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Spatial curvature and periodic boundary conditions in cosmology

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We discuss the role of spatial curvature and periodic boundary conditions in numerical studies of cosmology. In particular, we discuss simple spatially inhomogeneous G_2 models where some qualitative results are possible.

1. Introduction

Cosmology concerns the large scale behaviour of the Universe within a theory of gravity, which is usually taken to be General Relativity (GR) [1]. The "Cosmological Principle", which asserts that on large scales the Universe can be well–modeled by a solution to Einstein's field equations (EFE) which is spatially homogeneous and isotropic, leads to the background Friedmann-Lemaıtre-Robertson-Walker (FLRW) model (with constant spatial curvature) with the cosmological constant, Λ, representing dark energy and CDM is the acronym for cold dark matter (or so-called ΛCDM concordance cosmology or standard cosmology for short). Early universe inflation is often regarded as a part of the standard model. The background spatial curvature of the universe, characterized by the normalized curvature parameter, is predicted to be negligible by inflationary models [2]. Regardless of whether inflation is regarded as part of the standard model or not, spatial curvature is assumed zero [1].

One of the greatest challenges in cosmology is understanding the origin of the structure of the universe. Under the hypothesis that cosmic structure grew out of small initial fluctuations, we can then study their evolution on sufficiently large scales using linear perturbation theory (LPT). The spatially inhomogeneous perturbations exist on the uniform flat FLRW background spacetime. Cosmic inflation provides a causal mechanism for primordial cosmological perturbations, through the generation of quantum fluctuations in the inflaton field, which act as seeds for the observed anisotropies in the cosmic microwave background (CMB) [3] and large scale structure of our universe. At late times and sufficiently small scales (much smaller than the Hubble scale) fluctuations of the cosmic density are not small. LPT is then not adequate and clustering needs to be treated non-linearly. Usually this is studied with (non-relativistic) N-body simulations. Recently cosmological non-linear perturbations have been studied at second-order and non perturbative relativistic effects have been studied computationally [1].

Standard cosmology has been very successful in describing current observations up to various possible anomalies [4], which includes the tension between the recent determination of the local value of the Hubble constant based on direct measurements of supernovae [5] and the value derived from the most recent CMB data [3, 6]. Also, since the Universe is not isotropic or spatially homogeneous on local scales, the effective gravitational FE on large scales should be obtained by averaging the EFE of GR, after which a smoothed out macroscopic geometry and macroscopic matter fields is obtained. The averaging of the EFE for local inhomogeneities can lead to significant dynamical backreaction effects on the average evolution of the Universe [7], and can certainly affect precision cosmology at the level of 1 % [1].

2. Spatial curvature

In standard cosmology the spatial curvature is assumed to be constant and zero (or at least very small). But there is, as yet, no fully independent constraint with an appropriate accuracy

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that gaurentees a value for the magnitude of the effective normalized spatial curvature Ω_k of less than approximately 0.01. Moreover, a small non-zero measurement of Ω_k at such a level perhaps indicates that the assumptions in the standard model are not satisfied. It has also been increasingly emphasised that spatial curvature is, in general, evolving in relativistic cosmological models [7, 8].

It is necessary to make assumptions to derive models to be used for cosmological predictions and comparison with observational data. But it is important to check whether the assumptions "put in" affect the results that "come out". In addition, we can only confirm the consistency of assumptions and we cannot rule out alternative explanations. The assumption of a FLRW background on cosmological scales presents a number of problems [1]. In particular, the assumptions that underscore the use of a 1+3 spacetime split and a global time and a background inertial coordinate system (Gaussian normal coordinates which are approximately Cartesian and orthogonal) over a complete Hubble scale 'background' patch in the standard model lead to the simple conditions that the spatial curvature (and the vorticity) must be very small. The assumptions for the existence of exact periodic boundary conditions (PBC) (appropriate on scales comparable to the homogeneity scale) imply necessarily that the spatial curvature is exactly zero. In the actual standard model the Universe is taken to be simply connected and hence the background is necessarily flat. Any appropriate approximation will amount to Ω_k being less than the perturbation (e.g., LPT) scale.

There are also assumptions behind the weak field approach, the applicability of perturbation theory, Gaussian initial conditions, etc., that include neglecting spatial curvature. It is often claimed that backreaction can be neglected, but in LPT the fluctuations are assumed Gaussian, which means that at the linear level all averages are zero by construction. We should also be cogniscent that any intuition based on Newtonian theory may be misleading; e.g., the average spatial curvature of voids and clustered matter (with negative and positive curvature, respectively) is not necessarily zero within GR. Thus, in standard cosmology the spatial curvature is assumed to be zero, or at least very small and at most first order in terms of the perturbation approximation, in order for any subsequent analysis to be valid. Any prediction larger than this indicates an inconsistency in the approach. The standard model cannot be used to predict a small spatial curvature.

Indeed, current constraints on the background spatial curvature within the standard cosmology are often used to "demonstrate" that it is dynamically negligible, primarily based on CMB data. However, the recently measured temperature and polarization power spectra of the CMB provides a 99% confidence level detection of a negative (non-zero) Ω_k [6]. Direct measurements of the spatial curvature Ω_k using low-redshift data such as supernovae, baryon acoustic oscillations and Hubble constant observations do not place tight constraints on the spatial curvature and allow for a large range of possible values (but generally do include spatial flatness). Attempts at a consistent analysis of CMB anisotropy data in the non-flat case suggest a closed model with $\Omega_k \sim 1\%$. Including low-redshift data [6] provides weak evidence in favor of a closed spatial geometry, with stronger evidence for closed spatial hypersurfaces coming from dynamical dark energy models [9]. In principle, and as noted above, a small non-zero measurement of Ω_k perhaps indicates that the assumptions in the standard model are not met, thereby motivating models with curvature at the level of a few percent. In addition, if the geometry of the universe does deviate, even slightly, from the standard FLRW geometry, then the spatial curvature will no longer necessarily be constant and any effective spatial flatness may not be preserved [8].

