathcal{P}^t_{23} = \mathsf{P}_{23}\mathsf{P}_{12}\mathcal{P}^t_{23}, \qquad D\,\mathsf{P}_{23}\mathcal{P}^t_{12}\mathcal{P}^t_{23} = \mathsf{P}_{12}\mathcal{P}^t_{23}, \tag{21}$$

$$D\,\mathcal{P}^t_{23}\mathcal{P}^t_{12} = \mathcal{P}^t_{23}\mathsf{P}_{12}\mathsf{P}_{23}, \qquad D\,\mathcal{P}^t_{23}\mathcal{P}^t_{12}\mathsf{P}_{23} = \mathcal{P}^t_{23}\mathsf{P}_{12}; \tag{22}$$

Eq. (19-22) hold also for $l > 3$, e.g. for all $2 \le h \le l-1$

$$A_{(h-1)h}\,\mathsf{P}_{h(h+1)}\,\mathsf{P}_{(h-1)h} = \mathsf{P}_{h(h+1)}\,\mathsf{P}_{(h-1)h}\,A_{h(h+1)}. \tag{23}$$

The *completely symmetric trace-free* projectors $\mathcal{P}^l$ generalize $\mathcal{P}^2 \equiv \mathcal{P}$ to all $l > 2$. $\mathcal{P}^l$ projects $\mathcal{E}^{\otimes^l}$ to the carrier space of the l-fold completely symmetric irrep of $Uso(D)$, isomorphic to $\check{V}^l_D, V^l_D$, therein contained. It is uniquely characterized by the following properties: for $n = 1, ..., l-1$,

$$\mathcal{P}^l\mathcal{P}^-_{n(n+1)} = 0, \qquad \mathcal{P}^-_{n(n+1)}\mathcal{P}^l = 0, \tag{24}$$

$$\mathcal{P}^l\mathcal{P}^t_{n(n+1)} = 0, \qquad \mathcal{P}^t_{n(n+1)}\mathcal{P}^l = 0, \tag{25}$$

$$\left(\mathcal{P}^l\right)^2 = \mathcal{P}^l, \tag{26}$$

Eq.s (25) amount to $\mathcal{P}^{l\,i_1\ldots i_l}_{j_1\ldots j_l}\delta^{j_nj_{n+1}}=0$, $\delta_{i_ni_{n+1}}\mathcal{P}^{l\,i_1\ldots i_l}_{j_1\ldots j_l}=0$. Proposition 3.2 of [1] yields a recursive construction of the projectors $\mathcal{P}^l$ (mimicking that of the quantum group $U_qso(D)$ covariant symmetric projectors of Proposition 1 of [26]): $\mathcal{P}^{l+1}$ can be expressed as a polynomial in the permutators $\mathsf{P}_{12},\ldots,\mathsf{P}_{(l-1)l}$ and trace projectors $\mathcal{P}^t_{12},\ldots,\mathcal{P}^t_{(l-1)l}$ through either recursive relation

$$\mathcal{P}^{l+1} = \mathcal{P}^l_{12\ldots l}M_{l(l+1)}\mathcal{P}^l_{12\ldots l}, \tag{27}$$

$$= \mathcal{P}^l_{2\ldots(l+1)}M_{12}\mathcal{P}^l_{2\ldots(l+1)}, \tag{28}$$

$M\equiv M(l+1)=\frac{1}{l+1}\left[\mathbf{1}_{D^2}+l\mathsf{P}-\frac{2Dl}{D+2l-2}\mathcal{P}^t\right]$. All $\mathcal{P}^l$ are symmetric, $(\mathcal{P}^l)^T=\mathcal{P}^l$. Let

$$X^{i_1\ldots i_l}_l\equiv\mathcal{P}^{l\,i_1\ldots i_l}_{j_1\ldots j_l}x^{j_1}\ldots x^{j_l},\qquad T^{i_1i_2\ldots i_l}_l\equiv r^{-l}X^{i_1i_2\ldots i_l}_l=\mathcal{P}^{l\,i_1\ldots i_l}_{j_1\ldots j_l}t^{j_1}\ldots t^{j_l}. \tag{29}$$

Using (25) one easily shows that $\Delta X^{i_1\ldots i_l}_l=0$: the harmonic homogeneous x^i-polynomials $X^{i_1\ldots i_l}_l$ make up a complete set of $\check{V}^l_D$ (not a basis, because they are invariant under permutations of $(i_1\ldots i_l)$ and fulfill $\delta_{i_ni_{n+1}}X^{i_1\ldots i_l}_l=0$, $n=1,\ldots,l-1$). Similarly, the t^i-polynomials $T^{i_1\ldots i_l}_l$ make up a complete set $\mathcal{T}_l$ (but not a basis) of V^l_D that is easier to work with than the basis of spherical harmonics. Moreover, $\boldsymbol{L}^2$, iL_{hk} and the multiplication operators $t^h\cdot$ act on the $T^{i_1\ldots i_l}_l$ as follows:

$$\boldsymbol{L}^2\,T^{i_1\ldots i_l}_l=E_l\,T^{i_1\ldots i_l}_l, \tag{30}$$

$$\begin{aligned} iL_{hk}T^{i_1\ldots i_l}_l &= (l+1)\,\frac{D+2l-2}{D+2l}\left(\mathcal{P}^{l+1\,hi_1\ldots i_l}_{\;\;kj_1\ldots j_l}-\mathcal{P}^{l+1\,ki_1\ldots i_l}_{\;\;hj_1\ldots j_l}\right)T^{j_1\ldots j_l}_l,\\ &= l\,\mathcal{P}^{l\,i_1\ldots i_l}_{j_1\ldots j_l}\left(\delta^{kj_1}T^{hj_2\ldots j_l}_l-\delta^{hj_1}T^{kj_2\ldots j_l}_l\right), \end{aligned} \tag{31}$$

$$t^h\,T^{i_1\ldots i_l}_l=T^{hi_1\ldots i_l}_{l+1}+\frac{l}{D+2l-2}\,\mathcal{P}^{l\,i_1i_2\ldots i_l}_{hj_2\ldots j_l}T^{j_2\ldots j_l}_{l-1}\;\in\;V^{l+1}_D\oplus V^{l-1}_D, \tag{32}$$

$$t^iT^{ii_2\ldots i_l}_l=\frac{1}{D+2l-2}\left[D+l-1-\frac{2l-2}{D+2l-4}\right]T^{i_2\ldots i_l}_{l-1}\;\in\;V^{l-1}_D. \tag{33}$$

