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A consistent limit of 11D Supergravity

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Motivated by old and new developments in non-relativistic string theory, we show that there exists a consistent non-relativistic limit of eleven-dimensional supergravity. Before taking the limit we give a short review of the underlying Membrane Newton-Cartan geometry. This geometry is a particular extension of the Newton-Cartan geometry in the sense that the two nondegenerate metrics of Newton-Cartan geometry (one to measure time intervals and another one to measure spatial distances) are replaced by two nondegenerate metrics of rank 3 and rank 8, respectively. An important role in describing this geometry and in the consistency of the limit is played by the so-called intrinsic torsion tensor components. These are the components of the torsion tensor that are independent of the spin-connection.
After expanding the action of eleven-dimensional supergravity as a power series of a contraction parameter, we show how the different divergences that arise when taking the limit can be tamed. We furthermore show how the divergences that arise in expanding the supersymmetry rules can be controlled by imposing a supersymmetric set of constraints. This leads to a Membrane Newton-Cartan supergravity theory where the Newton potential can be identified with the component of the 3-form of eleven-dimensional supergravity that points in the three directions corresponding to the rank 3 degenerate metric.

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1. Motivation

The motivation for the research described here goes back to the first proposal for a non-relativistic closed string theory at the turn of the century [1, 2]. This proposal made use of the observation of [3] that, assuming a compact spatial direction, strings with a positive winding can survive a critical limit of relativistic string theory whereas strings with a negative winding behave like anti-particles and become infinitely heavy. In the supersymmetric case such a string theory should lead to a non-relativistic version of $\mathcal{N} = 1$ supergravity in ten spacetime dimensions. Indeed, about twenty years later it was shown that $\mathcal{N} = 1$ supergravity in ten dimensions allows for a consistent non-relativistic limit with a particular degenerate geometric structure [4]. This so-called String Newton-Cartan geometry is a particular extension of the Newton-Cartan geometry in the sense that the two nondegenerate metrics of Newton-Cartan geometry (one to measure time intervals and another one to measure spatial distances) are replaced by two nondegenerate metrics of rank 2 and rank 8, respectively. These two non-degenerate metrics refer to the two directions longitudinal to the string and to the eight directions transverse to the string.

Soon after the proposal of [1, 2], the possibility to define a non-relativistic closed supermembrane theory in eleven dimensions was considered [5]. This suggests that it might also be possible to define a consistent non-relativistic limit of eleven-dimensional supergravity. Recently, the first steps in defining such a limit for the bosonic part of eleven-dimensional supergravity were undertaken [6]. The underlying geometry in this case is a Membrane Newton-Cartan geometry with two nondegenerate metrics of rank 3 and rank 8 that refer to the three directions longitudinal to the membrane and to the eight directions transverse to the membrane.

We will show how supersymmetry can be incorporated by imposing a supersymmetric set of constraints. These constraints involve certain torsion tensor components, called 'intrinsic torsion' tensor components, that are independent of the spin-connection fields. Before describing the limit of eleven-dimensional supergravity, we will first review a few relevant details of this Membrane Newton-Cartan geometry.

2. Membrane Newton-Cartan Geometry

Before introducing Membrane Newton-Cartan geometry as the geometry underlying non-relativistic eleven-dimensional supergravity, we will first define Membrane Galilean geometry since that is the part which can be defined in a mathematically rigorous way. After that we will explain which further ingredients need to be added to extend the Membrane Galilean geometry to a Membrane Newton-Cartan geometry.

Our starting point for defining a Membrane Galilean geometry is an 11-dimensional manifold with a degenerate metric structure that reduces the local structure group to [7, 8]

$$\mathrm{SO}(2,1) \times \mathrm{SO}(8) \ltimes \mathbb{R}^{24} \,. \tag{1}$$

The Minkowskian worldvolume of a non-relativistic membrane at rest divides up the tangent space directions of this manifold in 3 'longitudinal' directions and 8 'transversal' ones. The $\mathrm{SO}(2,1)$ and $\mathrm{SO}(8)$ factors of the structure group then correspond to Lorentz transformations of the 3 longitudinal directions and rotations of the 8 transversal directions, respectively. The $\mathbb{R}^{24}$ factor represents boost transformations that can transform transversal directions into longitudinal ones, but not vice versa. We will refer to these as 'membrane Galilean boosts'.

In analogy to particle Galilean geometry, the Cartan formulation of Membrane Galilean geometry includes two different types of one-forms (also called soldering forms): a 'longitudinal Vielbein' ${\tau_\mu}^A$ ($A = 0, 1, 2$) and a 'transversal Vielbein' ${e_\mu}^a$ ($a = 3, \cdots, 10$). The flat longitudinal index A can be freely raised and lowered with a 3-dimensional Minkowski metric $\eta_{AB} = \mathrm{diag}(-1, 1, 1)$, whereas for the flat transversal index a this is done using a 8-dimensional Euclidean metric δ_{ab}. These one-forms transform under the structure group (1) in a reducible, indecomposable manner according to the following local transformation rules:

$$\delta {\tau_\mu}^A = {\lambda^A}_B {\tau_\mu}^B \,, \qquad\qquad \delta {e_\mu}^a = {\lambda^a}_b {e_\mu}^b - {\lambda_A}^a {\tau_\mu}^A \,. \tag{2}$$

Here, $\lambda^{AB} = -\lambda^{BA}$ corresponds to the parameters of longitudinal SO(2, 1) Lorentz transformations, $\lambda^{ab} = -\lambda^{ba}$ to that of transversal SO(8) rotations, while the λ^{Aa} are the 24 Membrane Galilean boost parameters. Similar to the particle case, one introduces an 'inverse longitudinal Vielbein' ${\tau_A}^\mu$ and an 'inverse transversal Vielbein' ${e_a}^\mu$ (both are also called frame fields) such that the matrices $\begin{pmatrix} {\tau_\mu}^A & {e_\mu}^a \end{pmatrix}$ and the matrices $\begin{pmatrix} {\tau_A}^\mu \\ {e_a}^\mu \end{pmatrix}$ are each other's inverse.