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3. Periodic boundary conditions

In the general inhomogeneous cosmological case, the formulation of the evolution eqns. consists of utilizing global Cartesian coordinates and metric variables, where initial conditions on the metric are specified. In the actual N-body simulations a cell of length L (a fraction of the Hubble scale) is taken and the system is integrated where the PBC at L is applied at all times (and is used at every step to compute spatial derivatives near the edges of the cell). Hence spatial curvature is held zero on the boundary, and in this sense the growth of spatial curvature is suppressed, relatively speaking. It is of importance to verify numerically that initial PBC constrain the spatial curvature to be zero at boundary for all time, which is maintained at zero even below numerical errors that begin to grow.

In addition, if in N-body simulations the cell is restricted to a finite size L, then periodicity implies that longer wavelength inhomogeneities than L are effectively neglected. It is a general concern as to what effect this may have on numerical simulations. Finally, it is not absolutely clear whether periodic initial conditions guarantee periodic evolution for all times. There is no general mathematical theorem on the preservation of discrete symmetries (that periodicity implies). But note that if it were true that periodic initial conditions are propagated, then it follows that the spatial curvature is indeed constrained at L for all times.

In the few current relativistic cosmological simulations, the spatial curvature can have large local fluctuations but remains small when averaged over large scales. It becomes extremely small, $\Omega_R \sim 10^{-8}$, when averaged over the simulation's whole periodic box [10]. Work is in progress for a more in-depth investigation of average spatial curvature from numerical relativity, along with the constraints due to the PBC, in a more frame-independent formalism [11]. However, the evidence is that these numerical solutions have an exceptionally and unnaturally small spatial curvature in a neighbourhood of the periodic boundary, which affects the average spatial curvature in the whole cell, and can perhaps be ascribed to the use of PBC.

In different formulations of the EFE (e.g., the G_2 case below), it is difficult to compare with the standard numerical results of the full 3D code and the usual N-body simulations. Indeed, since the EFE are hyperbolic, and depending on the particular formulation of the evolution equations, once the initial conditions are fully specified it is not clear whether the application of spatial PBC are even required.

Because we wish to numerically study non-PBC, we shall consequently use zooming techniques [12], in which we use a coordinate system adapted to the structure that is under investigation. In particular, we can choose a coordinate system that shrinks exponentially with time, in which the new coordinates zoom in on the worldline, and where the rate of focus is controlled [13]. The numerical grid ends at a fixed coordinate value $X = L$. Ordinarily, that would call for a boundary condition at L, but we can use the method of excision, which can be applied to any hyperbolic equations where the outer boundary is chosen so that all modes are outgoing. In that case the equations of motion are simply implemented at the outer boundary; no boundary condition is needed (or even allowed).

4. G_2 models

The FLRW cosmologies describe an expanding Universe that is exactly spatially homogeneous and isotropic [14]. It is widely believed that on a sufficiently large spatial scale the Universe can be described by such a model, at least since the time of last scattering. However, there are good reasons for studying cosmological models more general than the FLRW models. The observable part of the Universe is not exactly spatially homogeneous and isotropic on any spatial scale but, from a practical point of view, we are interested in models that are "close to FLRW" in some appropriate dynamical sense. The usual way to study deviations from an FLRW model is to apply LPT. But it is not known how reliable the linear theory is and, moreover, in using it one is a priori excluding the possibility of finding important non-linear effects.

Therefore, it is necessary to consider more general models, and especially spatially inhomogeneous models, in order to investigate the constraints that observations impose [15]. Indeed, perhaps it could be argued that we should not study exotic ideas until conventional GR (with inhomogeneities) has been fully explored. In the first instance, it is perhaps advisable to investigate the effect of spatial curvature and PBC in a special inhomogeneous model. In the specific case of G_2 models some analytical and qualitative results are possible.

The simplest spatially inhomogeneous cosmological models have two spacelike commuting Killing vector fields and thus have one degree of freedom as regards spatial inhomogeneity. This class of G_2 cosmologies is governed by the EFE evolution eqns., which are partial differential equations (PDE) in two independent variables [14]. Since we are interested in late time evolution, we focus on G_2 cosmologies with perfect fluid matter content, and especially dust [16]. Formally, the evolution of a G_2 cosmology is described by an orbit in an infinite-dimensional dynamical state space. Following [14, 17], we use the dynamical orthonormal frame formalism adapted to the G_2 orbits, which has proved effective in studying such cosmologies. And we shall define scale-invariant dependent variables by normalisation with the area expansion rate of the G_2–orbits, in order to obtain the evolution equations as a system of PDE in first-order symmetric hyperbolic (FOSH) format, which provides a natural framework for numerical studies.

In the G_2 FOSH formulation of the evolution eqns., the variables are not metric variables and consequently the initial conditions are not specified on the metric functions. Hence the meaning of periodic initial data has a different meaning to that in the usual standard sense. Indeed, in this formulation once the initial conditions are fully specified the application of spatial boundary conditions are not even required.

Preliminary numerical results [12] show that as the worldlines generally evolve (drift) away from the flat FLRW model (regarded as a "saddle point" in the dynamical state space), Ω_k is clearly suppressed as a result of the application of PBC. The growth of Ω_k can perhaps be attributed to a term whose wavelength is very large.

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