These formulae immediately follow from analogous ones for the $X^{i_1\ldots i_l}_l$. More generally, the product $T^{i_1\ldots i_l}_lT^{j_1\ldots j_m}_m$ decomposes as follows into V^n_D components:

$$T^{i_1\ldots i_l}_lT^{j_1\ldots j_m}_m=\sum_{n\in\mathcal{I}^{lm}}S^{i_1\ldots i_l,j_1\ldots j_m}_{k_1\ldots k_n}T^{k_1\ldots k_n}_n, \tag{34}$$

where $\mathcal{I}^{lm}\equiv\{|l-m|,|l-m|+2,\ldots,l+m\}$ and, defining $s\equiv\dfrac{l+m-n}{2}\in\{0,1,\ldots,m\}$,

$$\begin{aligned} S^{i_1\ldots i_l,j_1\ldots j_m}_{k_1\ldots k_n}&=N^{lm}_n\,V^{i_1\ldots i_l,j_1\ldots j_m}_{k_1\ldots k_n},\qquad N^{lm}_n=\frac{(D+2n-2)!!\,l!\,m!}{(D+2n+2s-2)!!\,(l-s)!\,(m-s)!}\\ V^{i_1\ldots i_l,j_1\ldots j_m}_{k_1\ldots k_n}&=\mathcal{P}^{l\,i_1\ldots i_l}_{a_1\ldots a_sc_1\ldots c_{l-s}}\mathcal{P}^{m\,j_1\ldots j_sj_{s+1}\ldots j_m}_{a_1\ldots a_sc_{l-s+1}\ldots c_n}\mathcal{P}^{n\,k_1\ldots k_n}_{c_1\ldots c_n}. \end{aligned} \tag{35}$$

Thus the $S^{i_1\ldots i_l,j_1\ldots j_m}_{k_1\ldots k_n}$ play the role of Clebsch-Gordon coefficients in the decomposition of a product of spherical harmonics. Finally, $\left\langle T^{i_1\ldots i_l}_l,T^{j_1\ldots j_n}_n\right\rangle\propto\delta_{ln}\mathcal{P}^{l\,j_1\ldots j_l}_{i_1\ldots i_l}$ w.r.t. the scalar product of $\mathcal{L}^2(S^d)$.

2.2 Embedding in $\mathbb{R}^{D+1}$, isomorphism End$\big(Pol_D^\Lambda\big) \simeq \pi_{D+1}^\Lambda\big[Uso(D{+}1)\big]$

Henceforth we abbreviate $\mathbf{D} \equiv D+1$. We naturally embed $\mathbb{C}[\mathbb{R}^D] \hookrightarrow \mathbb{C}[\mathbb{R}^{\mathbf{D}}]$; we use real Cartesian coordinates (x^i) for $\mathbb{R}^D$ and (x^I) for $\mathbb{R}^{\mathbf{D}}$; $h,i,j,k \in \{1,...,D\}$, $H,I,J,K \in \{1,...,\mathbf{D}\}$. We naturally embed $O(D) \hookrightarrow SO(\mathbf{D})$ identifying $O(D)$ as the subgroup of $SO(\mathbf{D})$ that is the little group of the $\mathbf{D}$-th axis; its Lie algebra, isomorphic to $so(D)$, is generated by the L_{hk}. We shall add $\mathbf{D}$ as a subscript to distinguish objects in dimension $\mathbf{D}$ from their counterparts in dimension D, e.g. the distance $r_{\mathbf{D}}$ from the origin in $\mathbb{R}^{\mathbf{D}}$, from its counterpart $r \equiv r_D$ in $\mathbb{R}^D$, $\mathcal{P}^l_{\mathbf{D}}$ from $\mathcal{P}^l \equiv \mathcal{P}^l_D$, and so on. Setting $t^I \equiv r_{\mathbf{D}}^{-1}x^I$, for $\Lambda \in \mathbb{N}_0$ $\check{V}_{\mathbf{D}}^\Lambda$, $V_{\mathbf{D}}^\Lambda = r_{\mathbf{D}}^{-\Lambda}\check{V}_{\mathbf{D}}^\Lambda$ are respectively spanned by the

$$X_{\mathbf{D},\Lambda}^{I_1...I_\Lambda} = \mathcal{P}_{\mathbf{D}}^{\Lambda}{}^{I_1...I_\Lambda}_{J_1...J_\Lambda} x^{J_1}...x^{J_\Lambda}, \qquad T_{\mathbf{D},\Lambda}^{I_1...I_\Lambda} = r_{\mathbf{D}}^{-\Lambda} X_{\mathbf{D},\Lambda}^{I_1...I_\Lambda} = \mathcal{P}_{\mathbf{D}}^{\Lambda}{}^{I_1...I_\Lambda}_{J_1...J_\Lambda} t^{J_1}...t^{J_\Lambda}. \tag{36}$$

The following combinations of the latter factorize into $X_l^{i_1...i_l}$ (resp. $T_l^{i_1...i_l}$) times a $O(D)$-scalar:

$$\check{F}_{\mathbf{D},\Lambda}^{i_1...i_l} \equiv \mathcal{P}^{l}{}^{i_1...i_l}_{j_1...j_l} X_{\mathbf{D},\Lambda}^{j_1...j_l\mathbf{D}...\mathbf{D}} = \check{p}_{\Lambda,l}\, X_l^{i_1...i_l}, \qquad F_{\mathbf{D},\Lambda}^{i_1...i_l} \equiv r_{\mathbf{D}}^{-\Lambda}\check{F}_{\mathbf{D},\Lambda}^{i_1...i_l} = p_{\Lambda,l} T_l^{i_1...i_l} \tag{37}$$

where $\check{p}_{\Lambda,l}$ is the homogeneous polynomial of degree $\Lambda - l$ in $x^{\mathbf{D}}, r_{\mathbf{D}}$

$$\check{p}_{\Lambda,l} = \left(x^{\mathbf{D}}\right)^{\Lambda-l} + \left(x^{\mathbf{D}}\right)^{\Lambda-l-2} r_{\mathbf{D}}^2\, b_{\Lambda,l+2} + \left(x^{\mathbf{D}}\right)^{\Lambda-l-4} r_{\mathbf{D}}^4\, b_{\Lambda,l+4} + \dots\ , \tag{38}$$

$$b_{\Lambda,l+2k} = (-)^k \frac{(\Lambda{-}l)!\,(2\Lambda{-}4{-}2k{+}\mathbf{D})!!}{(\Lambda{-}l{-}2k)!\,(2k)!!\,(2\Lambda{-}4{+}\mathbf{D})!!}, \qquad k = 1,2,....\left[\frac{\Lambda{-}l}{2}\right], \tag{39}$$

and $p_{\Lambda,l} \equiv \check{p}_{\Lambda,l}(x^{\mathbf{D}}, r_{\mathbf{D}})\, r_{\mathbf{D}}^{l-\Lambda}$ is a polynomial of degree $h = \Lambda - l$ in $t^{\mathbf{D}}$ only. Hence the $F_{\mathbf{D},\Lambda}^{i_1...i_l}$ are eigenvectors of $\boldsymbol{L}^2$ with eigenvalue E_l, transform under L_{hk} as the $T_l^{i_1...i_l}$ and under $L_{h\mathbf{D}}$ as follows:

$$iL_{h\mathbf{D}}F_{\mathbf{D},\Lambda}^{i_1...i_l} = (\Lambda{-}l)\, F_{\mathbf{D},\Lambda}^{hi_1...i_l} - \frac{l(\Lambda{+}l{+}D{-}2)}{D{+}2l{-}2}\, \mathcal{P}^{l}{}^{i_1 i_2...i_l}_{h j_2...j_l}\, F_{\mathbf{D},\Lambda}^{j_2...j_l}\,. \tag{40}$$

