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# Resolving Hubble Tension with New Gravitational Scalar Tensor Theories

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We investigate the cosmological applications of new gravitational scalar-tensor theories and we analyze them in the light of $H_0$ tension. In these theories the Lagrangian contains the Ricci scalar and its first and second derivatives in a specific combination that makes them free of ghosts, thus corresponding to healthy bi-scalar extensions of general relativity. We examine two specific models, and for particular choices of the model parameters we find that the effect of the additional terms is negligible at high redshifts, obtaining a coincidence with ΛCDM cosmology, however as time passes the deviation increases and thus at low redshifts the Hubble parameter acquires increased values ($H_0 \approx 74 km/s/Mpc$) in a controlled way. The mechanism behind this behavior is the fact that the effective dark-energy equation-of-state parameter exhibits phantom behavior, which implies faster expansion, which is one of the theoretical requirements that are capable of alleviating the $H_0$ tension. Lastly, we confront the models with Cosmic Chronometer (CC) data showing full agreement within $1\sigma$ confidence level.

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## 1. Introduction

Although the concordance ΛCDM paradigm is very successful in describing early- and late-time cosmological evolution at both background and perturbation levels, nevertheless the last years there have appeared some potential tensions with specific datasets, such as the $H_0$ and $\sigma_8$ ones. In particular, the estimation for the present Hubble parameter $H_0$ according to the Planck collaboration and assuming ΛCDM scenario is $H_0 = (67.27 \pm 0.60)$ km/s/Mpc [1], which is in tension at about $4.4\sigma$ with the direct measurement of the 2019 SH0ES collaboration (R19), namely $H_0 = (74.03 \pm 1.42)$ km/s/Mpc, obtained using long-period Cepheids [2]. On the other hand, the $\sigma_8$ tension arises from the fact that the parameter that quantifies the matter clustering within spheres of radius $8h^{-1}$Mpc, is found to be different from Cosmic Microwave Background (CMB) estimation [1] and from SDSS/BOSS measurement [3–5]. These tensions, and especially the $H_0$ one, progressively seem not to be related to unknown systematics, opening the road to many modifications of the standard lore [6, 7] (for a review see [8]).

One may follow two ways to alleviate the $H_0$ tension. The first is to modify the universe content and/or particle interactions while keeping general relativity as the underlying gravitational theory [9–44]. The second way is to construct gravitational modifications, which applied to cosmological framework would lead to altered expansion rate [45–71]. We mention here that modified gravity has additional advantages too, such as the improvement of the renormalizability behavior of general relativity as well as the description of inflationary and/or dark-energy phases, and thus it might be more preferable.

In the present work we are interested in alleviating the $H_0$ tension in the framework of new gravitational scalar-tensor theories [72–74]. In such constructions one uses Lagrangians with the Ricci scalar as well as its first and second derivatives, nevertheless in combinations that result to ghost-free theories. These theories are found to have $2 + 2$ propagating degrees of freedom, and thus falling outside Horndeski/Galileon [75–77] and beyond-Horndeski theories [78]. However, although they are bi-scalar extensions of general relativity, they were named "new gravitational scalar-tensor theories" since they can still be expressed in pure geometrical terms [72].

The plan of the work is the following: In Section 2 we briefly review the new gravitational scalar-tensor theories, and in Section 3 we apply them to a cosmological framework, extracting the modified Friedmann equations. Then, in Section 4 we construct specific models that can alleviate the $H_0$ tension, and we compare the induced behavior to that of ΛCDM scenario as well as to Cosmic Chronometers (CC) data. Finally, in Section 5 we provide the conclusions.

## 2. Overview

In this section we give a brief overview of the gravitational scalar-tensor theories. The action of such constructions is given as [72, 73]

$$S = \int d^4\sqrt{-g}\, f\left(R, (\nabla R)^2, \Box R\right), \tag{1}$$

with $(\nabla R)^2 = g^{\mu\nu}\nabla_\mu R \nabla_\nu R$. In the following we set the Planck mass $M_{pl}$ to one. One can rewrite the above action by converting the Lagrangian using double Lagrange multipliers, resulting to actions of multi-scalar fields coupled minimally to gravity. In order to achieve it, one fixes the dependence of $f$ on $\Box R = \beta$.