The longitudinal and inverse transversal Vielbeine can be 'squared' to obtain two degenerate symmetric (covariant and contravariant) two-tensors that are invariant under local SO(2, 1), SO(8) and Membrane Galilean boost transformations:

$$\tau_{\mu\nu} \equiv {\tau_\mu}^A {\tau_\nu}^B \eta_{AB} \,, \qquad\qquad h^{\mu\nu} \equiv {e_a}^\mu {e_b}^\nu \delta^{ab} \,. \tag{3}$$

These two tensors constitute a degenerate metric structure on the manifold. The covariant metric $\tau_{\mu\nu}$ is referred to as the 'longitudinal metric'. Its kernel is spanned by the 8 vectors ${e_a}^\mu$ and it thus has rank 3. The contravariant metric $h^{\mu\nu}$ is called the 'transversal metric' and has rank 8, since its kernel is spanned by the 3 one-forms ${\tau_\mu}^A$.

To define a metric compatible affine connection in Membrane Galilean geometry, we first introduce a structure group connection Ω_μ that takes values in the Lie algebra of (1)

$$\Omega_\mu = \frac{1}{2}{\omega_\mu}^{AB} J_{AB} + \frac{1}{2}{\omega_\mu}^{ab} J_{ab} + {\omega_\mu}^{Aa} G_{Aa} \,, \tag{4}$$

where $J_{AB} = -J_{BA}$, $J_{ab} = -J_{ba}$ and G_{Aa} are generators of the Lie algebras of SO(2, 1), SO(8) and $\mathbb{R}^{24}$, respectively. We will refer to ${\omega_\mu}^{AB} = -{\omega_\mu}^{BA}$, ${\omega_\mu}^{ab} = -{\omega_\mu}^{ba}$ and ${\omega_\mu}^{Aa}$ as spin connections for longitudinal Lorentz transformations, transversal rotations and Membrane Galilean boosts, respectively. We next introduce an affine connection $\Gamma^\rho_{\mu\nu}$ by imposing the following 'Vielbein postulates':

$$\begin{aligned} &\partial_\mu {\tau_\nu}^A - {\omega_\mu}^A{}_B {\tau_\nu}^B - \Gamma^\rho_{\mu\nu} {\tau_\rho}^A = 0 \,, \\ &\partial_\mu {e_\nu}^a - {\omega_\mu}^{ab} e_{\nu b} + {\omega_\mu}^{Aa} \tau_{\nu A} - \Gamma^\rho_{\mu\nu} {e_\rho}^a = 0 \,. \end{aligned} \tag{5}$$

The form of these Vielbein postulates is motivated by the requirement that the affine connection $\Gamma^\rho_{\mu\nu}$ should be invariant under the local structure group transformation rules of the Vielbein fields and spin connections. Using (5), one can express $\Gamma^{\rho}_{\mu\nu}$ in terms of the Vielbeine $\tau_\mu{}^A$, $e_\mu{}^a$, their inverses and the spin connections $\omega_\mu{}^{AB}$, $\omega_\mu{}^{ab}$, $\omega_\mu{}^{Aa}$ as follows:

$$\Gamma^{\rho}_{\mu\nu} = \tau_A{}^\rho \partial_\mu \tau_\nu{}^A + e_a{}^\rho \partial_\mu e_\nu{}^a - \omega_\mu{}^A{}_B \tau_\nu{}^B \tau_A{}^\rho - \omega_\mu{}^a{}_b e_\nu{}^b e_a{}^\rho + \omega_\mu{}^{Aa} \tau_{\nu A} e_a{}^\rho \,. \tag{6}$$

We will view the torsion $2\Gamma^{\rho}_{[\mu\nu]}$ of the affine connection as an independent and a priori arbitrary geometric ingredient. We will split it into 'longitudinal torsion' components $T_{\mu\nu}{}^A$ along $\tau_A{}^\rho$ and 'transversal torsion' components $E_{\mu\nu}{}^a$ along $e_a{}^\rho$:

$$2\Gamma^{\rho}_{[\mu\nu]} = \tau_A{}^\rho T_{\mu\nu}{}^A + e_a{}^\rho E_{\mu\nu}{}^a \,. \tag{7}$$

These equations imply that under local SO(2, 1), SO(8) and Membrane Galilean boosts, $T_{\mu\nu}{}^A$ and $E_{\mu\nu}{}^a$ transform as follows:

$$\delta T_{\mu\nu}{}^A = \lambda^A{}_B T_{\mu\nu}{}^B \,, \qquad \delta E_{\mu\nu}{}^a = \lambda^a{}_b E^b_{\mu\nu} - \lambda_A{}^a T_{\mu\nu}{}^A \,. \tag{8}$$

By antisymmetrizing the Vielbein postulates (5), one obtains the following equations that are covariant with respect to the local structure group transformations:

$$T_{\mu\nu}{}^A = 2\partial_{[\mu}\tau_{\nu]}{}^A - 2\omega_{[\mu}{}^A{}_B \tau_{\nu]}{}^B \,, \tag{9}$$

$$E_{\mu\nu}{}^a = 2\partial_{[\mu} e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab} e_{\nu]b} + 2\omega_{[\mu}{}^{Aa} \tau_{\nu]A} \,. \tag{10}$$

We now wish to investigate which components of the above torsion two-forms T^A and E^a are independent of a spin-connection. We will call these the 'intrinsic torsion' tensor components. For this purpose, we first decompose the curved indices μ of the torsion two-forms into longitudinal indices A and transversal indices a according to the following decomposition rule for any one-form V_μ:

$$V_\mu = \tau_\mu{}^A V_A + e_\mu{}^a V_a \quad \text{or} \quad V_A = \tau_A{}^\mu V_\mu \;\text{ and }\; V_a = e_a{}^\mu V_\mu \,. \tag{11}$$

Decomposing the 2-forms (9) and (10) in this way, we find that the following tensor components correspond to intrinsic torsion:

$$T_a{}^{\{AB\}} \,, \qquad T_a{}^A{}_A \,, \qquad T_{ab}{}^A \,. \tag{12}$$

We use here a notation where $\{AB\}$ indicates the symmetric traceless part of AB.