These relations follow from exactly the same relations for the $\check{F}_{\mathbf{D},\Lambda}^{i_1...i_l}$. As a consequence, $\check{V}_{\mathbf{D}}^\Lambda$, $V_{\mathbf{D}}^\Lambda$ decompose into irreducible components of $Uso(D)$ as follows:

$$\check{V}_{\mathbf{D}}^\Lambda = \bigoplus_{l=0}^{\Lambda} \check{V}_{D,\Lambda}^l, \qquad V_{\mathbf{D}}^\Lambda = \bigoplus_{l=0}^{\Lambda} V_{D,\Lambda}^l, \tag{41}$$

where $\check{V}_{D,\Lambda}^l \simeq V_D^l$, $V_{D,\Lambda}^l \simeq V_D^l$ are resp. spanned by the $\check{F}_{\mathbf{D},\Lambda}^{i_1...i_l}$, $F_{\mathbf{D},\Lambda}^{i_1...i_l}$. For $\Lambda = 0,1,2$ we have: $\check{V}_{\mathbf{D}}^0 \simeq V_{\mathbf{D}}^0 \simeq \mathbb{C} \simeq V_D^0$. $\check{V}_{D,1}^0, V_{D,1}^0$ are isomorphic to V_D^0 and resp. spanned by $x^{\mathbf{D}}, t^{\mathbf{D}}$; $\check{V}_{D,1}^1, V_{D,1}^1$ are isomorphic to V_D^1 and resp. spanned by the x^i, t^i. $\check{V}_{D,2}^0, V_{D,2}^0$ are isomorphic to V_D^0 and resp. spanned by $X_{\mathbf{D},2}^{\mathbf{DD}} = x^{\mathbf{D}}x^{\mathbf{D}} - r_{\mathbf{D}}^2/\mathbf{D}$, $F_{\mathbf{D},2} = T_{\mathbf{D},2}^{\mathbf{DD}} = t^{\mathbf{D}}t^{\mathbf{D}} - 1/\mathbf{D} = D/\mathbf{D} - \sum_{h=0}^{D} t^h t^h$; $\check{V}_{D,2}^1, V_{D,2}^1$ are isomorphic to V_D^0 and resp. spanned by the $\check{F}_{\mathbf{D},2}^i = X_{\mathbf{D},2}^{i\mathbf{D}} = x^i x^{\mathbf{D}}$, $F_{\mathbf{D},2}^i = T_{\mathbf{D},2}^{i\mathbf{D}} = t^i t^{\mathbf{D}}$; $\check{V}_{D,2}^2, V_{D,2}^2$ are isomorphic to V_D^2 and resp. spanned by the $\check{F}_{D,2}^{ij} = X_{\mathbf{D},2}^{ij} + X_{\mathbf{D},2}^{\mathbf{DD}}\delta^{ij}/D = X_2^{ij}$, $F_{D,2}^{ij} = T_{\mathbf{D},2}^{ij} + \frac{\delta^{ij}}{D}T_{\mathbf{D},2}^{\mathbf{DD}} = T_2^{ij}$; the last equalities follow from $X_2^{ij} = x^i x^j - r^2\frac{\delta^{ij}}{D}$, $X_{\mathbf{D},2}^{ij} = x^i x^j - r_{\mathbf{D}}^2\frac{\delta^{ij}}{\mathbf{D}}$, $T_2^{ij} = t^i t^j - \frac{\delta^{ij}}{D}$, $T_{\mathbf{D},2}^{ij} = t^i t^j - \frac{\delta^{ij}}{\mathbf{D}}$.

3. Relations among the $\overline{x}^i$, $\overline{L}_{hk}$, isomorphisms of $\mathcal{H}_\Lambda$, $\mathcal{A}_\Lambda$, $*$-automorphisms of $\mathcal{A}_\Lambda$

The functions $\psi_l^{i_1i_2...i_l} \equiv T_l^{i_1i_2...i_l} f_l$ with fixed l make up a complete set $\mathcal{S}^l_{D,\Lambda}$ in the eigenspace $\mathcal{H}^l_\Lambda$ of H, $\boldsymbol{L}^2$ with eigenvalues $E_{0,l}$, E_l. $\mathcal{S}_{D,\Lambda} \equiv \cup_{l=0}^{\Lambda} \mathcal{S}^l_{D,\Lambda}$ is complete in $\mathcal{H}_\Lambda$. The $\overline{L}_{hk}, \overline{x}^i$ act as

$$i\overline{L}_{hk}\,\psi_l^{i_1i_2...i_l} = l\,\mathcal{P}^{l\,i_1...i_l}_{j_1...j_l}\left(\delta^{kj_1}\psi_l^{hj_2...j_l} - \delta^{hj_1}\psi_l^{kj_2...j_l}\right), \tag{42}$$

$$\overline{x}^i\,\psi_l^{i_1i_2...i_l} = c_{l+1}\,\psi_{l+1}^{ii_1...i_l} + \frac{c_l\,l}{D+2l-2}\,\mathcal{P}^{l\,i_1i_2...i_l}_{ij_2...j_l}\,\psi_{l-1}^{j_2...j_l}, \tag{43}$$

$$\text{where}\qquad c_l \equiv \begin{cases} \sqrt{1+\frac{(2D-5)(D-1)}{2k}+\frac{(l-1)(l+D-2)}{k}} & \text{if } 1\le l\le\Lambda,\\ 0 & \text{otherwise.}\end{cases}$$