In the present work, we consider theories with the following $f$ form:

$$f(R, (\nabla R)^2, \Box R) = \mathcal{K}((R, (\nabla R)^2) + \mathcal{G}(R, (\nabla R)^2)\Box R, \tag{2}$$

thus maintaining a linear form in $\Box R = \beta$. Generalizations to non-linear forms are straightforward, although more complicated. In this case, the action (1) transforms to

$$\begin{aligned} S = \int d^4x\sqrt{-\hat{g}}\Bigg[&\frac{1}{2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi \\ &-\frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\hat{g}^{\mu\nu}\mathcal{G}\nabla_\mu\chi\nabla_\nu\phi + \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\ &+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\hat{\Box}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\phi\Bigg], \end{aligned} \tag{3}$$

where $\mathcal{K} = \mathcal{K}(\phi, B)$ and $\mathcal{G} = \mathcal{G}(\phi, B)$, with $B = 2e^{\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The $\chi$ and $\phi$ fields are introduced through the conformal transformations $g_{\mu\nu} = \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\hat{g}_{\mu\nu}$, $\varphi \equiv f_\beta$, and they enter in a specific combination in a way that the final form of the action is equivalent to the original higher-derivative gravitational action.

Varying the action (3) with respect to the metric leads to the following field equations in Einstein frame [72, 73]:

$$\begin{aligned} \mathcal{E}_{\mu\nu} \;=\; &\frac{1}{2}G_{\mu\nu} + \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi - \frac{1}{2}\nabla_\mu\chi\nabla_\nu\chi \\ &+\frac{1}{4}g_{\mu\nu}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}g^{\alpha\beta}\mathcal{G}\nabla_\alpha\chi\nabla_\beta\phi \\ &-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\nabla_{(\mu}\chi\nabla_{\nu)}\phi \\ &-\sqrt{\frac{2}{3}}g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\phi\,\mathcal{G}_B\nabla_\mu\phi\nabla_\nu\phi \\ &-\frac{1}{4}g_{\mu\nu}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\Box\phi + \mathcal{G}_B(\Box\phi)\nabla_\mu\phi\nabla_\nu\phi \\ &+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\nabla_\mu\nabla_\nu\phi - \frac{1}{2}\nabla_\kappa\left(e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\delta^\lambda_{(\mu}\delta^\kappa_{\nu)}\nabla_\lambda\phi\right) \\ &+\frac{1}{4}\nabla_\kappa\left(e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}g_{\mu\nu}\nabla^\kappa\phi\right) - \frac{1}{8}g_{\mu\nu}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\ &+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{K}_B\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{8}g_{\mu\nu}e^{-\sqrt{\frac{2}{3}}\chi}\phi = 0. \end{aligned} \tag{4}$$

Additionally, varying (3) with respect to $\chi$ and $\phi$ gives rise to field equations as

$$\begin{aligned}
\mathcal{E}_\chi &= \Box\chi + \frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}\nabla_\mu\chi\nabla_\nu\phi \\
&-\frac{2}{3}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\phi\,\mathcal{G}_B g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi \\
&+\frac{1}{2}\sqrt{\frac{2}{3}}\nabla_\mu\left(e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}\nabla_\nu\phi\right) \\
&-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\Box\phi + \sqrt{\frac{2}{3}}\mathcal{G}_B\nabla_\mu\phi\nabla_\nu\phi\, g^{\mu\nu}\Box\phi \\
&-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} + \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{K}_B\sqrt{\frac{2}{3}}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\
&+\frac{1}{4}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\phi = 0,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\mathcal{E}_\phi &= -\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}_\phi\nabla_\mu\chi\nabla_\nu\phi \\
&+2\sqrt{\frac{2}{3}}\nabla_\beta\left(g^{\mu\nu}\mathcal{G}_B g^{\alpha\beta}\nabla_\alpha\phi\nabla_\mu\chi\nabla_\nu\phi\right) \\
&+\frac{1}{2}\sqrt{\frac{2}{3}}\nabla_\nu\left(e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}\nabla_\mu\chi\right) \\
&+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}_\phi\Box\phi - 2\mathcal{G}_B(\Box\phi)^2 - 2\nabla_\nu\mathcal{G}_B\Box\phi\nabla^\nu\phi \\
&-\frac{1}{2}\sqrt{\frac{2}{3}}\nabla^\mu\left(e^{-\sqrt{\frac{2}{3}}\chi}\nabla_\mu\chi\,\mathcal{G}\right) + \frac{1}{2}\nabla^\mu\left(e^{-\sqrt{\frac{2}{3}}\chi}\,\mathcal{G}_\phi\nabla_\mu\phi\right) \\
&-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\nabla^\mu\chi\mathcal{G}_B\nabla_\mu B + \frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\nabla^\mu\mathcal{G}_B\nabla_\mu B \\
&+\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}_B\nabla^\mu\left(e^{\sqrt{\frac{2}{3}}\chi}\nabla_\mu\chi\nabla^\nu\phi\nabla_\nu\phi\right) \\
&+2e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}_B\nabla^\mu\left(e^{\sqrt{\frac{2}{3}}\chi}\nabla^\nu\phi\right)\nabla_\mu\nabla_\nu\phi \\
&+2\mathcal{G}_B R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}_\phi \\
&-\nabla_\nu\left(e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{K}_B g^{\mu\nu}\nabla_\mu\phi\right) - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi} = 0,
\end{aligned} \tag{6}$$