To classify the different constraints that one may impose on the intrinsic torsion tensor components given in (12), it is important to realize that under Galilean boosts, some components of the intrinsic torsion tensors transform to other components, and hence, those torsion tensors cannot be set to zero independently from other torsion components. The way that these boost transformations act on the torsion tensor components are displayed in Figure 1. This Figure shows that, besides zero intrinsic torsion, there also exist other boost-invariant sets of constraints on the geometry. One may systematically derive these sets of constraints by starting in Figure 1 with no constraints (generic intrinsic torsion) and, next, by adding more constraints starting from below by first setting the boost-invariant constraint $T_{ab}{}^A = 0$. Continuing in this way, one may add to this constraint a second constraint by setting one of the two upper components given in Figure 1 to zero such that one again obtains a boost-invariant set of constraints. The fifth possibility is that one sets both of

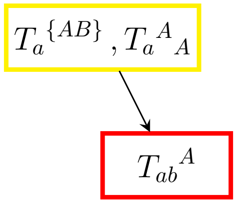

Figure 1: This Figure indicates the different intrinsic torsion tensor components. The arrow indicates the direction in which the Membrane Galilean boost transformations act. For instance, the boost transformation of $T_a{}^{\{AB\}}$ gives $T_{ab}{}^A$ but not the other way around.

the upper components to zero which leads to zero intrinsic torsion. These five distinct Membrane Galilean geometries that one obtains in this way are shortly discussed below (for more details, see [8]).

1. The intrinsic torsion is unconstrained.

2. $\mathbf{T_{ab}{}^A = 0}$: According to the Frobenius theorem this constraint implies that the foliation by transverse submanifolds of dimension 8 is integrable.

3. $\mathbf{T_{ab}{}^A = T_a{}^{\{AB\}} = 0}$: the foliation is integrable and one can show that the vectors $e_a{}^\mu$ are homothetic Killing vectors with respect to the longitudinal metric $\tau_{\mu\nu}$.

4. $\mathbf{T_{ab}{}^A = T_a{}^A{}_A = 0}$: the foliation is integrable and the worldvolume 3-form

$$\Omega_{\mu\nu\rho} = \epsilon_{ABC}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C \tag{13}$$

is closed, i.e. $d\Omega = 0$.

5. $\mathbf{T_{\mu\nu}{}^A = 0}$: all constraints mentioned above are valid.

This finishes our classification of the Membrane Galilean geometries. To extend these geometries to a Membrane Newton-Cartan geometry underlying non-relativistic eleven-dimensional supergravity the following three ingredients need to be added that will be discussed in more detail in the next section:

1. The geometry underlying non-relativistic eleven-dimensional supergravity contains an additional 3-form $c_{\mu\nu\rho}$ whose mathematical description seems to require, according to some of the literature, the introduction of the notion of gerbes. It transforms under Membrane boost transformations and plays an important role in describing the geometry.

2. The frame fields of Membrane Newton-Cartan geometry transform under an emergent an-isotropic local scale symmetry [6, 9] which requires an additional dilatation gauge field b_μ beyond the spin-connections. It has the effect that the would-be intrinsic torsion tensor components $T_a{}^A{}_A$ of Figure 1 contain a dilatation gauge field and therefore are not intrinsic anymore.

3. In the presence of supersymmetry the membrane Newton-Cartan geometry needs to be embedded into a so-called supergeometry which also contains fermionic intrinsic torsion tensor components. We will see that to describe the geometry underlying eleven-dimensional supergravity, we need to impose constraints on both the bosonic and fermionic intrinsic torsion tensor components as well as on the (super-covariant) extrinsic derivative of the 3-form.

All these additional features will play a role in the next section where we will define a consistent non-relativistic limit of eleven-dimensional supergravity.

3. A Consistent Limit of 11D Supergravity

We will first discuss a consistent limit of the bosonic part of 11D supergravity. This part has already been discussed in [6]. Next, we will discuss what happens if one extends to the supersymmetric case. Since this second part is work in progress [10], we will be brief there and only give an overview of the situation.

3.1 The Bosonic Case

Our starting point is the bosonic part of 11D supergravity with as basic fields the Elfbein $E^{\hat{a}}{}_\mu$ and three-form $C_{\mu\nu\rho}$ with $\hat{a}, \mu = 0, 1, \cdots 10$. The Lagrangian describing the dynamics of these basic fields is given by

$$\mathcal{L}(\text{bosonic}) = E\left(R(\Omega) - \tfrac{1}{2\cdot 4!}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\right) + \tfrac{1}{144^2}\epsilon^{\mu_1\dots\mu_{11}}F_{\mu_1\dots\mu_4}F_{\mu_5\dots\mu_8}C_{\mu_9\mu_{10}\mu_{11}}\,. \tag{14}$$

Here, E is the determinant of the elfbein $E^{\hat{a}}{}_\mu$ and $F_{\mu\nu\rho\sigma}$ is the external derivative of the three-form $C_{\mu\nu\rho}$:

$$F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}C_{\nu\rho\sigma]}\,. \tag{15}$$

To define a non-relativistic limit, we write $\hat{a} = (A, a)$ and make the following redefinition of the basic fields:

$$E_\mu{}^{\hat{a}} = (c\tau_\mu{}^A, c^{-1/2}e_\mu{}^a)\,, \tag{16}$$

$$C_{\mu\nu\rho} = -c^3\epsilon_{ABC}\tau_\mu{}^A\tau_\nu{}^B\tau_\rho{}^C + c_{\mu\nu\rho}\,. \tag{17}$$