Eq. (42) follows from (31), while (43) holds up to $O\left(k^{-3/2}\right)$ corrections that depend on the terms proportional to $(r-1)^k$, $k>2$, in the Taylor expansion of V and could be made vanish by suitably choosing V. Henceforth we adopt (42-43) as exact *definitions* of $\overline{L}_{hk}, \overline{x}^i$. By Proposition 4.1 in [1], the $\overline{L}_{hk}, \overline{x}^i$ defined by (42-43) are self-adjoint operators generating the N^2-dimensional $*$-algebra $\mathcal{A}_\Lambda \equiv End(\mathcal{H}_\Lambda) \simeq M_N(\mathbb{C})$ of observables on $\mathcal{H}_\Lambda$; here $N \equiv \frac{(D+\Lambda-2)...(\Lambda+1)}{(D-1)!}(D+2\Lambda-1)$. Abbreviating $\overline{\boldsymbol{x}}^2 \equiv \overline{x}^i\,\overline{x}^i$, $\overline{\boldsymbol{L}}^2 \equiv \overline{L}_{ij}\,\overline{L}_{ij}/2$, $B \equiv (2D-5)(D-1)/2$, they fulfill the relations

$$\left[i\overline{L}_{ij}, \overline{x}^h\right] = \overline{x}^i\delta^h_j - \overline{x}^j\delta^h_i, \tag{44}$$

$$\left[i\overline{L}_{ij}, i\overline{L}_{hk}\right] = i\left(\overline{L}_{ik}\delta^j_h - \overline{L}_{jk}\delta^i_h - \overline{L}_{ih}\delta^j_k + \overline{L}_{jh}\delta^i_k\right), \tag{45}$$

$$\varepsilon^{i_1i_2i_3....i_D}\overline{x}^{i_1}\overline{L}_{i_2i_3} = 0, \qquad D\ge 3, \tag{46}$$

$$(\overline{x}^h \pm i\overline{x}^k)^{2\Lambda+1} = 0, \quad (\overline{L}^{hj} + i\overline{L}^{kj})^{2\Lambda+1} = 0, \qquad \text{if } h\neq j\neq k\neq h, \tag{47}$$

$$\left[\overline{x}^i, \overline{x}^j\right] = i\overline{L}_{ij}\left(-\frac{I}{k} + K\,P^\Lambda_\Lambda\right), \qquad K \equiv \tfrac{1}{k} + \tfrac{1}{D+2\Lambda-2}\left[1+\tfrac{B}{k}+\tfrac{(\Lambda-1)(\Lambda+D-2)}{k}\right], \tag{48}$$

$$\overline{\boldsymbol{x}}^2 = 1 + \frac{\overline{\boldsymbol{L}}^2}{k} + \frac{B}{k} - \frac{\Lambda+D-2}{2\Lambda+D-2}\left[1+\frac{B}{k}+\frac{\Lambda(\Lambda+D-1)}{k}\right]P^\Lambda_\Lambda =: \chi(\boldsymbol{L}^2). \tag{49}$$

A fuzzy sphere is obtained choosing k as a function $k(\Lambda)$ fulfilling (12), e.g. $k = \Lambda^2(\Lambda+D-2)^2/4$; the commutative limit is $\Lambda\to\infty$. We remark that:

3.a Eq. (46) is the analog of (7b). By (48), it can be reformulated also as $\varepsilon^{i_1i_2i_3....i_D}\overline{x}^{i_1}\overline{x}^{i_2}\overline{x}^{i_3} = 0$.

3.b By (49), $(15)_{l=\Lambda}$ $\overline{\boldsymbol{x}}^2$ is not a constant, but can be expressed as a polynomial χ in $\overline{\boldsymbol{L}}^2$ only, with the same eigenspaces $\mathcal{H}^l_\Lambda$. All its eigenvalues r_l^2, except r_Λ^2, are close to 1, slightly (but strictly) grow with l and collapse to 1 as $\Lambda\to\infty$. Conversely, $\overline{\boldsymbol{L}}^2$ can be expressed as a polynomial υ in $\overline{\boldsymbol{x}}^2$, via $\overline{\boldsymbol{L}}^2 = \sum_{l=0}^{\Lambda} E_l P^l_\Lambda$ and $P^l_\Lambda = \prod_{n=0,n\neq l}^{\Lambda}\frac{\overline{\boldsymbol{x}}^2 - r_n^2}{r_l^2 - r_n^2}$.

3.c By (48), $(15)_{l=\Lambda}$ the commutators $[\overline{x}^i, \overline{x}^j]$ are Snyder-like, i.e. of the form $\alpha\overline{L}_{ij}$; also α depends only on the $\overline{L}_{hk}$, more precisely can be expressed as a polynomial in $\overline{\boldsymbol{L}}^2$.

3.d Using (44), (45), (48), all polynomials in $\overline{x}^i, \overline{L}_{hk}$ can be expressed as combinations of monomials in $\overline{x}^i, \overline{L}_{hk}$ in any prescribed order, e.g. in the natural one

$$(\overline{x}^1)^{n_1}...(\overline{x}^D)^{n_D}(\overline{L}_{12})^{n_{12}}(\overline{L}_{13})^{n_{13}}...(\overline{L}_{dD})^{n_{dD}}, \qquad n_i, n_{ij} \in \mathbb{N}_0; \tag{50}$$

the coefficients, which can be put at the right of these monomials, are complex combinations of 1 and P_Λ^Λ. Also P_Λ^Λ can be expressed as a polynomial in $\overline{\boldsymbol{L}}^2$ via $(15)_{l=\Lambda}$. Hence a suitable subset of such ordered monomials makes up a basis of the N^2-dim vector space $\mathcal{A}_\Lambda$.

3.e Actually, $\overline{x}^i$ *generate* the $*$-algebra $\mathcal{A}_\Lambda$, because also the $\overline{L}_{ij}$ can be expressed as *non-ordered* polynomials in the $\overline{x}^i$: by (48) $\overline{L}_{ij} = [\overline{x}^j, \overline{x}^i]/\alpha$, and also $1/\alpha$, which depends only on P_Λ^Λ, can be expressed itself as a polynomial in $\overline{\boldsymbol{x}}^2$, as shown above.

3.f Eq. (44-49) are equivariant under the whole group $O(D)$, including the inversion $\overline{x}^i \mapsto -\overline{x}^i$ of one axis, or more (e.g. parity), contrary to Madore's and Hoppe's FS.

We slightly enlarge $Uso(D)$ by introducing the new generator $\lambda = \left[\sqrt{(D-2)^2 + 4\boldsymbol{L}^2} - D + 2\right]/2$, which fulfills $\lambda(\lambda+D-2) = \boldsymbol{L}^2$, so that V_D^l is a $\lambda = l$ eigenspace, and $\lambda F_{\mathbf{D},\Lambda}^{i_1...i_l} = l\, F_{\mathbf{D},\Lambda}^{i_1...i_l}$. Theorem 5.1 in [1] states that there exist a $O(D)$-module isomorphism $\varkappa_\Lambda : \mathcal{H}_\Lambda \to V_{\mathbf{D}}^\Lambda$ and a $O(D)$-equivariant algebra map $\kappa_\Lambda : \mathcal{A}_\Lambda \equiv \mathrm{End}(\mathcal{H}_\Lambda) \to \boldsymbol{\pi}_{\mathbf{D}}^\Lambda\big[Uso(\mathbf{D})\big]$, $\mathbf{D} \equiv D+1$, such that

$$\varkappa_\Lambda(a\boldsymbol{\psi}) = \kappa_\Lambda(a)\varkappa_\Lambda(\boldsymbol{\psi}), \qquad \forall\, \boldsymbol{\psi} \in \mathcal{H}_\Lambda, \quad a \in \mathcal{A}_\Lambda\,. \tag{51}$$

