PROCEEDINGS OF SCIENCE

Learning from superstring massive states

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We discuss the first massive level of the open charged superstring in a flat four-dimensional spacetime. We show how the Rarita-Schwinger and Fierz-Pauli Lagrangians are retrieved for spin-3/2 and 2, respectively. In the absence of an electromagnetic background, We derive explicit equations of motion that describe the propagation of fields of charged spin-3/2 in flat space-time, which has been a long-standing problem. We briefly comment on use of amplitude computation to investigate different forms of the Weak Gravity Conjectures. This talk is based on [1–5]

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1. Introduction

The field theory of high spin particles has been a longstanding and difficult problem in physics. Specifically, for the case of massive particles, a challenge arises as soon as one tries to propagate states of spin greater than 1 in an electromagnetic background. In a 1936 paper, Dirac stressed the importance of formulating equations of motion for these states [6]. Formulating equations of motion for these states proved to be a challenging problem, as noted by Fierz and Pauli who quickly took up the challenge [7], but derived in this work their famous Fierz-Pauli Lagrangian for a neutral massive spin-2.

However, it was only several decades later that the most significant aspect of the problem's difficulty was revealed, thanks to the works of various researchers such as Johnson and Sudarshan [8], and Velo and Zwanziger [9–11]. When Johnson and Sudarshan tried to canonically quantize minimally coupled spin-3/2 fields, they discovered that the equal-time commutators were not compatible with the relativistic covariance of the theory. Later, Velo and Zwanziger demonstrated that the minimally coupled Lagrangians for spin-3/2 and spin-2 fields exhibited pathological behavior at the classical level. Interestingly, both problems arose at a specific value of the electromagnetic field strength, indicating a shared origin. In fact, it was later understood that a sign of the problems is that the set of secondary constraints becomes degenerate, which signals a loss of invertibility. This means that the constraints no longer determine all the components of the fields leading to acausality and loss of hyperbolicity.

The Lagrangian that leads to systems of Fierz-Pauli equations use additional fields of lower spin s-1, s-2, etc. In the free case, these are as they should be auxiliary non-propagating fields. They are projected out by the constraints in the systems of Fierz-Pauli equations. Unfortunately, known tentative Lagrangian tend to mix the different components of the higher spin fields, mixing what was physical and auxiliary in a non-trivial way, leading to propagation that are not causal..At present, the Federbush Lagrangian [12] which remains the only four-dimensional Lagrangian describing an isolated charged massive spin-2 state, has superluminal propagating modes and thus suffers from the causality loss problem.

The Regge trajectories of String theory contain arbitrarily high spin states. Following the solution of the string propagation in an electromagnetic field [13–15], Argyres and Nappi utilized String Field Theory to investigate the first massive level of the open bosonic string [16, 17] and established a Lagrangian for the massive charged spin-2 field. However, this Lagrangian exhibited pathologies in dimensions other than $d = 26$. Porrati and Rahman later investigated its reduction to four dimensions [18] and showed that it yields a spin-2 field coupled to a scalar. The study of the second mass level of bosonic strings resulted in the development of an action describing a charged massive spin-3 coupled to lower spin states [19].

Several works have allowed first to understand well the difficulties, then to make some notable progress as we will see below. Despite these developments, the problem of high spin particles remains unsolved to this day, when it comes to finding a Lagrangian with only the fields of higher spins. Here, we shall report on a substantial progress: a solution the original problem of finding the equations of motion [1–4].

Half of the presentation delivered in Corfu focused on the utilization of Kaluza-Klein states based on [5]. A brief overview of this topic is provided in the final section of this document.

2. Lagrangian for the neutral case

In this part, we take the usual string theory convention $\alpha' = 1/2$.

2.1 Bosons

One starts from the action obtained from Superstring Field Theory [4]. After expanding the superfields into components, the bosonic action has 80 degrees of freedom off-shell. The Lagrangian can be split into two independent pieces, each of which being gauge invariant separately:

$$\mathcal{L}_B \equiv \mathcal{L}_1 + \mathcal{L}_2 \tag{1}$$

where:

$$\begin{aligned}
\mathcal{L}_1 =& -6\bar{M}\left(4+3\partial^2\right)M + 6\left[(2\mathrm{i}M_m + q_m)\,\partial^m\bar{M} - \mathrm{i}\left(N+\frac{1}{2}s\right)\partial^2\bar{M} + \text{h.c.}\right]\\
&+ 4M_m\bar{M}^m + \left[M_m\left(4\partial^m\bar{N} + \partial^m\bar{s} + 2\partial_n\bar{s}^{mn} + 2\mathrm{i}\bar{q}^m\right) + \text{h.c.}\right]\\
&- \frac{1}{4}s\partial^2\bar{s} + q_m\bar{q}^m - 2\bar{N}\left(\partial^2-4\right)N - \left(2\mathrm{i}q_m\partial^m\bar{N} + s\partial^2\bar{N} + \text{h.c.}\right)\\
&+ \left(\partial_k s^{mk}\right)\left(\partial^n\bar{s}_{mn}\right) - \frac{1}{2}\mathrm{i}\left(q_m\partial^m\bar{s} + 2q_m\partial_n\bar{s}^{mn} + \text{h.c.}\right)
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
\mathcal{L}_2 =& -\frac{1}{2}v^{mn}(2-\partial^2)v_{mn} + \frac{1}{2}\partial^n v_{mn}\partial_k v^{mk} + \partial^n v_{mn}\partial_k f^{mk} - \frac{1}{2}f^{mn}(2-\partial^2)f_{mn} + \partial^n v_{mn}\omega_2^m\\
&+ \frac{1}{2}\partial^n f_{mn}\partial_k f^{mk} - 2c^m\left(\partial^n f_{mn} - \partial^n v_{mn}\right) + \varepsilon_{mnpq}f^{mn}\tau_2^{pq} - \left(\partial_n\partial_m v^{mn}\right)\left(\frac{1}{4}h + 6\phi\right)\\
&- 2c^m c_m - \frac{1}{2}(\partial^m c_m)^2 + \frac{1}{8}\omega_2^m\partial^2\omega_{2m} - \frac{1}{8}(\partial_m\omega_2^m)^2 - \partial^m c_m\left(6D + \frac{1}{2}h + \frac{3}{2}\partial^2\phi - \tau_1\right)\\
&+ \partial^m\omega_{2m}\left(3D - \frac{1}{2}\partial_n c^n + \frac{1}{4}h - \frac{9}{4}\partial^2\phi - \frac{1}{2}\tau_1\right) - \Omega^m\left(6a_m - \frac{1}{2}\omega_{1m}\right) - 66D^2\\
&- 3D\left(4h - 6\tau_1 + 8\phi - 5\partial^2\phi\right) - \frac{1}{4}h\left(1 - \frac{3}{8}\partial^2\right)h - \tau_1^2 - \tau_1\left(\frac{3}{2}\partial^2\phi - h\right) - \frac{33}{8}\phi\partial^4\phi\\
&+ \frac{3}{2}h\partial^2\phi + 2D_m^2 - \frac{1}{8}\varphi\partial^4\varphi - G\left(-8\varphi + \partial^2\varphi\right) - 2G^2 - \frac{1}{8}\omega_1^m\partial^2\omega_{1m} - \frac{1}{8}(\partial^m\omega_{1m})^2\\
&- D^m\left(12a_m - 4C_m + \partial^2 C_m - 4\partial_m\varphi - 2\omega_{1m}\right) - \varphi\left(\frac{1}{2}\partial^2\tau_2 - \frac{3}{2}\partial^2\partial^m a_m + \frac{1}{4}\partial^2\partial^m\omega_{1m}\right)\\
&- G\left(-6\partial^m a_m + 4\partial^m C_m + \partial^m\omega_{1m} + 2\tau_2\right) + 6(\partial^m a_m)(\partial^n C_n) - \tau_2\left(\partial^m C_m - 3\partial^m a_m\right)\\
&+ \omega_1^m\left(2\partial^n\tau_{2mn} + 3\partial^2 a_m\right) - \frac{1}{2}\tau_2\partial^m\omega_{1m} + \frac{9}{2}(\partial^m a_m)(\partial^n\omega_{1n}) + \tau_{2mn}\tau_2^{mn}\\
&- 2\partial^n\tau_{2mn}\left(6a^m - C^m\right) + \frac{1}{8}C^m\partial^4 C_m - 6a_m(2\partial^2 - 1)a^m - \frac{33}{2}a_m a^m + 3a_m\partial^2 C^m
\end{aligned} \tag{3}$$

However, several degrees of freedom are non-physical. These include auxiliary fields, which are integrated out before performing appropriate field redefinitions, gauge degrees of freedom that are totally fixed by the unitary gauge, and non-propagating fields like the transverse components of a_m

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and c_m, which disappear after dualisation. After eliminating these degrees of freedom, further field redefinitions are needed to decouple the fields in the Lagrangian.

For $\mathcal{L}_1$, the auxiliary field M_m is integrated out, which also eliminates s, s_{mn} and q_m. The redefinitions

$$N_1 \to \frac{1}{4}N_1 + 3M_2, \quad M_1 \to \frac{1}{12}M_1 - \frac{N_2}{3}. \tag{4}$$

render N_2 and M_2 auxiliary, so they are integrated out to give

$$\mathcal{L}_1 = \frac{1}{2}M_1\left(-2+\partial^2\right)M_1 + \frac{1}{2}N_1\left(-2+\partial^2\right)N_1. \tag{5}$$

For $\mathcal{L}_2$, the auxiliary fields $D, D_m, G, \tau_1, \tau_2, \tau_{2mn}$ are integrated out, then a gauge-fixed Lagrangian is obtained by making the redefinitions:

$$\begin{aligned}
&\omega_{1m} \to \omega_{1m} + 2a_m + \Omega_m - \frac{2}{\sqrt{3}}C_m, \quad \omega_{2m} \to \omega_{2m} - 4\partial_m\phi \\
&c_m \to c_m - \frac{1}{2}\partial^n f_{mn} - \frac{1}{8}\partial^2\omega_{2m} + \frac{1}{8}\partial_m\partial_n\omega_2^n \\
&h \to h - 8\phi - \frac{1}{2}\partial^m\omega_{2m}, \quad v_{mn} \to v_{mn} + \frac{1}{8}\eta_{mn}\partial^k\omega_{2k} - \frac{1}{4}\partial_m\omega_{2n} - \frac{1}{4}\partial_n\omega_{2m} \\
&C_m \to \frac{1}{\sqrt{3}}C_m - \frac{1}{2}\omega_{1m}, \quad a_m \to \frac{1}{2}a_m + \frac{1}{4}\Omega_m - \frac{1}{2\sqrt{3}}C_m
\end{aligned} \tag{6}$$

These lead to

$$\begin{aligned}
\mathcal{L}_2 =& \frac{1}{2}C^m\left(\partial^2-2\right)C_m + \frac{1}{2}\partial^m C_m\partial^n C_n + a^m a_m + \frac{1}{2}\partial^m a_m\partial^n a_n - \frac{1}{2}A^2 + \frac{2}{5}B^2 \\
&- 2c^m c_m - \frac{2}{5}\partial^m c_m\partial^n c_n - \frac{3}{20}h^2 + \frac{3}{32}h\partial^2 h - v^{mn}v_{mn} + \frac{3}{10}h\partial^m c_m \\
&+ 2c^m\partial^n v_{mn} - \frac{1}{4}h\partial^m\partial^n v_{mn} + \frac{1}{2}\partial^n v_{mn}\partial_k v^{mk} + \frac{1}{2}v^{mn}\partial^2 v_{mn}
\end{aligned} \tag{7}$$