where for simplicity we have neglected the hats. Here, the subscripts in $\mathcal{G}$ and $\mathcal{K}$ denote the partial derivatives and the symmetrization is indicated by the parentheses in spacetime indices. The above equations reduce to GR for $\mathcal{K} = \phi/2$ and $\mathcal{G} = 0$, with the conformal transformation in this case being $\chi = -\sqrt{\frac{3}{2}}\ln 2$. As we can see, the above equations do not contain any higher derivative terms, and therefore the present theory is well-behaved.

## 3. Cosmological Behaviour

We can now proceed to the study of the cosmological behaviour of the present model. For this we consider a flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \tag{7}$$

with $a(t)$ the scale factor. We further assume that the two scalars are time-dependent only.

Including the matter sector, considered to correspond to a perfect fluid, the metric field equations (4) become

$$\mathcal{E}_{\mu\nu} = \frac{1}{2}T_{\mu\nu}, \tag{8}$$

with $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ representing the matter energy-momentum tensor.

With the above substitutions into equations (4), we obtain the following Friedmann equations:

$$\begin{aligned} &3H^2 - \rho_m - \frac{1}{2}\dot{\chi}^2 + \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\ &+\frac{2}{3}\dot{\phi}^2\left[\dot{\phi}\left(\sqrt{6}\dot{\chi} - 9H\right) - 3\ddot{\phi}\right]\mathcal{G}_B \\ &-\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot{B}\dot{\phi}\mathcal{G}_B + \frac{\phi}{2} + \dot{\phi}^2\left(\mathcal{G}_\phi - 2\mathcal{K}_B\right)\right] = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &3H^2 + 2\dot{H} + p_m + \frac{1}{2}\dot{\chi}^2 + \frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\ &+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left(-\frac{\phi}{2} + \dot{B}\dot{\phi}\mathcal{G}_B + \dot{\phi}^2\mathcal{G}_\phi\right) = 0, \end{aligned} \tag{10}$$

with $B(t) = 2e^{\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = -2e^{\sqrt{\frac{2}{3}}\chi}\dot{\phi}^2$, and $H = \dot{a}/a$ the Hubble parameter, where dots denoting differentiation with respect to $t$. Similarly, the two scalar field equations (5) and (6) lead to:

$$\begin{aligned} \mathcal{E}_\chi = \ddot{\chi} + 3H\dot{\chi} &- \frac{1}{3}\dot{\phi}^2\left[\dot{\phi}\left(3\sqrt{6}H - 2\dot{\chi}\right) + \sqrt{6}\ddot{\phi}\right]\mathcal{G}_B \\ &+\frac{1}{2\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\left[2\dot{B}\dot{\phi}\mathcal{G}_B - \phi + 2\dot{\phi}^2\left(\mathcal{K}_B + \mathcal{G}_\phi\right)\right] \\ &+\frac{1}{\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} = 0, \end{aligned} \tag{11}$$

and

$$
\begin{aligned}
\mathcal{E}_\phi = &\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot\phi\left(-9H+\sqrt{6}\dot\chi\right)-3\ddot\phi\right]\mathcal{K}_B \\
&+\frac{1}{6}\dot B\left\{3e^{-\sqrt{\frac{2}{3}}\chi}\dot B+4\dot\phi\left[\dot\phi\left(9H-\sqrt{6}\dot\chi\right)+3\ddot\phi\right]\right\}\mathcal{G}_{BB} \\
&+\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot\phi\left(9H-\sqrt{6}\dot\chi\right)+3\ddot\phi\right]\mathcal{G}_\phi \\
&+\left\{e^{-\sqrt{\frac{2}{3}}\chi}\dot B\dot\phi+\frac{2}{3}\dot\phi^2\left[\dot\phi\left(9H-\sqrt{6}\dot\chi\right)+3\ddot\phi\right]\right\}\mathcal{G}_{B\phi} \\
&-e^{-\sqrt{\frac{2}{3}}\chi}\dot\phi^2\mathcal{K}_{B\phi}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\dot\phi^2\mathcal{G}_{\phi\phi}-e^{-\sqrt{\frac{2}{3}}\chi}\dot B\dot\phi\mathcal{K}_{BB} \\
&+\Bigg[\frac{4}{3}\dot\phi\left(9H-2\sqrt{6}\dot\chi\right)\ddot\phi-\frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\dot B\dot\chi \\
&\quad+\dot\phi^2\left(18H^2+6\dot H-3\sqrt{6}H\dot\chi-\frac{2}{3}\dot\chi^2-\sqrt{6}\ddot\chi\right)\Bigg]\mathcal{G}_B \\
&-\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}_\phi+\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}=0,
\end{aligned}
\tag{12}
$$