On the $\boldsymbol{\psi}_l^{i_1...i_l}$ (spanning $\mathcal{H}_\Lambda$) and on generators $L_{hi}, \overline{x}^i$ of $\mathcal{A}_\Lambda$ they respectively act as follows:

$$\varkappa_\Lambda(\boldsymbol{\psi}_l^{i_1...i_l}) \equiv a_{\Lambda,l} F_{\mathbf{D},\Lambda}^{i_1...i_l} = a_{\Lambda,l}\, p_{\Lambda,l}\, T_l^{i_1...i_l}, \qquad l = 0, 1, ..., \Lambda, \tag{52}$$

$$\kappa_\Lambda(\overline{L}_{hi}) \equiv \boldsymbol{\pi}_{\mathbf{D}}^\Lambda(L_{hi}), \qquad \kappa_\Lambda(\overline{x}^i) \equiv \boldsymbol{\pi}_{\mathbf{D}}^\Lambda\big[m_\Lambda^*(\lambda)\, X^i\, m_\Lambda(\lambda)\big], \tag{53}$$

where $X^i \equiv L_{\mathbf{D}i}$, $A \equiv \sqrt{k + (D-1)(D-3)3/4}$, Γ is Euler gamma function, and

$$a_{\Lambda,l} = a_{\Lambda,0}\, i^l \sqrt{\frac{\Lambda(\Lambda-1)...(\Lambda-l+1)}{(\Lambda+D-1)(\Lambda+D)...(\Lambda+l+D-2)}}, \tag{54}$$

$$m_\Lambda(s) = \sqrt{\frac{\Gamma\left(\frac{\Lambda+s+d}{2}\right)\Gamma\left(\frac{\Lambda-s+1}{2}\right)\Gamma\left(\frac{s+1+d/2+iA}{2}\right)\Gamma\left(\frac{s+1+d/2-iA}{2}\right)}{\Gamma\left(\frac{\Lambda+s+D}{2}\right)\Gamma\left(\frac{\Lambda-s}{2}+1\right)\Gamma\left(\frac{s+d/2+iA}{2}\right)\Gamma\left(\frac{s+d/2-iA}{2}\right)\sqrt{k}}}. \tag{55}$$

Finally, $*$-automorphisms ω of $\mathcal{A}_\Lambda \simeq M_N(\mathbb{C})$ are inner and make up a group $G \simeq SU(N)$, i.e.

$$\omega\; :\; a \in M_N(\mathbb{C}) \mapsto g\, a\, g^{-1} \in M_N(\mathbb{C}) \tag{56}$$

for some unitary $N\times N$ matrix g with $\det g = 1$. Consider the G-subgroup $G' \equiv \{g = \boldsymbol{\pi}_{\mathbf{D}}^\Lambda\left[e^{i\alpha}\right] \mid \alpha \in so(\mathbf{D})\} \simeq SO(\mathbf{D})$. Choosing $\alpha \in so(D) \subset so(\mathbf{D})$ the automorphism amounts to a $SO(D) \subset SO(\mathbf{D})$ transformation, i.e. a rotation in the $x \equiv (x^1, ..., x^D) \in \mathbb{R}^D$ space. The $O(D) \subset SO(\mathbf{D})$ transformations with determinant -1 keep the same form also in the $\overline{X} \equiv (X^1, ..., X^D)$ and [by (53)] in the $\overline{x} \equiv (\overline{x}^1, ..., \overline{x}^D)$ spaces. In particular, those inverting one or more axes of $\mathbb{R}^D$ (i.e. changing the sign of one or more x^i, and thus also of $X^i, \overline{x}^i$), e.g. parity, can be also realized as $SO(\mathbf{D})$ transformations, i.e. rotations in $\mathbb{R}^{\mathbf{D}}$. This shows that (53) is equivariant under the whole $O(D)$, which plays the role of isometry group of this fuzzy sphere.

4. Fuzzy spherical harmonics, and limit $\Lambda \to \infty$

It's simpler to work with the $T_l^{i_1...i_l}$ than spherical harmonics, their combinations. In $\mathcal{H}_s = \mathcal{L}^2(S^d)$ we have $\psi_l^{i_1...i_l} \propto T_l^{i_1...i_l}$, $\psi_0 \propto 1$. The $T_l^{i_1...i_l} \in C(S^d)$ act on $\mathcal{H}_s$ as multiplication operators fulfilling $T_l^{i_1...i_l} \cdot \psi_0 \propto \psi_l^{i_1...i_l}$. We define their Λ-th fuzzy analogs replacing $t^i \cdot \mapsto \overline{x}^i$ in (29b), i.e.

$$\widehat{T}_l^{i_1...i_l} \equiv \mathcal{P}^{l\, i_1...i_l}_{\quad j_1...j_l} \overline{x}^{j_1}...\overline{x}^{j_l}, \qquad \Rightarrow \qquad \widehat{T}_l^{i_1...i_l}\psi_0 \propto \psi_l^{i_1...i_l} \tag{57}$$