Although each vector a_m and c_m has four components, only one component represents a physical degree of freedom for each vector. They are therefore scalar fields in disguise. To exhibit a Lagrangian with only physical degrees of freedom, we introducing auxiliary scalars A and B. Field redefinitions are then made, specifically A is shifted by $\partial^m a_m$ and B is shifted by $\partial^m c_m - \frac{3}{8}h$, then a_m and c_m can be eliminated to leave behind their dual scalars. Finally, the redefinition $B \to \frac{5\sqrt{2}}{4}B + \frac{15}{8}h$ and $h \to h + 2\sqrt{2}B$ is made, which leads to the decoupled bosonic Lagrangian. Further redefinitions:

$$\mathfrak{h}'_{mn} \equiv v_{mn} + \eta_{mn}(\frac{1}{4}h + \partial_k c^k) \tag{8}$$

and

$$h_{mn} \equiv v_{mn} + \frac{1}{4}\eta_{mn}h', \tag{9}$$

allow finally to get the Lagrangian

$$\begin{aligned}
\mathcal{L}_B =& \frac{1}{2}h^{mn}\left(\partial^2-2\right)h_{mn} - \frac{1}{2}h\left(\partial^2-2\right)h + h_{mn}\partial^m\partial^n h + \partial^n h_{mn}\partial_k h^{mk} \\
&+ \frac{1}{2}C^m\left(\partial^2-2\right)C_m + \frac{1}{2}(\partial^m C_m)^2 + \frac{1}{2}A\left(\partial^2-2\right)A + \frac{1}{2}B\left(\partial^2-2\right)B \\
&+ \frac{1}{2}M_1\left(\partial^2-2\right)M_1 + \frac{1}{2}N_1\left(\partial^2-2\right)N_1
\end{aligned} \tag{10}$$

which contains a Fierz-Pauli part for the massive spin-2 field.

2.2 Fermions

For the fermionic fields, the expansion of the Superstring Field Theory Lagrangian [4] gives:

$$
\begin{aligned}
-\mathcal{L}_F =& \mathrm{i}(\lambda^m\sigma^n\partial_n\bar\lambda_m)+\frac{\mathrm{i}}{4}\left(\bar\chi^m\bar\sigma^n\partial_n\partial^2\chi_m\right)-\frac{1}{2}\left[\left(\lambda^m\partial^2\chi_m\right)+\left(\bar\chi^m\partial^2\bar\lambda_m\right)\right]+2\left[\lambda^m\chi_m+\bar\chi^m\bar\lambda_m\right]\\
&-\frac{33}{4}\mathrm{i}\left[\left(\bar\xi\bar\sigma^m\partial_m\partial^2\xi\right)+4\left(\psi\sigma^m\partial_m\bar\psi\right)\right]+\frac{15}{2}\left[\left(\psi\partial^2\xi\right)+\left(\bar\xi\partial^2\bar\psi\right)\right]-12\left[(\psi\xi)+(\bar\xi\bar\psi)\right]\\
&+3\left[\mathrm{i}\left(\chi^m\partial_m\psi\right)-\mathrm{i}\left(\lambda^m\partial_m\xi\right)+2\left(\lambda^m\sigma_m\bar\psi\right)+\frac{1}{2}\left(\chi^m\sigma_m\partial^2\bar\xi\right)+\text{h.c.}\right]\\
&-6\left[\mathrm{i}(\partial^m\psi\sigma_{mn}\chi^n)+\frac{1}{2}(\chi^m\sigma^n\partial_m\partial_n\bar\xi)+\mathrm{i}(\lambda^m\sigma_{mn}\partial^n\xi)+\text{h.c.}\right]\\
&+\frac{3}{4}\mathrm{i}\left[(\nu\partial^2\psi)-(\bar\psi\partial^2\bar\nu)\right]+\frac{9}{8}\left[\left(\nu\sigma^m\partial_m\partial^2\bar\xi\right)+\text{h.c.}\right]+9\mathrm{i}\left[(\mu\psi)-(\bar\psi\bar\mu)\right]\\
&+\frac{3}{2}\left[\left(\mu\sigma^m\partial_m\bar\xi\right)+\text{h.c.}\right]-3\mathrm{i}\left[(\zeta\partial^2\xi)-(\bar\xi\partial^2\bar\zeta)\right]-6\left[(\zeta\sigma^m\partial_m\bar\psi)+\text{h.c.}\right]\\
&+\frac{3}{4}\left[\mathrm{i}(r_m\sigma^m\partial^2\bar\xi)-3\mathrm{i}(r_m\sigma^n\partial_n\partial^m\bar\xi)-4(r_m\sigma^{mn}\partial_n\psi)+4(r_m\partial^m\psi)+\text{h.c.}\right]\\
&+\frac{1}{2}\left[(\lambda^m\partial_m\nu)+2(\lambda_m\sigma^{mn}\partial_n\nu)-2\mathrm{i}(\chi^m\sigma_m\bar\mu)+\text{h.c.}\right]\\
&+\frac{1}{2}\left[-(\chi^m\partial_m\zeta)-2(\chi_m\sigma^{mn}\partial_n\zeta)+2\mathrm{i}(\lambda^m\sigma_m\bar\zeta)+\text{h.c.}\right]+\frac{1}{4}\left[(\chi^m\sigma^n\partial_n\bar r_m)\right.\\
&\quad\left.+(\bar\chi^m\bar\sigma_m\partial^n r_n)+(\bar\chi^m\bar\sigma^n\partial_m r_n)-\mathrm{i}\varepsilon_{mlkn}(\chi^m\sigma^l\partial^k\bar r^n)-2\mathrm{i}(\lambda_m\sigma^n\bar\sigma^m r_n)+\text{h.c.}\right]\\
&+\frac{1}{2}\left[(\nu\partial^2\zeta)+(\bar\nu\partial^2\bar\zeta)\right]+2\left[(\mu\zeta)+(\bar\zeta\bar\mu)\right]+\mathrm{i}\left[(\zeta\partial^m r_m)+(\bar r^m\partial_m\bar\zeta)\right]\\
&+\frac{1}{8}\left[-4\mathrm{i}(\nu\sigma^m\partial_m\bar\mu)+(r_m\sigma^m\partial^2\bar\nu)-2(\nu\sigma^m\partial_m\partial_n\bar r^n)+\text{h.c.}\right]-\mathrm{i}(\zeta\sigma^m\partial_m\bar\zeta)\\
&-\frac{1}{2}\left[(r_m\sigma^m\bar\mu)+\text{h.c.}\right]-\frac{1}{8}\mathrm{i}\left[(r_m\sigma^k\bar\sigma^n\sigma^m\partial_k\bar r_n)+(r_m\sigma^k\bar\sigma^m\sigma^n\partial_n\bar r_k)\right]\\
&+\frac{3}{2}\left[\left(\rho+\frac{1}{2}\mathrm{i}\bar\gamma\bar\sigma^m\partial_m\right)\left(\mathrm{i}\partial^2\xi-2\sigma^n\partial_n\bar\psi\right)+\text{h.c.}\right]+2\left[(\chi^m\partial_m\rho)+(\lambda^m\partial_m\gamma)+\text{h.c.}\right]\\
&+\left[\left(\mu-\frac{1}{4}\partial^2\nu\right)\left(\rho+\frac{1}{2}\mathrm{i}\sigma^m\partial_m\bar\gamma\right)+\text{h.c.}\right]+\left[\mathrm{i}r_m\sigma^{mn}\partial_n\left(\rho+\frac{1}{2}\mathrm{i}\sigma^k\partial_k\bar\gamma\right)+\text{h.c.}\right]\\
&-\mathrm{i}\left(\rho+\frac{1}{2}\mathrm{i}\bar\gamma\bar\sigma^m\partial_m\right)\sigma^n\partial_n\left(\bar\rho+\frac{1}{2}\mathrm{i}\bar\sigma^k\partial_k\gamma\right)+4\left[(\gamma\rho)+(\bar\rho\bar\gamma)\right]
\end{aligned}
\tag{11}
$$