with $\mathcal{G}_{B\phi}=\mathcal{G}_{\phi B}\equiv\frac{\partial^2\mathcal{G}}{\partial B\partial\phi}$, etc.

The above Friedmann equations (9),(10) can be rewritten as

$$
H^2=\frac{1}{3}(\rho_{DE}+\rho_m) \tag{13}
$$

$$
2\dot H+3H^2=-(p_{DE}+p_m), \tag{14}
$$

with the effective dark energy and pressure defined as

$$
\begin{aligned}
\rho_{DE}\equiv &\frac{1}{2}\dot\chi^2-\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\
&-\frac{2}{3}\dot\phi^2\left[\dot\phi\left(\sqrt{6}\dot\chi-9H\right)-3\ddot\phi\right]\mathcal{G}_B \\
&+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot B\dot\phi\mathcal{G}_B+\frac{\phi}{2}+\dot\phi^2\left(\mathcal{G}_\phi-2\mathcal{K}_B\right)\right],
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
p_{DE}\equiv &\frac{1}{2}\dot\chi^2+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K} \\
&+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left(\dot B\dot\phi\mathcal{G}_B+\dot\phi^2\mathcal{G}_\phi-\frac{\phi}{2}\right).
\end{aligned}
\tag{16}
$$

Hence, one can show that in the new gravitational scalar-tensor theories the effective dark-energy density satisfies

$$
\dot\rho_{DE}+3H(\rho_{DE}+p_{DE})=0, \tag{17}
$$

while one can define the corresponding dark-energy equation-of-state parameter as

$$
w_{DE}\equiv\frac{p_{DE}}{\rho_{DE}}. \tag{18}
$$

## 4. Hubble Tension

In this section we construct specific models of the theory in order to be able to alleviate the $H_0$ tension. In particular, we will consider specific ansatzes for the functions $\mathcal{K}(\phi, B)$ and $\mathcal{G}(\phi, B)$ that lead to higher Hubble function at low redshifts, while introducing negligible deviations in the Hubble parameter at high redshifts as compared to $\Lambda$CDM.

### 4.1 Model I

As a first example we consider the following forms for $\mathcal{K}(\phi, B)$ and $\mathcal{G}(\phi, B)$:

$$\mathcal{K}(\phi, B) = \frac{1}{2}\phi - \frac{\zeta}{2}B \quad \text{and} \quad \mathcal{G}(\phi, B) = 0, \tag{19}$$

with $\zeta$ a coupling constant. The corresponding Friedmann equations (9),(10) read as

$$3H^2 - \rho_m - \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi + \zeta\dot{\phi}^2\right) = 0, \tag{20}$$

$$\begin{aligned}3H^2 + 2\dot{H} + p_m + \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi \\ -\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi - \zeta\dot{\phi}^2\right) = 0,\end{aligned} \tag{21}$$

while the two scalar field equations (11) and (12) become

$$\ddot{\chi} + 3H\dot{\chi} + \frac{1}{2\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{2\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi - \zeta\dot{\phi}^2\right) = 0, \tag{22}$$

$$\zeta\ddot{\phi} + \frac{1}{3}\zeta\dot{\phi}\left(9H - \sqrt{6}\dot{\chi}\right) - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi} + \frac{1}{2} = 0. \tag{23}$$

The corresponding effective dark-energy energy density and pressure (15),(16) become

$$\rho_{DE} = \frac{1}{2}\dot{\chi}^2 - \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi + \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi + \zeta\dot{\phi}^2\right), \tag{24}$$

$$p_{DE} = \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\left(\phi - \zeta\dot{\phi}^2\right). \tag{25}$$