for $l \leq \Lambda$. Since ψ_0 is a scalar, $\psi_l^{i_1...i_l}, \widehat{T}_l^{i_1...i_l}, T_l^{i_1...i_l}$ transform under $O(D)$ exactly in the same way, consistently with $\mathcal{H}_\Lambda \simeq Pol_D^\Lambda$. As $\Lambda \to \infty$ the decomposition of $\mathcal{H}_\Lambda \simeq Pol_D^\Lambda$ into irreducible components under $O(D)$ becomes isomorphic to the decomposition of $\mathcal{H}_s \simeq Pol_D$. We define the $O(D)$-equivariant embedding $\mathcal{I} : \mathcal{H}_\Lambda \hookrightarrow \mathcal{H}_s$ by setting $\mathcal{I}\left(\psi_l^{i_1...i_l}\right) \equiv T_l^{i_1...i_l}$ and applying the linear extension. Below we drop $\mathcal{I}$ and identify $\psi_l^{i_1...i_l} = T_l^{i_1...i_l}$ as elements of the Hilbert space $\mathcal{H}_s$. For all $\phi \equiv \sum_{l=0}^{\infty} \phi^l_{i_1...i_l} T_l^{i_1...i_l} \in \mathcal{L}^2(S^2)$ and $\Lambda \in \mathbb{N}$ let $\phi_\Lambda \equiv P_\Lambda \phi = \sum_{l=0}^{\Lambda} \phi^l_{i_1...i_l} T_l^{i_1...i_l}$ be its projection to $\mathcal{H}_\Lambda$ (or Λ-th truncation). Clearly $\phi_\Lambda \to \phi$ in the $\mathcal{H}_s$-norm $\|\;\|$: in this simplified notation, $\mathcal{H}_\Lambda$ 'invades' $\mathcal{H}_s$ as $\Lambda \to \infty$. $\mathcal{I}$ induces the $O(D)$-equivariant embedding of operator algebras $\mathcal{J} : \mathcal{A}_\Lambda \hookrightarrow B(\mathcal{H}_s)$ by setting $\mathcal{J}(a)\,\mathcal{I}(\psi) \equiv \mathcal{I}(a\psi)$; here $B(\mathcal{H}_s)$ stands for the $*$-algebra of bounded operators on $\mathcal{H}_s$. By construction, $\mathcal{A}_\Lambda$ annihilates $\mathcal{H}_\Lambda^\perp$. In particular, $\mathcal{J}\left(\overline{L}_{hk}\right) = L_{hk}P^\Lambda$, and $\overline{L}_{hk}\phi \overset{\Lambda\to\infty}{\longrightarrow} L_{hk}\phi$ for all $\phi \in D(L_{hk}) \equiv$ the domain of L_{hk}. More generally, $f(\overline{L}_{hk}) \to f(L_{hk})$ strongly on $D[f(L_{hk})] \subset \mathcal{H}_s$, for all measurable functions $f(s)$. Continuous functions f on S^d, acting as multiplication operators $f\cdot : \phi \in \mathcal{H}_s \mapsto f\phi \in \mathcal{H}_s$, make up a subalgebra $C(S^d)$ of $B(\mathcal{H}_s)$. Clearly, f belongs also to $\mathcal{H}_s$. Since Pol_D is dense in both $\mathcal{H}_s$, $C(S^d)$, f_N converges to f as $N \to \infty$ in both the $\mathcal{H}_s$ and the $C(S^d)$ norm. Identifying $\psi_l^{i_1...i_l} \equiv T_l^{i_1...i_l}$, eq. (32), (43) become

$$t^h\, T_l^{i_1...i_l} = T_{l+1}^{hi_1...i_l} + d_l\, \mathcal{P}^{l\, i_1 i_2...i_l}_{\quad hj_2...j_l} T_{l-1}^{j_2...j_l}, \qquad d_l \equiv \frac{l}{D+2l-2} \tag{58}$$

$$\overline{x}^h T_l^{i_1 i_2...i_l} = c_{l+1}\, T_{l+1}^{hi_1...i_l} + c_l\, d_l\, \mathcal{P}^{l\, i_1 i_2...i_l}_{\quad hj_2...j_l}\, T_{l-1}^{j_2...j_l}. \tag{59}$$

Theorem 6.1 in [1] states that the action of the $\widehat{T}_l^{i_1...i_l}$ on $\mathcal{H}_\Lambda$ is determined by

$$\widehat{T}_l^{i_1...i_l} T_m^{j_1...j_m} = \sum\nolimits_{n\in L} \widehat{N}_n^{lm}\, \mathcal{P}^{l\, i_1...i_l}_{\quad a_1...a_r c_1...c_{l-r}} \mathcal{P}^{m\, j_1...j_r j_{r+1}...j_m}_{\quad a_1...a_r c_{l-r+1}...c_n} \mathcal{P}^{n\, k_1...k_n}_{\quad c_1...c_n}\, T_n^{k_1...k_n}, \tag{60}$$

with suitable coefficients $\widehat{N}_n^{lm}$, cf. (34-35). As a fuzzy analog of the vector space $C(S^d)$ we adopt

$$C_\Lambda \equiv \left\{ \hat{f}_{2\Lambda} \equiv \sum\nolimits_{l=0}^{2\Lambda} f^l_{i_1...i_l} \widehat{T}_l^{i_1...i_l} \mid f^l_{i_1...i_l} \in \mathbb{C} \right\} \subset \mathcal{A}_\Lambda \subset B(\mathcal{H}_s); \tag{61}$$

here the highest l is 2Λ because the $\widehat{T}_l^{i_1...i_l}$ annihilate $\mathcal{H}_\Lambda$ if $l > 2\Lambda$. By construction,

$$C_\Lambda = \bigoplus\nolimits_{l=0}^{2\Lambda} \widehat{V}_D^l, \qquad \widehat{V}_D^l \equiv \left\{ f^l_{i_1...i_l} \widehat{T}_l^{i_1...i_l}\;,\; f^l_{i_1...i_l} \in \mathbb{C} \right\} \tag{62}$$

is the decomposition of C_Λ into irreducible components under $O(D)$. $\widehat{V}_D^l$ is trace-free for all $l > 0$. In the limit $\Lambda \to \infty$ (62) becomes the decomposition of $C(S^d)$. As a fuzzy analog of $f \in C(S^d)$ we adopt the sum $\hat{f}_{2\Lambda}$ appearing in (61) with the coefficients of the expansion $f = \sum_{l=0}^{\infty} \sum_{i_1,...,i_l} f^l_{i_1...i_l} T_l^{i_1...i_l}$ up to $l = 2\Lambda$. Theorem 6.2 in [1] states that for all $f, g \in C(S^d)$ the following strong $\Lambda \to \infty$ limits hold: $\hat{f}_{2\Lambda} \to f\cdot, \widehat{(fg)}_{2\Lambda} \to fg$ and $\hat{f}_{2\Lambda}\hat{g}_{2\Lambda} \to fg\cdot$. However $\hat{f}_{2\Lambda}$ *does not* converge to f *in operator norm*, because the operator $\hat{f}_{2\Lambda}$ (a polynomial in the $\overline{x}^i$) annihilates $\mathcal{H}_\Lambda^\perp$ (the orthogonal complement of $\mathcal{H}_\Lambda$), since so do the $\overline{x}^i = P^\Lambda x^i \cdot P^\Lambda$.