We first perform the redefinitions:

$$
\begin{aligned}
&\lambda_{m\alpha}\to\lambda_{m\alpha}+\frac{1}{2}\mathrm{i}(\sigma^n\partial_n\bar\chi_m)_\alpha+\frac{1}{8}(\sigma^k\bar\sigma^n\sigma_m\partial_k\bar r_n)_\alpha+\frac{1}{4}(\sigma^n\bar\sigma_m\partial_n\zeta)_\alpha+\partial_m\zeta_\alpha+2(\sigma_m\bar\xi)_\alpha\\
&\mu_\alpha\to\mu_\alpha+\frac{1}{4}\partial^2\nu_\alpha-\frac{1}{2}\mathrm{i}\partial^m r_{m\alpha}+\mathrm{i}(\sigma^m\partial_m\bar\zeta)_\alpha+4\mathrm{i}\xi_\alpha\\
&r_{m\alpha}\to r_{m\alpha}+\mathrm{i}\partial_m\nu_\alpha+4\partial_m\xi_\alpha,\qquad \rho_\alpha\to\rho_\alpha+\frac{1}{4}\partial^2\zeta_\alpha-\frac{1}{2}\mathrm{i}(\sigma^m\partial_m\bar\gamma)_\alpha+\frac{1}{2}(\sigma^m\partial_m\bar\xi)_\alpha\\
&\zeta_\alpha\to\zeta_\alpha-2(\sigma^m\partial_m\bar\xi)_\alpha,\qquad \gamma_\alpha\to\gamma_\alpha-\frac{1}{2}\mathrm{i}(\sigma^m\partial_m\bar\zeta)_\alpha-\mathrm{i}\xi_\alpha\\
&\psi_\alpha\to\psi_\alpha-\frac{1}{2}\mathrm{i}(\sigma^m\partial_m\bar\xi)_\alpha,\qquad \chi_{m\alpha}\to\chi_{m\alpha}+\frac{1}{4}\mathrm{i}(\sigma_n\bar\sigma_m r^n)_\alpha-\frac{1}{2}\mathrm{i}(\sigma_m\bar\zeta)_\alpha
\end{aligned}
\tag{12}
$$

In the result μ_α appears as a Lagrange multiplier for the constraint:

$$\rho_\alpha = -9\mathrm{i}\psi_\alpha - \mathrm{i}(\sigma_m \bar{\chi}^m)_\alpha \tag{13}$$

which eliminates ρ_α. We further redefine:

$$\begin{aligned}
&\chi_{m\alpha} \longrightarrow \sqrt{2}(\sigma_{mn}\chi^n)_\alpha - \mathrm{i}(\sigma_m\bar{\psi})_\alpha \\
&\bar{\lambda}^{\dot{\alpha}}_m \longrightarrow \bar{\lambda}^{\dot{\alpha}}_m - \frac{1}{\sqrt{2}}\mathrm{i}(\bar{\sigma}_m\gamma)^{\dot{\alpha}} + \partial_m\bar{\psi}^{\dot{\alpha}} + \frac{1}{\sqrt{2}}\mathrm{i}(\bar{\sigma}^n\partial_m\chi_n)^{\dot{\alpha}} \\
&\gamma_\alpha \longrightarrow \frac{1}{\sqrt{2}}\gamma_\alpha - \frac{\mathrm{i}}{2}(\sigma^m\partial_m\bar{\psi})_\alpha - \frac{1}{2}\mathrm{i}(\sigma^m\bar{\lambda}_m)_\alpha + \frac{1}{2\sqrt{2}}(\sigma^m\bar{\sigma}^n\partial_m\chi_n)_\alpha \\
&\bar{\psi}^{\dot{\alpha}} \longrightarrow \frac{\mathrm{i}}{2}\bar{\psi}^{\dot{\alpha}} - \frac{1}{2\sqrt{2}}(\bar{\sigma}^m\chi_m)^{\dot{\alpha}}
\end{aligned} \tag{14}$$