Here, $\tau_\mu{}^A, e_\mu{}^a$ and $c_{\mu\nu\rho}$ are the fields that become basic non-relativistic fields after taking the limit $c \to \infty$. Substituting the redefinitions (16) and (17) into the Lagrangian (14), we obtain the following expansion:

$$S = c^3 S_3 + c^0 S_0 + c^{-3}S_{-3} + \dots \tag{18}$$

There are divergences of order c^3 arising from all three terms in the Lagrangian (14). The kinetic term of the three-form even gives rises to two different types of divergences. Fortunately, the divergence coming from the Einstein-Hilbert term cancels against a similar divergence originating form the kinetic term of the three-form. Note that this cancellation requires a fine-tuning between these two terms. A similar cancellation takes place when one considers a so-called critical limit of the sigma model action describing the coupling of a membrane to the basic fields $E_\mu{}^{\hat{a}}$ and $C_{\mu\nu\rho}$ via a kinetic term and a Wess-Zumino term. The remaining divergences coming from the three-form kinetic term and the Chern-Simons term do not cancel but, surprisingly, they combine into the following form:

$$S_3 = \int d^{11}x\ -\frac{e}{4!}f^{(-)abcd}f^{(-)}{}_{abcd}\,, \tag{19}$$

where $e = \det(\tau_\mu{}^A, e_\mu{}^a)$ and $f^{(-)}{}_{abcd}$ is the anti-self-dual part of f_{abcd} which are the SO(8) transverse components of the field-strength of the non-relativistic three-form $c_{\mu\nu\rho}$:

$$f_{\mu\nu\rho\sigma} = 4\partial_{[\mu} c_{\nu\rho\sigma]} \,. \tag{20}$$

A divergence of the form (19) can be tamed by introducing an anti-self-dual auxiliary field λ_{abcd} (with flat SO(8) indices) and replacing S_3 by the following two terms which contribute to S_0 and S_{-3}:

$$S(\lambda) = \int d^{11}x \frac{e}{4!}\Big(-2\lambda_{abcd} f^{(-)abcd} + \frac{1}{c^3}\lambda_{abcd}\lambda^{abcd}\Big) \,. \tag{21}$$

After solving for this auxiliary field from its equation of motion:

$$\lambda_{abcd} = c^3 f^{(-)}{}_{abcd} \tag{22}$$

and substituting this solution back into the action $S(\lambda)$ one re-obtains the original action (18) with the divergent term S_3. Note that the solution (22) determines how the auxiliary field transforms under all the symmetries of the theory before taking the limit. From that one can then derive how this auxiliary field transforms after taking the limit.

Making the redefinitions (16) and (17) and using the auxiliary field λ_{abcd} we are now able to take the limit $c \longrightarrow \infty$ of the relativistic action (14). We thus end up with the following non-relativistic action

$$\begin{aligned} \mathcal{L} = R^{(0)} &+ \tfrac{1}{2} T_{aA}{}^A T^a{}_B{}^B - \tfrac{1}{12} f_{Aabc} f^{Aabc} + \tfrac{1}{4} f^{abAB}\epsilon_{ABC} T_{ab}{}^C \\ &- \tfrac{2}{4!}\lambda_{abcd} f^{(-)abcd} + \frac{e^{-1}}{144^2}\epsilon^{\mu_1\dots\mu_{11}} f_{\mu_1\dots\mu_4} f_{\mu_5\dots\mu_8} c_{\mu_9\mu_{10}\mu_{11}} \end{aligned} \tag{23}$$

in terms of the basic non-relativistic fields

$$\{\tau_\mu{}^A \,, e_\mu{}^a \,, c_{\mu\nu\rho} \,, \lambda_{abcd}\} \,. \tag{24}$$

Here, $R^{(0)}$ is defined in eq. (2.23) of [6] as the term linear in c in the expansion of the Ricci scalar. We note that after taking the limit the first term in $S(\lambda)$ vanishes and the auxiliary field becomes a Lagrange multiplier imposing the constraint

$$f^{(-)abcd} = 0 \,. \tag{25}$$

The action (23) is invariant under the following local boost transformations with parameter $\lambda_A{}^a$, longitudinal Lorentz rotations with parameter $\lambda^A{}_B$, transversal rotations with parameter $\lambda^a{}_b$, gauge transformations with a 2-form parameter $\lambda_{\mu\nu}$ and under an additional emergent dilatation with parameter α:

$$\delta\tau_\mu{}^A = -\lambda^A{}_B \tau_\mu{}^B + \alpha\tau_\mu{}^A \,, \tag{26}$$

$$\delta e_\mu{}^a = \lambda_A{}^a \tau_\mu{}^A - \lambda^a{}_b e_\mu{}^b - \tfrac{1}{2}\alpha e_\mu{}^a \,, \tag{27}$$

$$\delta c_{\mu\nu\rho} = -3\epsilon_{ABC}\lambda^A{}_a e_{[\mu}{}^a \tau_\nu{}^B \tau_{\rho]}{}^C + 3\partial_{[\mu}\lambda_{\nu\rho]} \,, \tag{28}$$

$$\delta\lambda_{abcd} = \tfrac{2}{4!}\big(\lambda^A{}_{[a} f_{|A|bcd]} - \text{dual}\big) + 4\lambda^e{}_{[a}\lambda_{|e|bcd]} - \alpha\lambda_{abcd} \,. \tag{29}$$

The emergent dilatations are in line with the fact that the non-relativistic sigma model describing a membrane in a curved background [5] is invariant under the an-isotropic dilatations of ${\tau_\mu}^A$ and ${e_\mu}^a$.

A few words are in order on the Membrane Newton-Cartan geometry underlying the non-relativistic action (23). First of all, the formerly intrinsic tensor ${T_a}^A{}_A$ is not invariant under the emergent dilatations. It should therefore be replaced by a dilatation-covariant version that contains a dilatation gauge field b_μ. Therefore, this tensor should be considered as a conventional tensor that can be used to solve for the transverse components b_a of the dilatation gauge field. Secondly, as we explained in the previous section, the Membrane Newton-Cartan geometry is characterized by the additional three-form $c_{\mu\nu\rho}$. Like ${\tau_\mu}^A$ and ${e_\mu}^a$ and their inverses, this field transforms under boost transformations as in eq. (28). It's exterior derivative therefore contains a spin-connection. Like we did for the torsion tensors one can now consider different flat components of this exterior derivative and see whether or not they contain a spin-connection. Those, that do not contain a spin-connection we call intrinsic components. Therefore, on top of the intrinsic torsion tensor components given in eq. (12), excluding ${T_a}^A{}_A$, we have the following additional intrinsic exterior derivatives:

$$f_{abcd}\,, \qquad\qquad f_{Abcd}\,. \tag{30}$$

The other components, f_{ABcd} and f_{ABCd}, are conventional tensors containing a spin-connection. These tensors are set to zero in order to solve for some of the spin-connection components.