In order to obtain the behaviour of the Hubble parameter, we first set $z = -1 + a_0/a$, with the current value of the scale factor being set to $a_0 = 1$. It is well know that the behaviour of the Hubble parameter in $\Lambda$CDM cosmology is given by

$$H_{\Lambda CDM}(z) = H_0\sqrt{\Omega_{m_0}(1+z)^3 + 1 - \Omega_{m_0}}, \tag{26}$$

where $H_0$ is the present value of the Hubble parameter and $\Omega_{m_0}$ is the present value of matter density parameter defined as $\Omega_{m_0} = \frac{8\pi G\rho_m}{3H^2}$. We set $\Omega_{m_0} = 0.31$ and $H_0 = 67.3 km/s/Mpc$. We then solve Eq. (20)-(23) numerically to obtain the solutions for the scale factor and hence for the Hubble parameter. In order to achieve this we set the initial conditions such that the evolution of $H(z)$ that we obtain for $z = z_{CMB} \approx 1100$ coincides with $H_{\Lambda CDM}$, namely $H(z \rightarrow z_{CMB}) \approx H_{\Lambda CDM}$ while

$H(z \to 0) > H_{\Lambda CDM}(z \to 0)$. For our present analysis we have one model parameter, i.e. $\zeta$, which determines the late-time deviation of the model from ΛCDM scenario.

In Fig. 1 we plot the evolution of the dark-energy equation-of-state parameter in terms of the redshift. As we can see from the figure, $w_{DE} < -1$ most of the time, thereby depicting phantom evolution which implies faster expansion. The phantom behavior is one of the mechanisms that can lead to the Hubble tension alleviation [79, 80] (see also the discussion in [8]), and as we will see in the following, this is exactly what happens.

In Fig. 2, we plot the normalised combination $H(z)/(1+z)^{3/2}$ as a function of the redshift for ΛCDM cosmology, and for Model I for different values of $\zeta$. Here we used $\zeta = -8, -10, -12$ in $H_0$ units. As $\chi$ is dimensionless and $\phi$ has the dimensions of $GeV^2$, performing dimensional analysis we find that $\zeta$ has dimensions of $GeV^{-4}$. We find that the present value of $H_0$ depends on the model parameter $\zeta$ as expected. For $\zeta = -10$, the present value of the Hubble parameter is around $H_0 \approx 74km/s/Mpc$, which is consistent with the direct measurement of the present Hubble parameter. Values of $\zeta$ higher or lower than this give higher or lower values for $H_0$ respectively. Positive $\zeta$ corresponds to $H_0$ values lower than the value of $H_0$ in ΛCDM, hence they are not relevant for our present analysis. Also, as we consider more and more negative values of $\zeta$, the value of $H_0$ goes much beyond 74 km/s/Mpc, hence are again irrelevant with respect to the direct measurements of the present Hubble parameter. We have concentrated in and around the value of $\zeta = -10$ which is consistent with the direct measurements of the present Hubble parameter.

In Fig. 3, we have further shown the behaviour of the deceleration parameter $q$ with redhsift for model I for $\zeta = -10$. As we see, the redshift at which transition from deceleration to acceleration occurs is around $z = 0.68$, in agreement with current observations.

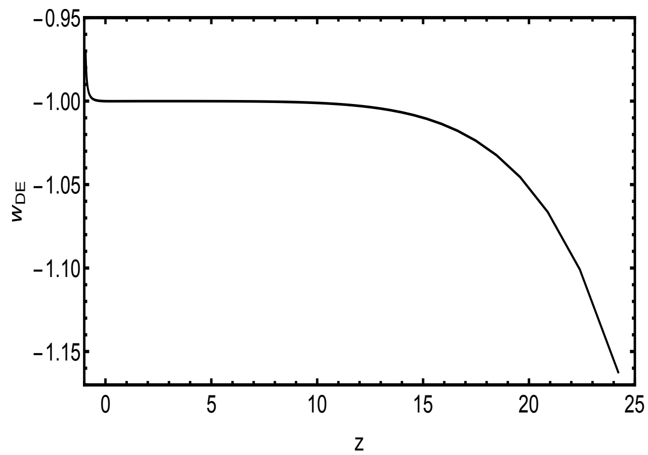


**Figure 1:** *The effective dark-energy equation-of-state parameter $w_{DE}$ as a function of the redshift, for Model I for $\zeta = -10$ in $H_0$ units.*

In summary, as we observe, there exist a range of the free model parameter $\zeta$ that is able to reproduce a Hubble function evolution that coincides with ΛCDM cosmology at high redshifts, but at late times it alleviates the $H_0$ tension. The reason that this happens is the fact that the effective dark-energy equation-of-state parameter exhibits a phantom behavior (following the general requirements of [8, 80]).