which at end results in

$$\begin{aligned}
\mathcal{L}_F = &- \varepsilon^{mnkl}(\lambda_m\sigma_n\partial_k\bar{\lambda}_l) + \varepsilon^{mnkl}(\bar{\chi}_m\bar{\sigma}_n\partial_k\chi_l) - 2\sqrt{2}\,[(\lambda^m\sigma_{mn}\chi^n) + \text{h.c.}] \\
&- \mathrm{i}\left(\psi\sigma^m\partial_m\bar{\psi}\right) - \mathrm{i}(\gamma\sigma^m\partial_m\bar{\gamma}) - \sqrt{2}\,[(\psi\gamma) + \text{h.c.}]
\end{aligned} \tag{15}$$

where we recognize the Rarita-Schwinger Lagrangian for the massive spin-3/2 $(\chi_m, \bar{\lambda}_m)$ and Dirac Lagrangian for the massive spin-1/2 $(\gamma, \bar{\psi})$. The corresponding equations of motion and constraints read

$$\begin{aligned}
&\mathrm{i}\bar{\sigma}^{n\dot{\alpha}\alpha}\partial_n\gamma_\alpha = -\sqrt{2}\bar{\psi}^{\dot{\alpha}}, \quad \mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} = -\sqrt{2}\gamma_\alpha \\
&\mathrm{i}\bar{\sigma}^{n\dot{\alpha}\alpha}\partial_n\chi_{m\alpha} = -\sqrt{2}\bar{\lambda}^{\dot{\alpha}}_m, \quad \mathrm{i}\sigma^n_{\alpha\dot{\alpha}}\partial_n\bar{\lambda}^{\dot{\alpha}}_m = -\sqrt{2}\chi_{m\alpha} \\
&\bar{\sigma}^{m\dot{\alpha}\alpha}\chi_{m\alpha} = 0, \quad \partial^m\chi_{m\alpha} = 0, \quad \sigma^m_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_m = 0, \quad \partial^m\bar{\lambda}^{\dot{\alpha}}_m = 0
\end{aligned} \tag{16}$$

3. Charged states propagation in a constant electromagnetic background

The open superstrings states carry total charges $Q = q_0 + q_\pi$, with a matrix $\mathfrak{M}$ satisfying

$$\mathfrak{M}\cdot\mathfrak{M}^T = \frac{\epsilon}{QF} \tag{17}$$

The matrix ϵ depends on the field strength F_{mn} and the fundamental scale Λ of the theory, it appears in the commutator of the dressed covariant derivatives [13, 16]

$$\mathfrak{D}_m = -\mathrm{i}\,\mathfrak{M}_{mn}\,D^n, \qquad [\mathfrak{D}_m, \mathfrak{D}_n] = \mathrm{i}\epsilon_{mn} \tag{18}$$

Notably, our analysis continues to hold even if we take the limit $\epsilon_{mn} \to QF_{mn}$ and $\mathfrak{D}_m \to D_m$, as shown by the consistency of the Lagrangian and the derivation of the equations of motion.

In [4], the computation of the superspace Lagrangian and equations of motion for the first massive level of the open superstring [21, 22] was extended to the charged case, thereby generalizing the work of [16, 20] for the bosonic case.

3.1 Charged spin 2

Among the bosonic 12 degrees of freedom (d.o.f.) with mass M, 5 form a spin-2 field. It will be described by the rank-2 tensor h_{mn}. The on-shell conditions of symmetry, tracelessness, and vanishing divergence are imposed to reduce the d.o.f. in h_{mn} to 5. Additionally, 3 correspond to a massive vector field C_m and 4 d.o.f. are taken up by four scalar fields $\mathcal{M}_1, \mathcal{N}_1$, A and B. The Lagrangian reads:

$$\begin{aligned}
\mathcal{L} =&\bar{C}^m \left(\mathfrak{D}^2 - M^2\right) C_m + \mathfrak{D}^m \bar{C}_m \mathfrak{D}^n C_n + 2\mathrm{i}\epsilon_{mn}\bar{C}^m C^n \\
&+ \bar{\mathcal{M}}_1 \left(\mathfrak{D}^2 - M^2\right) \mathcal{M}_1 + \bar{\mathcal{N}}_1 \left(\mathfrak{D}^2 - M^2\right) \mathcal{N}_1 \\
&+ \bar{a}^m \left(M^2 \eta_{mn} - \mathrm{i}\epsilon_{mn}\right) a^n + \mathfrak{D}^m \bar{a}_m \mathfrak{D}^n a_n - M^2 \bar{c}^m c_m - \frac{2}{5}\mathfrak{D}^m \bar{c}_m \mathfrak{D}^n c_n \\
&+ \frac{1}{\sqrt{2}} \left[M\bar{c}^m \left(-\frac{2}{5}\mathfrak{D}_m \mathcal{H} + \mathfrak{D}^n \mathcal{H}_{nm}\right) + \bar{\tilde{F}}^{mn}(a) \left(F_{mn}(c) - \frac{M}{\sqrt{2}}\mathcal{H}_{[mn]}\right) + \mathrm{h.c.}\right] \\
&+ \frac{1}{2}\bar{\mathcal{H}}_{mn}\mathfrak{D}^2 h^{mn} + \frac{1}{2}\mathfrak{D}^n \bar{\mathcal{H}}_{mn}\mathfrak{D}_k h^{mk} - \frac{M^2}{2}\bar{\mathcal{H}}^{(mn)}\mathcal{H}_{(mn)} + \frac{M^2}{20}\bar{\mathcal{H}}\mathcal{H} + \mathrm{i}\epsilon^{nk}\bar{\mathcal{H}}_{mn} h_k{}^m
\end{aligned} \tag{19}$$