At this point we have 573 conventional tensors components

$${T_a}^A{}_A\,,\quad {T_a}^{[AB]}\,,\quad {T_{AB}}^C\,,\quad {E_{\mu\nu}}^a\,,\quad f_{ABcd}\,,\quad f_{ABCd} \tag{31}$$

to solve for the 616 spin-connections and dilatation gauge fields:

$${\omega_\mu}^{AB}\,,\quad {\omega_\mu}^{Aa}\,,\quad {\omega_\mu}^{ab}\,,\quad b_\mu\,. \tag{32}$$

This implies that we can not solve for 43 dilatation gauge field and spin-connection components. It turns out that these 43 independent components are given by

$$b_A \qquad \text{and} \qquad {\omega_a}^{\{AB\}}\,. \tag{33}$$

Since such independent components were absent in the relativistic action and transformation rules we started from, it implies that they cannot arise in the action and transformations rules after taking the non-relativistic limit. This has implications for the spin-connection curvatures [1]

$$R_{\mu\nu} = 2\,\partial_{[\mu} b_{\nu]}\,, \tag{34a}$$
$${R_{\mu\nu}}^{AB} = 2\,\partial_{[\mu}\omega_{\nu]}{}^{AB} + 2\,\omega_{[\mu}{}^{AC}\omega_{\nu]C}{}^B\,, \tag{34b}$$
$${R_{\mu\nu}}^{ab} = 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b\,, \tag{34c}$$
$${R_{\mu\nu}}^{Aa} = 2\,\partial_{[\mu}\omega_{\nu]}{}^{Aa} + 2\,\omega_{[\mu}{}^{AB}\omega_{\nu]B}{}^a + 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^{Ab} + 3\,b_{[\mu}\omega_{\nu]}{}^{Aa}\,. \tag{34d}$$

[1]The expressions for these curvatures contain additional terms involving the intrinsic torsion tensors. We have not given these terms here since, as we will see in the next section, these intrinsic torsion tensors will be set to zero by hand. For the full expressions, see Appendix B of [9].

that occur after taking the non-relativistic limit. In particular, it implies that the field equations can only contain the following curvature components that do not contain these independent spin-connections:

$$\begin{aligned} & R_{ab}\,, && \widetilde{R}_{Aa}{}^{BC} \equiv R_{Aa}{}^{BC} + 2\,\delta_A{}^{[B}R^{C]}{}_a\,, \\ & R_{ab}{}^{AB}\,, && \widetilde{R}_{ab}{}^{Cc} \equiv R_{ab}{}^{Cc} - R^C{}_{[a}\delta^c{}_{b]}\,, \\ & R_{\mu\nu}{}^{ab}\,, && \widetilde{R}_{Aa}{}^{Ab} \equiv R_{Aa}{}^{Ab} + \tfrac{1}{8}\,R_{AB}{}^{AB}\delta_a{}^b\,. \end{aligned} \tag{35}$$

Besides this, there are also Bianchi identities relating different curvature components.

This finishes our discussion of the bosonic case. Our task is now to extend these results by including supersymmetry. Since this concerns work in progress we will be less detailed in the next subsection.

3.2 Introducing Supersymmetry

In the supersymmetric case we introduce an additional fermionic gravitino field $\Psi_\mu(x)$. There are then extra fermionic terms that need to be added to the bosonic action given in eq. (14). Ignoring quartic fermions these fermionic terms are given by

$$\mathcal{L}(\text{fermionic}) = \sqrt{|g|}\,\Big(-2\bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_\nu\Psi_\rho + \tfrac{1}{48}F_{\rho\sigma\lambda\tau}(\bar{\Psi}_\mu\gamma^{\mu\nu\rho\sigma\lambda\tau}\Psi_\nu + 12\bar{\Psi}^\rho\gamma^{\sigma\lambda}\Psi^\tau)\Big)\,, \tag{36}$$

where the covariant derivative in the gravitino kinetic term is given by

$$D_\mu\Psi_\nu = \partial_\mu\Psi_\nu + \tfrac{1}{4}\omega_{\mu\hat{a}\hat{b}}\gamma^{\hat{a}\hat{b}}\Psi_\nu\,, \tag{37}$$

The whole action (bosonic plus fermionic terms) is now invariant under the following supersymmetry rules (with spinor parameter ϵ) of the basic fields $E_\mu{}^{\hat{a}}, C_{\mu\nu\rho}$ and Ψ_μ:

$$\begin{aligned} \delta E_\mu{}^{\hat{a}} &= \bar{\epsilon}\gamma^{\hat{a}}\Psi_\mu\,, \\ \delta C_{\mu\nu\rho} &= +3\bar{\epsilon}\gamma_{\hat{a}\hat{b}}E_{[\mu}{}^{\hat{a}}E_\nu{}^{\hat{b}}\Psi_{\rho]} \\ \delta\Psi_\mu &= D_\mu(\hat{\omega})\epsilon - \tfrac{1}{2}\tfrac{1}{144}(\gamma^{\nu\rho\sigma\lambda}{}_\mu - 8\gamma^{\rho\sigma\lambda}\delta^\nu_\mu)\hat{F}_{\nu\rho\sigma\lambda}\epsilon\,. \end{aligned} \tag{38}$$

with the super-covariant spin-connection $\hat{\omega}_\mu{}^{\hat{a}\hat{b}}$ and the supercovariant exterior derivative $\hat{F}_{\mu\nu\rho\sigma}$ of $C_{\mu\nu\rho}$ given by

$$\begin{aligned} \hat{\omega}_\mu{}^{\hat{a}\hat{b}} &= \omega_\mu{}^{\hat{a}\hat{b}} - \tfrac{1}{2}(\Psi_\mu\gamma^{\hat{b}}\Psi^{\hat{a}} - \Psi^{\hat{a}}\gamma_\mu\Psi^{\hat{b}} + \Psi^{\hat{b}}\gamma^{\hat{a}}\Psi_\mu)\,, \\ \hat{F}_{\mu\nu\rho\sigma} &= F_{\mu\nu\rho\sigma} - 6\Psi_{[\mu}\gamma_{\nu\rho}\Psi_{\sigma]}\,. \end{aligned} \tag{39}$$