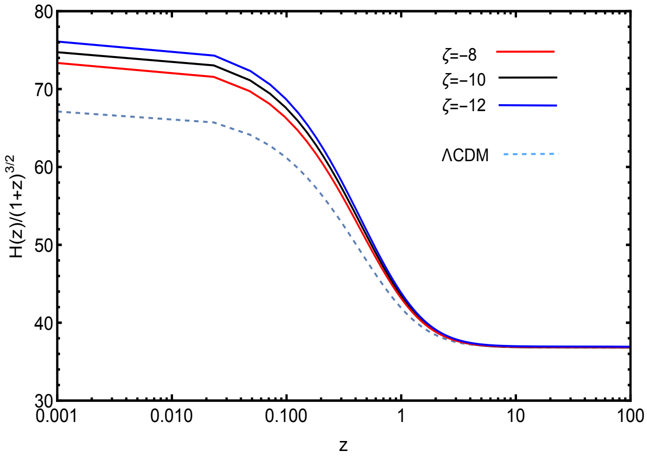


**Figure 2:** *The normalized combination $H(z)/(1+z)^{3/2}$ as a function of the redshift, for ΛCDM cosmology (blue dotted line), and for Model I for $\zeta = -12$ (solid blue line), for $\zeta = -10$ (solid black line), and for $\zeta = -8$ (solid red line), in $H_0$ units.*

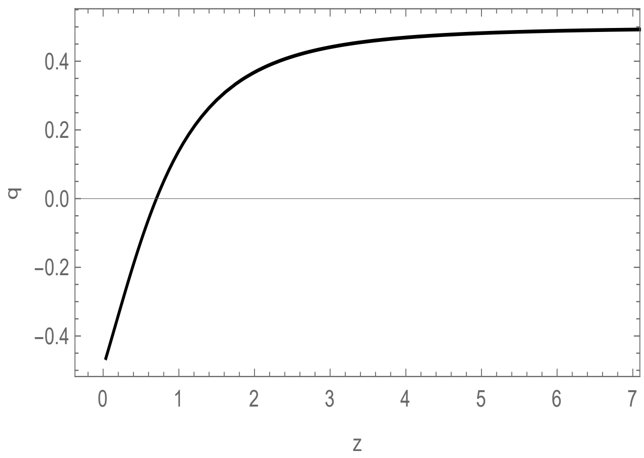


**Figure 3:** *The deceleration parameter $q$ as a function of redshift $z$ for Model I for $\zeta = -10$*



### 4.2 Model II

As a next we consider the case where

$$\mathcal{K}(\phi, B) = \frac{1}{2}\phi \quad \text{and} \quad \mathcal{G}(\phi, B) = \xi B, \tag{27}$$

with $\xi$ the corresponding coupling constant. Thus, the Friedmann equations (9),(10) become

$$3H^2 - \rho_m - \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\left(1 - 2e^{\sqrt{\frac{2}{3}}\chi}\right)\phi$$
$$+\xi\dot{\phi}^3\left(\sqrt{6}\dot{\chi} - 6H\right) = 0, \tag{28}$$

$$
\begin{aligned}
3H^2 + 2\dot{H} + p_m + \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\left(1 - 2e^{\sqrt{\frac{2}{3}}\chi}\right)\phi \\
-\frac{1}{3}\xi\dot{\phi}^2\left(\sqrt{6}\dot{\phi}\dot{\chi} + 6\ddot{\phi}\right) = 0,
\end{aligned} \tag{29}
$$

while the two scalar field equations (11) and (12) read as

$$
\begin{aligned}
\ddot{\chi} + 3H\dot{\chi} + \frac{1}{2\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\chi}\left(1 - e^{\sqrt{\frac{2}{3}}\chi}\right)\phi \\
-\sqrt{6}\,\xi\dot{\phi}^2\left(H\dot{\phi} + \ddot{\phi}\right) = 0,
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&\xi\dot{\phi}\Big\{2\left(-6H + \sqrt{6}\dot{\chi}\right)\ddot{\phi} \\
&\quad +\dot{\phi}\left[-6\dot{H} + 3H\left(-6H + \sqrt{6}\dot{\chi}\right) + \sqrt{6}\ddot{\chi}\right]\Big\} \\
&+\frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\left(1 - 2e^{\sqrt{\frac{2}{3}}\chi}\right) = 0.
\end{aligned} \tag{31}
$$

Therefore, in this case the effective dark-energy energy density and pressure (15),(16) write as

$$
\begin{aligned}
\rho_{DE} = \frac{1}{2}\dot{\chi}^2 - \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\left(1 - 2e^{\sqrt{\frac{2}{3}}\chi}\right)\phi \\
-\xi\dot{\phi}^3\left(\sqrt{6}\dot{\chi} - 6H\right),
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
p_{DE} = \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\left(1 - 2e^{\sqrt{\frac{2}{3}}\chi}\right)\phi \\
-\frac{1}{3}\xi\dot{\phi}^2\left(\sqrt{6}\dot{\phi}\dot{\chi} + 6\ddot{\phi}\right).
\end{aligned} \tag{33}
$$

Let us now proceed to the numerical investigation of the above equations. Similarly to the previous Model I, we choose the initial conditions such that our scenario matches $\Lambda$CDM cosmology for $z \approx 1100$. In Fig. 4, we depict the evolution of the dark-energy equation-of-state parameter with the redshift. As in the case of the previous subsection, here we also see that $w_{DE} < -1$ for most redshifts, thereby depicting phantom evolution, thus serving as a mechanism for Hubble tension alleviation.