5. Discussion and conclusions

We have obtained a sequence $\{(\mathcal{H}_\Lambda,\mathcal{A}_\Lambda)\}_{\Lambda\in\mathbb{N}}$ of $O(D)$-equivariant approximations of quantum mechanics of a particle on S^d; $\mathcal{H}_\Lambda$ is the Hilbert space of states, $\mathcal{A}_\Lambda \equiv$End$(\mathcal{H}_\Lambda)$ is the associated $*$-algebra of observables, $H_\Lambda\in\mathcal{A}_\Lambda$ is the free Hamiltonian (this may be modifed by adding interaction terms $H_I\in\mathcal{A}_\Lambda$, so that the new Hamiltonian still maps $\mathcal{H}_\Lambda$ into iself). $\mathcal{A}_\Lambda$ is spanned by ordered monomials (50) in $\overline{x}^i,\overline{L}_{ij}$ (of appropriately bounded degrees), in the same way as the algebra $\mathcal{A}_s$ of observables on $\mathcal{H}_s$ is spanned by ordered monomials in t^i,L_{ij}. However, while $\overline{x}^i$ generate the whole $\mathcal{A}_\Lambda$ because $[\overline{x}^i,\overline{x}^j]\propto\overline{L}_{ij}$ (as in Snyder spaces [4]), this has no analog in $\mathcal{A}_s$, because $[t^i,t^j]=0$. The square distance $\overline{\boldsymbol{x}}^2$ from the origin is not 1, but a function of $\boldsymbol{L}^2$ with a spectrum very close to 1, collapsing to 1 as $\Lambda\to\infty$. Each pair $(\mathcal{H}_\Lambda,\mathcal{A}_\Lambda)$ is isomorphic to $\big(V_{\mathbf{D}}^\Lambda,\boldsymbol{\pi}_\Lambda[Uso(\mathbf{D})]\big)$, $\mathbf{D}\equiv D{+}1$, also as $O(D)$-modules; $\boldsymbol{\pi}_\Lambda$ is the irrep of $Uso(\mathbf{D})$ on the space $V_{\mathbf{D}}^\Lambda$ of harmonic polynomials of degree Λ on $\mathbb{R}^{\mathbf{D}}$, restricted to S^D. We have also described (section 4) the subspace $C_\Lambda\subset\mathcal{A}_\Lambda$ of completely symmetrized trace-free polynomials in the $\overline{x}^i$; this is also spanned by the fuzzy analogs of spherical harmonics. $\mathcal{H}_\Lambda,\mathcal{A}_\Lambda,C_\Lambda$ carry reducible representations of $O(D)$; as $\Lambda\to\infty$ their decompositions into irreps respectively go to the decompositions of $\mathcal{H}_s\equiv\mathcal{L}^2(S^d)$, of $\mathcal{A}_s$ and of $C(S^d)\subset\mathcal{A}_s$ (the continuous functions on S^d act on $\mathcal{H}_s$ as multiplication operators). There are natural embeddings $\mathcal{H}_\Lambda\hookrightarrow\mathcal{H}_s$, $C_\Lambda\hookrightarrow C(S^d)$ and $\mathcal{A}_\Lambda\hookrightarrow\mathcal{A}_s$ such that $\mathcal{H}_\Lambda\to\mathcal{H}_s$ in the norm of $\mathcal{H}_s$, while $C_\Lambda\to C(S^d)$, $\mathcal{A}_\Lambda\to\mathcal{A}_s$ strongly as $\Lambda\to\infty$.

Reintroducing the physical angular momentum components $l_{ij}\equiv\hbar L_{ij}$, then in the $\hbar\to 0$ limit $\mathcal{A}_s$ endowed with the usual quantum Poisson bracket $\{f,g\}=[f,g]/i\hbar$ goes to the (commutative) Poisson algebra $\mathcal{F}$ of (polynomial) functions on the classical phase space T^*S^d, generated by t^i,l_{ij}. We can directly obtain $\mathcal{F}$ from $\mathcal{A}_\Lambda$ adopting a suitable Λ-dependent $\hbar$ going to zero as $\Lambda\to\infty$[1]. More formally, we can regard $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ as a fuzzy quantization of a coadjoint orbit of $O(\mathbf{D})$ that goes to the classical phase space T^*S^d. We recall that coadjoint orbits $O_\lambda=\mathrm{Ad}^*_G\lambda$ of a Lie group G are orbits of the coadjoint action Ad^*_G inside the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G passing through $\lambda\in\mathfrak{g}^*$, or equivalently homogeneous spaces G/G_λ, where G_λ is the stabilizer of λ w.r.t. Ad^*_G. They have a natural symplectic structure. If G is compact semisimple, identifying $\mathfrak{g}^*\simeq\mathfrak{g}$ via the (nondegenerate) Killing form, we can resp. rewrite these definitions in the form

$$O_\lambda\equiv\left\{g\lambda g^{-1}\mid g\in G\right\}\subset\mathfrak{g}^*,\qquad O_\lambda\equiv G/G_\lambda\quad\text{where }\;G_\lambda\equiv\left\{g\in G\mid g\lambda g^{-1}=\lambda\right\}.\tag{63}$$

Clearly, $G_{\Lambda\lambda}=G_\lambda$ for all $\Lambda\in\mathbb{C}\setminus\{0\}$. Denoting as $\mathcal{H}_\lambda$ the (necessarily finite-dimensional) carrier space of the irrep with highest weight λ, one can regard (see e.g. [27]) the sequence of $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$, with $\mathcal{A}_\Lambda\equiv\mathrm{End}\,(\mathcal{H}_{\Lambda\lambda})$, as a fuzzy quantization of the symplectic space $O_\lambda\simeq G/G_\lambda$. The Killing form B of $so(\mathbf{D})$ gives $B(L_{HI},L_{JK})=2(\mathbf{D}-2)\left(\delta^H_J\delta^I_K-\delta^H_K\delta^I_J\right)$ for all $H,I,J,K\in\{1,2,...,\mathbf{D}\}$. As a basis of the Cartan subalgebra $\mathfrak{h}$ of $so(\mathbf{D})$ we pick $\{H_a\}_{a=1}^{\sigma}$, where $\sigma\equiv\left[\frac{\mathbf{D}}{2}\right]=$ rank of $so(\mathbf{D})$,

$$H_\sigma\equiv L_{D\mathbf{D}},\quad H_{\sigma-1}\equiv L_{(d-1)d},\quad\ldots,\quad H_1=\begin{cases}L_{12} & \text{if }\mathbf{D}=2\sigma,\\ L_{23} & \text{if }\mathbf{D}=2\sigma{+}1.\end{cases}\tag{64}$$

We choose the irrep of $Uso(\mathbf{D})$ on $V_{\mathbf{D}}^\Lambda\simeq\mathcal{H}_\Lambda$ and $\Omega_{\mathbf{D}}^\Lambda\equiv(t^D+it^{\mathbf{D}})^\Lambda\in V_{\mathbf{D}}^\Lambda$ as the highest weight vector. The joint spectrum $\boldsymbol{\Lambda}=(0,...,0,\Lambda)$ of $H\equiv(H_1,...,H_\sigma)$ is the weight associated to the

[1]It suffices that $\hbar(\Lambda)k(\Lambda)$ diverges; if e.g. $k=\Lambda^2(\Lambda{+}D{-}2)^2/4$, then $\hbar(\Lambda)=O(\Lambda^{-\alpha})$ with $0<\alpha<4$ is enough.