To define the non-relativistic limit in the supersymmetric case we redefine the bosonic fields as before, see eqs. (16) and (17), and we furthermore redefine the gravitino as follows:

$$\Psi_\mu = c^{-1}\psi_{+\mu} + c^{1/2}\psi_{-\mu}\,, \tag{40}$$

where the longitudinal projections $\psi_{\pm\mu}$ are defined by:

$$\psi_{\pm\mu} \equiv \tfrac{1}{2}(1 \pm \gamma_{012})\psi_\mu\,. \tag{41}$$

Taking the limit of the 11D supergravity action goes rather similar to the bosonic case we discussed in the previous section. Again, there is a cancellation of divergences originating from the Einstein-Hilbert term and the kinetic term for the three-form. The only difference is that, including the fermions in the action, the divergent terms become super-covariant expressions. A similar thing happens when we try to control the divergent terms that arise from the kinetic term of the three-form and the Chern-Simons term. One can again tame this divergence by introducing an anti-selfdual auxiliary field λ_{abcd} whose equation of motion, instead of (22), is now given by

$$\lambda_{abcd} = c^3 \hat{f}^{(-)}{}_{abcd}\,, \tag{42}$$

where $\hat{f}^{(-)}{}_{abcd}$ is the supercovariant version of $f^{(-)}{}_{abcd}$:

$$\hat{f}^{(-)}{}_{abcd} = f^{(-)}{}_{abcd} - \tfrac{1}{4}\bar{\psi}_{-e}\gamma^{[e}\gamma_{abcd}\gamma^{f]}\psi_{-f}\,. \tag{43}$$

Note that the solution (42) can be used to determine how λ_{abcd} transforms under supersymmetry.

It turns out to be convenient to define an auxiliary field $\hat{\lambda}_{abcd}$ that differs from λ_{abcd} by terms that are bilinear in the gravitino field:

$$\hat{\lambda}_{abcd} = \lambda_{abcd} - \tfrac{1}{8}\big(\bar{\psi}_{+[a}\gamma_{bc}\psi_{+d]} - \text{dual}\big)\,. \tag{44}$$

Only in this way one obtains, after taking the limit, a Lagrange multiplier field $\hat{\lambda}_{abcd}$ that transforms under supersymmetry without a term containing the derivative of the supersymmetry parameter.

An additional complication in the supersymmetric case is that we are also facing divergences of order c^3 in what from now on we will call the Q-supersymmetry rules. These additional divergences are tamed by (1) the emergence of what we call S-supersymmetries and by (2) imposing constraints on the (super-covariant version) of the intrinsic torsion tensors and the intrinsic 4-form curvature tensor components. To understand how this works we consider the expansion of the supersymmetric action $S = S(\text{bosonic}) + S(\text{fermionic})$ after introducing the auxiliary field λ_{abcd}:

$$S = c^0 S_0 + c^{-3} S_{-3} + \cdots \tag{45}$$

Expanding the SUSY transformations of the fields appearing in the non-relativistic theory as

$$\delta = c^3\delta_3 + c^0\delta_0 + c^{-3}\delta_{-3} + \ldots \tag{46}$$

the invariance of the relativistic theory, in the presence of the auxiliary field, implies the following identities:

$$\delta_3 S_0 = 0 \qquad \text{and} \qquad \delta_0 S_0 + \delta_3 S_{-3} = 0\,. \tag{47}$$

Since $S_3 = 0$ the form of δ_3 must be such that S_0 is automatically invariant under this variation: this implies the existence of an emergent fermionic shift symmetry which we will denominate S-supersymmetry. These shift symmetries guarantee that the components of the gravitino that give rise to the c^3 divergences, do not occur in S_0. We find two S-supersymmetries characterized by parameters ρ_{A+} and η_- with the following transformation rules:

$$\begin{aligned} \delta\tau_\mu{}^A &= 0\,, & \delta e_\mu{}^a &= 0\,,\\ \delta c_{\mu\nu\rho} &= 0\,, & \delta\hat{\lambda}_{abcd} &= 0\,,\\ \delta\psi_{-\mu} &= \tau_\mu{}^A\gamma_A\eta_-\,, & \delta\psi_{+\mu} &= \tau_\mu{}^A\rho_{A+} - \tfrac{1}{2}e_\mu{}^a\gamma_a\eta_-\,. \end{aligned} \tag{48}$$

Furthermore, for $c \to \infty$, we can only have $\delta_0 S_0 = 0$ if we can arrange for δ_3 to vanish identically. This necessitates the imposition of the following supercovariant geometric constraints:

$$\hat{T}_{ab}{}^{A} = 0\,, \qquad \hat{T}_{a}{}^{\{AB\}} = 0\,, \qquad \hat{f}_{abcd} = 0\,, \qquad \hat{f}_{Aabc} = 0\,, \tag{49}$$

which includes the supercovariant equation of motion $\hat{f}^{(-)}{}_{abcd} = 0$. Looking to the bosonic part of the T-tensors, this means that the full torsion tensor is zero, i.e. we are dealing with case 5 in the classification of the previous section.