In Fig. 5, we present the normalised $H(z)/(1+z)^{3/2}$ as a function of the redshift for $\Lambda$CDM cosmology, and for model II for different values of $\xi$, namely $\xi = -8, -10, -12$. Working as before, in energy units, the dimension of $\xi$ is $GeV^{-8}$. As expected, we find that the present Hubble value $H_0$ depends on the model parameter $\xi$. Specifically, for $\xi = -10$ it is around $H_0 \approx 74 km/s/Mpc$, which is consistent with the directly measured value of the Hubble parameter. Values of $\xi$ higher or lower than this give higher or lower values for $H_0$ respectively.

In Fig. 6, we have shown the evolution of the deceleration parameter $q$ with $z$ for model II for $\xi = -10$. The transition redshift between deceleration and acceleration for this case is around $z = 0.65$, also in agreement with current observations.

We close our analysis by confronting the two examined models with Cosmic Chronometer (CC) cosmological data. This datasets is based on the $H(z)$ measurements through the relative

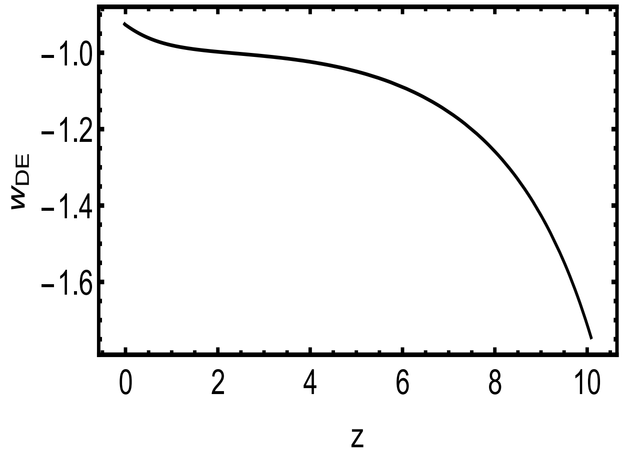


**Figure 4:** *The effective dark-energy equation-of-state parameter $w_{DE}$ as a function of the redshift, for Model II for $\xi = -10$ in $H_0$ units.*

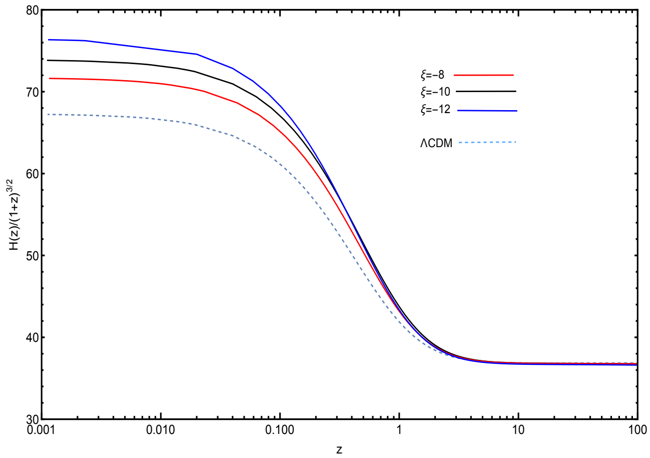


**Figure 5:** *The normalized combination $H(z)/(1+z)^{3/2}$ as a function of the redshift, for $\Lambda$CDM cosmology (blue dotted line), and for Model II for $\xi = -12$ (solid blue line), for $\xi = -10$ (solid black line), and for $\xi = -8$ (solid red line), in $H_0$ units.*

ages of passively evolving galaxies and the corresponding estimation of $dz/dt$ [81]. In Fig. 7 we confront the predicted $H(z)$ evolution of our models, alongside the one of $\Lambda$CDM scenario, with the $H(z)$ Cosmic Chronometer Data [82] at $1\sigma$ confidence level. As we deduce, the agreement is very good, and the theoretical $H(z)$ evolution lies within the direct measurements of the $H(z)$ from the CC data.