with the (dual) field strengths given by

$$F_{mn}(a) \equiv \mathfrak{D}_m a_n - \mathfrak{D}_n a_m, \quad \tilde{F}_{mn}(a) \equiv \frac{1}{2}\varepsilon_{mnpq} F^{pq}(a) \tag{20}$$

and similarly for $F_{mn}(c), \tilde{F}_{mn}(c)$. The equations of motion for the (decoupled) complex scalars $\mathcal{M}_1, \mathcal{N}_1$ and massive vector C_m, and the constraint for the latter, are straightforward to obtain:

$$\begin{aligned}
&\left(\mathfrak{D}^2 - M^2\right) \mathcal{M}_1 = 0, \\
&\left(\mathfrak{D}^2 - M^2\right) \mathcal{N}_1 = 0 \\
&\left(\mathfrak{D}^2 - M^2\right) C_m - \mathfrak{D}_m \mathfrak{D}_n C^n + 2\mathrm{i}\epsilon_{mn} C^n = 0, \quad \mathfrak{D}^m C_m = 0.
\end{aligned} \tag{21}$$

To derive the equations of motion and the constraints for the other fields is cumbersome. We get for the symmetric tensor h_{mn}:

$$\begin{aligned}
&\left(\mathfrak{D}^2 - M^2\right) h_{mn} - 2\mathrm{i}\left(\epsilon_{km} h^k{}_n + \epsilon_{kn} h^k{}_m\right) = 0 \\
&\mathfrak{D}^n h_{mn} + \sqrt{2} M c_m = 0, \quad M h = -4\sqrt{2}\mathfrak{D}^m c_m
\end{aligned} \tag{22}$$

The first line is the four-dimensional version of the same form of equations of motion obtained in [16] in 26 dimensions. The constraints are not in a satisfactory form and the equations of motion of the vectors $\{a_m, c_m\}$ are coupled. To proceed, we first make the on-shell redefinitions

$$\begin{aligned}
a'_m &\equiv a_m - \frac{\mathrm{i}}{M^2}\epsilon_{mn} a^n - \frac{\mathrm{i}}{M^3}\tilde{\epsilon}^{nk}\mathfrak{D}_k h_{mn} + \frac{\mathrm{i}}{M^3}\tilde{\epsilon}_{mn}\mathfrak{D}^n h + \frac{2\sqrt{2}}{M^2}\mathrm{i}\tilde{\epsilon}_{mn} c^n \\
c'_m &\equiv c_m - \frac{\sqrt{2}\mathrm{i}}{2M^2}\tilde{\epsilon}_{mn} a^n + \frac{\mathrm{i}}{\sqrt{2}M^3}\epsilon^{nk}\mathfrak{D}_n h_{mk}
\end{aligned} \tag{23}$$

their equations of motion become

$$\mathfrak{D}_m \mathfrak{D}_n a'^n = M^2 a'_m, \quad \mathfrak{D}_m \mathfrak{D}_n c'^n = M^2 c'_m \tag{24}$$

indicating that a'_m, c'_m count for only one d.o.f. on-shell each. Next, we consider:

$$\mathcal{H}'_{mn} \equiv \mathcal{H}_{(mn)} + \frac{\sqrt{2}}{M}\eta_{mn}\mathfrak{D}^k c_k \tag{25}$$

where $\mathcal{H}_{(mn)}$ is the symmetric part of the rescaled h_{mn}, defined by:

$$\mathcal{H}_{mn} \equiv (\eta_{mk} - \mathrm{i}\epsilon_{mk})\, h^k{}_n, \qquad \mathcal{H} = h \tag{26}$$

This new spin-2 satisfies

$$\begin{aligned}&\left(\mathfrak{D}^2 - M^2\right)\mathcal{H}'_{mn} + 2\mathrm{i}\left[(\epsilon\cdot\mathcal{H}')_{mn} - (\mathcal{H}'\cdot\epsilon)_{mn}\right] = 0,\\ &\mathfrak{D}^n\mathcal{H}'_{mn} = -\frac{\mathrm{i}}{M}\tilde{\epsilon}_{mn}a^n + \mathrm{i}\frac{\sqrt{2}}{M}\epsilon_{mn}c^n, \quad \mathcal{H}' = 0\end{aligned} \tag{27}$$

In the free case, $\epsilon = 0$, the above equations reduce to the Fierz-Pauli system. In order to get vanishing divergence and trace, we introduce:

$$\begin{aligned}\mathfrak{h}_{mn} \equiv &\frac{2}{3}h_{mn} - \frac{1}{6}\eta_{mn}h - \frac{\mathrm{i}}{M^2}\epsilon_m{}^k h_{kn} + \frac{\sqrt{2}}{3M}\mathfrak{D}_m c_n - \frac{1}{M^4}\left(\epsilon_{mk}\epsilon^{lk}h_{nl} + \epsilon_{mk}\epsilon_{nl}h^{kl} - \frac{1}{2}\eta_{mn}\epsilon^{kl}\epsilon^p{}_l h_{kp}\right)\\ &- \frac{\mathrm{i}\sqrt{2}}{M^3}\left(\epsilon_{mk}\mathfrak{D}^k c_n - \epsilon_{mk}\mathfrak{D}_n c^k + \frac{1}{2}\eta_{mn}\epsilon^{kl}\mathfrak{D}_k c_l\right)\\ &- \frac{1}{2M^4 + 4\epsilon\epsilon}\left[-\frac{2\mathrm{i}}{M}\tilde{\epsilon}_{mk}\mathfrak{D}^k\mathfrak{D}_n\mathfrak{D}_l a^l + \frac{5}{4M}(\epsilon\tilde{\epsilon})\,\eta_{mn}\mathfrak{D}^k a_k - \frac{2}{M^3}(\epsilon\tilde{\epsilon})\,\mathfrak{D}_m\mathfrak{D}_n\mathfrak{D}_k a^k\right.\\ &\qquad\left. + \frac{8}{M^3}\tilde{\epsilon}_{mk}\epsilon_{ln}\mathfrak{D}^k\mathfrak{D}^l\mathfrak{D}^p a_p\right]\\ &+ \frac{1}{2M^4 - 4\epsilon\epsilon}\left[\frac{M^2}{6}\mathfrak{D}_m\mathfrak{D}_n h - \frac{1}{4}\epsilon_{mk}\epsilon^k{}_n h + \frac{\mathrm{i}}{2}\epsilon_{mk}\mathfrak{D}^k\mathfrak{D}_n h - \frac{5\epsilon\epsilon}{24}\eta_{mn}h\right] + (m \leftrightarrow n)\end{aligned} \tag{28}$$