Taking the non-relativistic limit of the supersymmetry rules requires one more subtlety as far as the transformation rules of the gravitino and Lagrange multiplier are concerned. It involves a redefinition of these transformation rules using a field-dependent so-called trivial symmetry. We refrain from giving these rules here. For more details, see [10]. The other supersymmetry rules are not effected by this redefinition and are given by

$$\begin{aligned} \delta\tau_\mu{}^A &= \bar{\epsilon}_-\gamma^A\psi_{-\mu}\,, \\ \delta e_\mu{}^a &= \bar{\epsilon}_+\gamma^a\psi_{-\mu} + \bar{\epsilon}_-\gamma^a\psi_{+\mu}\,, \\ \delta c_{\mu\nu\rho} &= 6\,\bar{\epsilon}_+\epsilon_{ABC}\gamma^A\psi_{+[\mu}\tau_\nu{}^B\tau_{\rho]}{}^C + 3\bar{\epsilon}_-\gamma_{ab}\psi_{-[\mu}e_\nu{}^a e_{\rho]}{}^b \\ &\quad + 6\left(\bar{\epsilon}_+\gamma_{Aa}\psi_{-[\mu}\tau_\nu{}^A e_{\rho]}{}^a + \bar{\epsilon}_-\gamma_{Aa}\psi_{+[\mu}\tau_\nu{}^A e_{\rho]}{}^a\right)\,. \end{aligned} \tag{50}$$

The supersymmetry variation of the bosonic constraints (49) will lead to further fermionic constraints on the gravitino curvature components. Introducing S-supersymmetry gauge fields $\phi_{+\mu A}$ and $\phi_{-\mu}$ the Q-covariant and S-covariant gravitino curvatures are given by

$$\begin{aligned} r_{+\mu\nu} &= 2(\nabla_{[\mu}\psi_{+\nu]} + \tau_{[\mu}{}^A\varphi_{+\nu]A} - \tfrac{1}{2}e_{[\mu}{}^a\gamma_a\varphi_{-|\nu]})\,, \\ r_{-\mu\nu} &= 2(\nabla_{[\mu}\psi_{-\nu]} + \tau_{[\mu}{}^A\gamma_A\varphi_{-|\nu]})\,, \end{aligned} \tag{51}$$

where the derivatives ∇_μ are Q-supercovariant derivatives. The intrinsic fermionic torsion components that are independent of the S-supersymmetry gauge fields are given by

$$\begin{aligned} \check{r}_{-ab} &\equiv r_{-ab}\,, \\ \check{r}_{-Aa} &\equiv r_{-Aa} - \tfrac{1}{3}\gamma_A\gamma^B r_{-Ba}\,, \\ \check{r}_{+ab} &\equiv r_{+ab} + \tfrac{1}{3}\gamma_{[a|}\gamma^B r_{-B|b]}\,, \\ \check{r}_{+a} &\equiv \gamma^A r_{+Aa} - \tfrac{1}{8}\gamma_a\gamma^{BC}r_{-BC}\,. \end{aligned} \tag{52}$$

The supersymmetry variation of the bosonic constraints (49) leads to the following fermionic constraints

$$\check{r}_{-ab} = 0\,, \qquad \check{r}_{-Aa} = 0\,, \qquad \check{r}_{+ab} = 0\,. \tag{53}$$

but leaves the components $\check{r}_{+a}$ unconstrained. On it's turn, the supersymmetry variation of these fermionic constraints lead to constraints on the supercovariant spin-connection curvatures. These can only occur in the (super-covariant versions of the) combinations (35).

The 64 dollar question is whether the constraints we find do not lead to an over-constrained system. Based upon our experience with taking the limit of the 10D $\mathcal{N} = 1$ supergravity multiplet we expect that the Poisson equation is part of the supermultiplet of constraints and equations of motion but that, due to the emerging local scale symmetry, the Poisson equation itself will not be among the equations of motion corresponding to the non-relativistic supersymmetric action. We hope to come back to this soon in [10].

3.3 Gauge-fixing

All present calculations hint to the fact that we obtain a consistent 11D Membrane Newton-Cartan supergravity multiplet. Instead of verifying whether we have a finite orbit of constraints under supersymmetry, one can alternatively try to solve for the constraints by gauge-fixing and obtain an 11D Membrane Newtonian supergravity multiplet. In the case of 3D supergravity [11, 12] this approach has turned out to be very fruitful [11, 13]. In that case we found a particle Newton-Cartan supergravity multiplet with the basic fields

$$\{\tau_\mu\,, e_\mu{}^a\,, m_\mu; \psi_{\mu\pm}\}\,. \tag{54}$$

Since one of the constraints sets the curvature of spatial rotations equal to zero, we can assume that space is flat. This allows us to impose the following globally well-defined gauge-fixing [11, 13]:

$$\tau_\mu = \delta_\mu{}^\emptyset\,, \qquad e_\mu{}^a = (0, \delta_i{}^a)\,, \qquad m_\mu = (m_\emptyset, 0)\,, \qquad \psi_{\mu+} = \psi_{i-} = 0\,, \tag{55}$$

where we have decomposed the curved index μ as $\mu = (\emptyset, i = 1, 2, 3)$. After this gauge-fixing all constraints are solved and the multiplet reduces to a 3D particle Newtonian supergravity multiplet with basic fields

$$\{\Phi\,, \Psi\}\,, \tag{56}$$

where $\Phi = m_\emptyset$ is the Newton potential and $\Psi = \psi_{\emptyset-}$ is the Newtino potential. One may verify that the symmetries of the multiplet (56) realize a 3D supersymmetric Bargmann algebra.

We expect that a similar gauge-fixing in eleven dimensions will lead to a 11D Membrane Newtonian supergravity multiplet with basic fields

$$\{\Phi\,, \Psi_{\bar\mu-}\,, \lambda_{abcd}\}\,, \qquad \text{with} \quad \bar\mu = \emptyset, 1, 2\,, \tag{57}$$

where $\Phi = C_{\emptyset 12}$ is the Newton potential and $\Psi_{\bar\mu} = \psi_{\bar\mu -}$ is it's fermionic partner. There are a few notable differences with the 3D case. First of all, in 3D we are dealing with a particle while in 11D the basic object a membrane. Secondly, we find that in 11D the curvature of spatial rotations is restricted but not completely set to zero by the constraints. Therefore, we can not assume that space is flat. The actual existence and consistency of the multiplet (57) needs to be confirmed [10].