## 5. Conclusions

New gravitational scalar-tensor theories are novel modifications of gravity, consisting of a Lagrangian with the Ricci scalar and its first and second derivatives in a specific combination

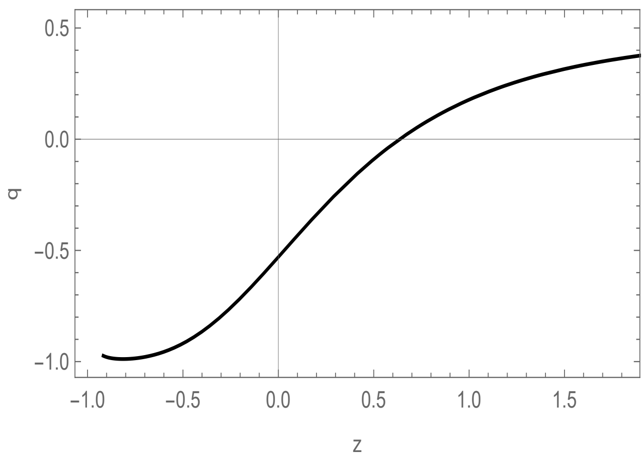


**Figure 6:** *The deceleration parameter $q$ as a function of redshift $z$ for Model II for $\xi = -10$*

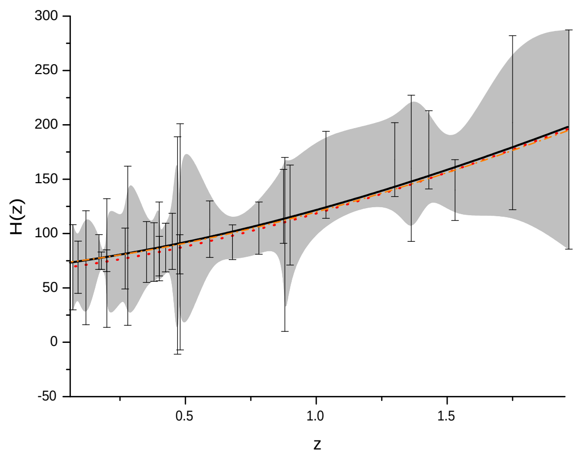


**Figure 7:** *The $H(z)$ in units of km/s/Mpc as a function of the redshift, for $\Lambda$CDM scenario (red dotted line), for Model I with $\zeta = -10$ (orange dashed-dotted), and for Model II with $\xi = -10$ (black solid line) in $H_0$ units, on top of the Cosmic Chronometers data points from [82] at $1\sigma$ confidence level. We have imposed* $\Omega_{m_0} = 0.31$.

that makes the theory free of ghosts. Such constructions propagate 2+2 degrees of freedom, thus forming a subclass of bi-scalar extensions of general relativity.

In the present work we investigated the possibility of resolving the Hubble tension using these new gravitational scalar tensor theories. Considering a homogenoeus and isotropic background, we extracted the Friedmann equations, as well as the evolution equations of the new extra scalar degrees of freedom. We obtained an effective dark energy sector that consists of both extra scalar degrees of freedom.

We then studied the cosmological behaviour of two specific models, imposing as initial conditions at high redshifts the coincidence of the behaviour of the Hubble function with that predicted by

$\Lambda$CDM cosmology. However, we showed that as time passes, the effect of bi-scalar modifications become important and thus at low redshifts the Hubble function acquires increased values in a controlled way. In particular, the present value of the Hubble parameter is sensitive to the choice of the model parameters.

In both models we showed that at high and intermediate redshifts the Hubble function behaves identically to that of $\Lambda$CDM scenario, however at low redshifts it acquires increased values, resulting to $H_0 \approx 74 km/s/Mpc$ for particular parameter choices. Hence, these new gravitational scalar tensor theories can alleviate the $H_0$ tension. The mechanism behind this behavior is the fact that the effective dark-energy equation-of-state parameter exhibits phantom behavior, which implies faster expansion, and it is one of the theoretical requirements that are capable of alleviating the $H_0$ tension [8, 80]. Finally, we further confronted our models with Cosmic Chromometer data and we found they are viable and in agreement with observations.

In conclusion, in this first work on the subject we deduced that the $H_0$ tension can be alleviated in the framework of new geometric gravitational theories. Definitely, the full verification of the above result requires a complete observational analysis, using data from Supernovae type Ia (SNIa), Baryonic Acoustic Oscillations (BAO), Redshift Space Distortion (RSD), and Cosmic Microwave Background (CMB) observations. Such a full and detailed observational confrontation, is left for a future project.

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