One can check that the above definition yields the following equations of motion and constraints:

$$\begin{aligned}&\left(\mathfrak{D}^2 - M^2\right)\mathfrak{h}_{mn} = 2\mathrm{i}\left(\epsilon_{km}\mathfrak{h}^k{}_n + \epsilon_{kn}\mathfrak{h}^k{}_m\right)\\ &\mathfrak{D}^n\mathfrak{h}_{mn} = 0, \quad \mathfrak{h} = 0\\ &\mathfrak{D}_m\mathfrak{D}_n a'^n = M^2 a'_m, \quad \mathfrak{D}_m\mathfrak{D}_n c'^n = M^2 c'_m\end{aligned} \tag{29}$$

which are the sought for equations of motion and constraints.

3.2 Charged spin3/2

The problem of how massive spin-3/2 charged states propagate in an electromagnetic background has been around for quite some time, with roots dating back to the 1930s. This problem is plagued by the issue of superluminal propagation, which leads to causality loss. Despite efforts to develop causal equations of motion and a Lagrangian, such as in the work of [24], a significant proportion of the modifications of the minimal theory have failed to restore causality. Nonetheless, a possible solution to this problem was proposed in [23]. A Lagrangian ansatz was written where the coefficients of the different terms can be obtained recursively order by order in the electromagnetic field strength. In [4], the computation of the superspace Lagrangian and equations of motion for the first massive level of the open superstring [21, 22] was extended to the charged case, thereby generalizing the work of [16, 20] for the bosonic case and allowing us to write explicit forms of the equations of motion.

The first massive level of open superstrings contains 12 complex fermionic physical degrees of freedom, with eight of them corresponding to massive spin-3/2 fields denoted $(\lambda_{mj}, \chi_{mj})$ and the remaining four to spin-1/2 ones ψ_j, γ_j, with $j = 1, 2$. The corresponding Lagrangian reads:

$$
\begin{aligned}
\mathcal{L}_F = & -\frac{\mathrm{i}}{2}\left[2\left(\lambda_1^m\sigma^n\mathfrak{D}_n\bar{\boldsymbol{\lambda}}_{1m}\right)+\left(\bar{\boldsymbol{\chi}}_{1m}\bar{\sigma}^n\sigma^k\bar{\sigma}^m\mathfrak{D}_k\boldsymbol{\chi}_{1n}\right)\right]-\sqrt{2}\left[\left(\boldsymbol{\lambda}_1^m\boldsymbol{\chi}_{1m}\right)+\text{h.c.}\right] \\
& +\left[\frac{3}{2\sqrt{2}}\left(\boldsymbol{\chi}_1^m\sigma_m\sigma_n\mathfrak{D}^n\psi_1\right)-\frac{\mathrm{i}}{2}\left(\boldsymbol{\lambda}_1^m\sigma_m\bar{\psi}_1\right)-\frac{3}{2}\mathrm{i}\left(\bar{\boldsymbol{\chi}}_1^m\bar{\sigma}_m\gamma_1\right)-\sqrt{2}\left(\boldsymbol{\lambda}_1^m\mathfrak{D}_m\gamma_1\right)+\text{h.c.}\right] \\
& +\left[\frac{\mathrm{i}}{4}\left(\psi_1\sigma^m\mathfrak{D}_m\bar{\psi}_1\right)+\mathrm{i}\left(\gamma_1\sigma^m\mathfrak{D}_m\bar{\gamma}_1\right)\right]+\frac{1}{\sqrt{2}}\left[\bar{\gamma}_1\mathfrak{D}^2\bar{\psi}_1+\text{h.c.}\right] \\
& -\frac{\mathrm{i}}{8}G^{mn}\bar{\gamma}_1\bar{\sigma}_m\left[1-\mathrm{i}\left(\epsilon\cdot\sigma\right)\right]\sigma_k\left[1-\mathrm{i}\left(\epsilon\cdot\bar{\sigma}\right)\right]\bar{\sigma}_n\mathfrak{D}^k\gamma_1 \\
& +\frac{\mathrm{i}}{4}G^{mn}\left[\eta_{mp}+\mathrm{i}\left(\epsilon_{mp}-\mathrm{i}\tilde{\epsilon}_{mp}\right)\right]\left[\eta_{nq}-\mathrm{i}\left(\epsilon_{nq}+\mathrm{i}\tilde{\epsilon}_{nq}\right)\right]\psi_1\sigma^k\mathfrak{D}^p\mathfrak{D}_k\mathfrak{D}^q\bar{\psi}_1 \\
& -\frac{\mathrm{i}}{4}\left(\epsilon_{mn}\epsilon^{mk}+\tilde{\epsilon}_{mn}\tilde{\epsilon}^{mk}\right)\psi_1\sigma_k\mathfrak{D}^n\bar{\psi}_1-\frac{\mathrm{i}}{2}\tilde{\epsilon}_{mn}\left(\psi_1\sigma^n\mathfrak{D}^m\bar{\psi}_1\right)+\mathrm{i}\tilde{\epsilon}_{mn}\left(\bar{\gamma}_1\bar{\sigma}^n\mathfrak{D}^m\gamma_1\right) \\
& +\frac{1}{4\sqrt{2}}\left\{G^{mn}\left[\eta_{kp}-\mathrm{i}\left(\epsilon_{kp}+\mathrm{i}\tilde{\epsilon}_{kp}\right)\right]\left[\eta_{nl}-\mathrm{i}\left(\epsilon_{nl}+\mathrm{i}\tilde{\epsilon}_{nl}\right)\right]\bar{\gamma}_1\bar{\sigma}_m\sigma^p\mathfrak{D}^k\mathfrak{D}^l\bar{\psi}_1+\text{h.c.}\right\} \\
& -\left\{\frac{\mathrm{i}}{2}\left[\eta_{mk}-\mathrm{i}\left(\epsilon_{mk}+\mathrm{i}\tilde{\epsilon}_{mk}\right)\right]\boldsymbol{\lambda}_1^m\sigma_n\mathfrak{D}^n\mathfrak{D}^k\bar{\psi}_1+\frac{1}{2\sqrt{2}}\boldsymbol{\lambda}_1^m\sigma_n\left[1-\mathrm{i}\left(\epsilon\cdot\bar{\sigma}\right)\right]\bar{\sigma}_m\mathfrak{D}^n\gamma_1+\text{h.c.}\right\} \\
& +\frac{1}{2}\left[\boldsymbol{\lambda}_1^m\left(\epsilon\cdot\sigma\right)\sigma_m\bar{\psi}_1+\frac{1}{\sqrt{2}}\left(\epsilon\epsilon-\mathrm{i}\epsilon\tilde{\epsilon}\right)\psi_1\gamma_1+\text{h.c.}\right]+\left(1\leftrightarrow 2,\epsilon\leftrightarrow-\epsilon\right)
\end{aligned}
\tag{30}
$$