$\mathfrak{h}$-basis. Identifying $\lambda \in \mathfrak{h}^*$ with $H_\lambda \in \mathfrak{h}$ via the Killing form, we find that $H_{\mathbf{\Lambda}} \propto H_\sigma = L_{D\mathbf{D}}$. The stabilizer of $H_{\mathbf{\Lambda}}$ in $SO(\mathbf{D})$ is $SO(2){\times}SO(d)$, where $so(2), so(d)$ are resp. spanned by $H_{\mathbf{\Lambda}}$, the L_{ij} with $i,j < D$. Thus the coadjoint orbit $O_{\mathbf{\Lambda}} = SO(\mathbf{D})/\big(SO(2){\times}SO(d)\big)$ has the dimension of T^*S^d,

$$\frac{D(D+1)}{2} - 1 - \frac{(D-2)(D-1)}{2} = 2(D-1) = 2d,$$

consistently with the interpretation of $\mathcal{A}_\Lambda$ as the algebra of observables (quantized phase space) on the fuzzy sphere. It would have not been the case with some other irrep of $Uso(\mathbf{D})$; O_λ would have been some other equivariant bundle over S^d [27]. For instance, the fuzzy spheres of dimension $d > 2$ of [17–20] are based on $End(V^\Lambda)$, where the spaces V^Λ carry *irreps* of both $Spin(D)$ and $Spin(\mathbf{D})$, hence of both $Uso(D)$ and $Uso(\mathbf{D})$. Then: i) for some Λ these may be only *projective* representations of $O(D)$; ii) in general (46) will not be satisfied; iii) as $\Lambda \to \infty$ V^Λ does not go to $\mathcal{L}^2(S^d)$ as a representation of $Uso(D)$, in contrast with our $\mathcal{H}_\Lambda \simeq V^\Lambda_{\mathbf{D}}$; iv) the central $\boldsymbol{x}^2 \equiv X^iX^i$ can be normalized to $\boldsymbol{x}^2 = 1$. Here $L_{i\mathbf{D}}$ play the role of fuzzy coordinates X^i. In [21, 22] $d = 4$ and $O_\lambda = \mathbb{C}P^3$, which has dimension 6 and can be seen as a $so(5)$-equivariant S^2 bundle over S^4. Ref. [21, 22] constructs also a fuzzy 4-sphere S^4_N based on based on a sequence of $End(V)$, where each V carries an irrep π of $Uso(6)$ which splits into the direct sum of a small number $m > 1$ of irreps of $Uso(5)$; the $O(5)$-scalar $\boldsymbol{x}^2 = X^iX^i$ is no longer central, but its spectrum is still very close to 1 provided. The associated coadjoint orbit is 10-dimensional and can be seen as a $so(5)$-equivariant $\mathbb{C}P^2$ bundle over $\mathbb{C}P^3$, or a $so(5)$-equivariant twisted bundle over either S^4_N or S^4_n.

$\mathcal{A}_s$ is generated by all the t^h, L_{ij} with $h \le D$, $i < j \le D$ (subject to the relations $t^it^h = t^ht^i$, $t^it^i = 1$, $[iL_{ij}, t^h] = t^i\delta^h_j - t^j\delta^h_i$, etc.), and $C(S^d)$ is generated by the t^h alone. On the contrary, by Remark 3.e the $\overline{x}^i$ *alone* generate the whole $\mathcal{A}_\Lambda \simeq \boldsymbol{\pi}^\Lambda_{\mathbf{D}}\big[Uso(\mathbf{D})\big]$, which contains C_Λ as a *proper* subspace, albeit not as a subalgebra; also the simpler generators $X^i = L_{\mathbf{D}i}$ alone generate $\mathcal{A}_\Lambda \simeq \boldsymbol{\pi}^\Lambda_{\mathbf{D}}\big[Uso(\mathbf{D})\big]$, because of $L_{ij} = i[X^j, X^i]$ and (53). Thus the Hilbert-Poincaré series of the algebra generated by the $\overline{x}^i$ (or X^i), $\mathcal{A}_\Lambda$, is larger than that of Pol^Λ_D and C_Λ. If by a "quantized space" we understand a noncommutative deformation of the *algebra* of functions on that space *preserving the Hilbert-Poincaré series*, then $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ is a ($O(D)$-equivariant, fuzzy) quantization of T^*S^d, the phase space on S^d, while $\{C_\Lambda\}_{\Lambda\in\mathbb{N}}$ is not a quantization of S^d, nor are the other fuzzy spheres, except the Madore-Hoppe fuzzy 2-dimensional sphere: all the others, as ours, have the same Hilbert-Poincaré series of a suitable equivariant bundle on S^d, i.e. a manifold with a dimension $n > d$ (in our case, $n = 2d$). (Incidentally, in our opinion also for the Madore-Hoppe fuzzy sphere the most natural interpretation is of a quantized phase space, because the $\hbar \longrightarrow 0$ limit of the quantum Poisson bracket endows its algebra with a nontrivial Poisson structure.)

We understand $\mathcal{H}_\Lambda, C_\Lambda$ as fuzzy "quantized" S^d in the following weaker sense. $\mathcal{H}_\Lambda, C_\Lambda$ are the quantizations of $\mathcal{L}^2(S^d)$, $C(S^d)$, because, by (57b), the whole $\mathcal{H}_\Lambda$ is obtained applying to the ground state $\boldsymbol{\psi}_0$ the polynomials in the $\overline{x}^i$ alone (or the subspace C_Λ), or equivalently [by (53)] the polynomials in the $X^i = L_{\mathbf{D}i}$ alone, in the same way as $\mathcal{L}^2(S^d)$ is obtained (modulo completion) by applying $C(S^d)$ or Pol_D, i.e. the polynomials in the $t^i = x^i/r$, to the ground state (the constant function on S^d). These quantizations are $O(D)$-equivariant because $\mathcal{H}_\Lambda$ (resp. C_Λ) carries the same reducible representation of $O(D)$ as the space Pol^Λ_D (resp. $Pol^{2\Lambda}_D$) of polynomials of degree Λ (resp. 2Λ) in the $t^i = x^i/r$. Identifying $\mathcal{H}_\Lambda, C_\Lambda$ with $Pol^\Lambda_D, Pol^{2\Lambda}_D$ as $O(D)$-modules, as $\Lambda \longrightarrow \infty$ the latter become dense in $\mathcal{L}^2(S^d)$, $C(S^d)$, and their decompositions into irreps of $O(D)$ become that (2) of both $\mathcal{L}^2(S^d)$, $C(S^d)$. This is not the case for the other fuzzy spheres.

We expect that space uncertainties and optimally localized/coherent states for $d = 1, 2$ [28] generalize to $d > 2$. It is also worth investigating about: distances between optimally localized states (as e.g. in [29]); extending our construction to particles with spin; QFT on S^d_Λ; their application to problems in quantum gravity, or condensed matter physics; etc. Finally, we mention that by using Drinfel'd twists one can construct [30, 31] a different kind of noncommutative submanifolds of noncommutative $\mathbb{R}^D$, equivariant with respect to a 'quantum group' (twisted Hopf algebra).

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