4. Generalizations

The establishment of a consistent non-relativistic limit of 11D supergravity, in line with earlier developments in non-relativistic string theory, paves the way for several generalizations relevant to non-relativistic string theory.

(1) The double dimensional reduction of our results establish a consistent non-relativistic limit of 10D IIA supergravity. This theory acts as the low-energy limit of non-relativistic IIA string

theory and extends the non-relativistic limit of 10D $\mathcal{N} = 1$ supergravity [4] to the case of $\mathcal{N} = 2$ supersymmetry.

(2) Using the same techniques as discussed here, one can also consider taking the limit of the 10D IIB supergravity theory. The limit of the bosonic part of this theory will be given in [14]. A noteworthy feature of this theory is that the SL(2,R) duality symmetry is realized in a polynomial way [14] and that this duality symmetry acts in a branched way on the (p, q)-string solutions of the IIB supergravity theory [15].

(3) An unsolved issue remains taking the limit of the 10D heterotic supergravity theory which should lead to a non-Lorentzian Chern-Simons term. It looks like that in redefining the supergravity and Yang-Mills fields these two type of fields start mixing but sofar the precise way to do this, avoiding divergences, has not been given.

(4) We expect that for each $(p + 1)$-form in a given supergravity multiplet there should exist a corresponding p-brane Newton-Cartan supergravity theory. For instance, there should exist an 11D five-brane Newton-Cartan supergravity multiplet corresponding to the 11D six-form C_6 which is the Poincare dual of the three-form C_3 we considered here.

All these new results on non-relativistic p-brane Newton-Cartan supergravity theories will enable us to investigate the half-supersymmetric brane solutions of these different theories. In particular, an interesting set to consider are the half-supersymmetric D-brane solutions since they can tell us more about the feasibility of a non-relativistic holographic principle with non-relativistic gravity in the bulk. We hope to come back to this issue in the nearby future.

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References

[1] J. Gomis and H. Ooguri, "Nonrelativistic closed string theory," J. Math. Phys. **42** (2001), 3127-3151 doi:10.1063/1.1372697 [arXiv:hep-th/0009181 [hep-th]].

[2] U. H. Danielsson, A. Guijosa and M. Kruczenski, "IIA/B, wound and wrapped," JHEP **10** (2000), 020 doi:10.1088/1126-6708/2000/10/020 [arXiv:hep-th/0009182 [hep-th]].

[3] I. R. Klebanov and J. M. Maldacena, "(1+1)-dimensional NCOS and its U(N) gauge theory dual," Adv. Theor. Math. Phys. **4** (2000), 283-302 doi:10.1142/S0217751X01004001 [arXiv:hep-th/0006085 [hep-th]].

[4] E. A. Bergshoeff, J. Lahnsteiner, L. Romano, J. Rosseel and C. Simsek, "Non-relativistic ten-dimensional minimal supergravity," JHEP **12** (2021), 123 doi:10.1007/JHEP12(2021)123 [arXiv:2107.14636 [hep-th]].

[5] J. A. Garcia, A. Guijosa and J. D. Vergara, "A Membrane action for OM theory," Nucl. Phys. B **630** (2002), 178-202 doi:10.1016/S0550-3213(02)00175-X [arXiv:hep-th/0201140 [hep-th]].

[6] C. D. A. Blair, D. Gallegos and N. Zinnato, "A non-relativistic limit of M-theory and 11-dimensional membrane Newton-Cartan geometry," JHEP **10** (2021), 015 doi:10.1007/JHEP10(2021)015 [arXiv:2104.07579 [hep-th]].

[7] E. Bergshoeff, K. van Helden, J. Lahnsteiner, L. Romano and J. Rosseel, "Generalized Newton–Cartan geometries for particles and strings," Class. Quant. Grav. **40** (2023) no.7, 075010 doi:10.1088/1361-6382/acbe8c [arXiv:2207.00363 [hep-th]].

[8] E. Bergshoeff, K. van Helden, J. Figueroa-O'Farrill, J. Rosseel and I. Rotko, "p-brane Newton-Cartan geometries with and without indices", work in progress.

[9] E. A. Bergshoeff, J. Lahnsteiner, L. Romano, J. Rosseel and C. Şimşek, "A non-relativistic limit of NS-NS gravity," JHEP **06** (2021), 021 doi:10.1007/JHEP06(2021)021 [arXiv:2102.06974 [hep-th]].

[10] E.A. Bergshoeff, C. Blair, J. Lahnsteiner and J. Rosseel, work in progress.

[11] R. Andringa, E. A. Bergshoeff, J. Rosseel and E. Sezgin, "3D Newton–Cartan supergravity," Class. Quant. Grav. **30** (2013), 205005 doi:10.1088/0264-9381/30/20/205005 [arXiv:1305.6737 [hep-th]].

[12] E. Bergshoeff, J. Rosseel and T. Zojer, "Newton–Cartan (super)gravity as a non-relativistic limit," Class. Quant. Grav. **32** (2015) no.20, 205003 doi:10.1088/0264-9381/32/20/205003 [arXiv:1505.02095 [hep-th]].

[13] E. A. Bergshoeff and J. Rosseel, "Non-Lorentzian Supergravity,", to appear as a Chapter for the "Handbook of Quantum Gravity" (Eds. C. Bambi, L. Modesto and I.L. Shapiro), Springer Singapore, expected in 2023, [arXiv:2211.02604 [hep-th]].

[14] E. A. Bergshoeff, K. T. Grosvenor, J. Lahnsteiner, Z. Yan and U. Zorba, "Non-Lorentzian IIB Supergravity from a Polynomial Realization of SL(2;R), to be submitted.

[15] E. A. Bergshoeff, K. T. Grosvenor, J. Lahnsteiner, Z. Yan and U. Zorba, "Branched SL(2,Z) duality," JHEP **10** (2022), 131 doi:10.1007/JHEP10(2022)131 [arXiv:2208.13815 [hep-th]].