where we denoted the inverse matrix by $G_{mn} \equiv (\eta_{mn} - \mathrm{i}\epsilon_{mn})^{-1}$. Under the sign flipping $\epsilon_{mn} \leftrightarrow -\epsilon_{mn}$, G_{mn} transforms as $G_{mn} \leftrightarrow G_{nm}$. The equations of motion and constraints are for the spin-3/2:

$$
\begin{aligned}
& \mathrm{i}\sigma^n\mathfrak{D}_n\bar{\boldsymbol{\lambda}}'_{1m} = -\sqrt{2}\left(\eta_{mn}-\mathrm{i}\epsilon_{mn}\right)\boldsymbol{\chi}_1'^n \\
& \mathrm{i}\bar{\sigma}^n\mathfrak{D}_n\boldsymbol{\chi}'_{1m} = -\sqrt{2}\bar{\boldsymbol{\lambda}}_{1m}, \\
& \mathfrak{D}^m\boldsymbol{\chi}'_{1m} = 0, \qquad \mathfrak{D}^m\bar{\boldsymbol{\lambda}}'_{1m} = -\frac{\sqrt{2}}{4}\bar{\sigma}^m\left(\epsilon\cdot\sigma\right)\boldsymbol{\chi}'_{1m} \\
& \bar{\sigma}^m\boldsymbol{\chi}'_{1m} = 0, \qquad \sigma^m\bar{\boldsymbol{\lambda}}'_{1m} = 0
\end{aligned}
\tag{31}
$$

as well as the Dirac equations for the spin-1/2 fields:

$$
\mathrm{i}\bar{\sigma}^m\mathfrak{D}_m\gamma_1 = -\sqrt{2}\bar{\psi}_1, \qquad \mathrm{i}\sigma^m\mathfrak{D}_m\bar{\psi}_1 = -\sqrt{2}\gamma_1 \tag{32}
$$

These can be written in four-component notations using the Dirac spinors:

$$
\Phi_1 \equiv \begin{pmatrix} \gamma_{1\alpha} \\ \bar{\psi}_1^{\dot{\alpha}} \end{pmatrix}, \quad \boldsymbol{\Psi}_{1m} \equiv \begin{pmatrix} \boldsymbol{\chi}'_{1m\alpha} \\ \bar{\boldsymbol{\lambda}}'^{\dot{\alpha}}_{1m} \end{pmatrix} \tag{33}
$$

where

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \tag{34}$$

and $\not{\mathfrak{D}} \equiv \gamma^m \mathfrak{D}_m$.

The spin-1/2 satisfies the Dirac equation:

$$\left(\mathrm{i}\not{\mathfrak{D}} + \sqrt{2}\right)\Phi_1 = 0 \tag{35}$$

The constraints take the form

$$\left[\mathfrak{D}^m - \frac{\sqrt{2}}{4}\left(\epsilon^{mn} + \mathrm{i}\tilde{\epsilon}^{mn}\right)\gamma_n\right]\mathbf{\Psi}_{1m} = 0, \quad \gamma^m \mathbf{\Psi}_{1m} = 0 \tag{36}$$

Projection operators are defined as $P_L = (1 + \mathrm{i}\gamma^5)/2$, $P_R = (1 - \mathrm{i}\gamma^5)/2$, then the equations of motion can be written as

$$\left(\mathrm{i}\not{\mathfrak{D}} + \sqrt{2}\right)\mathbf{\Psi}_{1m} = \sqrt{2}\mathrm{i}\epsilon_{mn}\mathbf{\Psi}^n_{1L} \tag{37}$$

The fermions of index 2 correspond to the conjugates of those of index 1 in the neutral case. Their equations of motion and constraints can be obtained by the sign flip $\epsilon \to -\epsilon$.

4. Kaluza-Klein states for the Weak Gravity Conjecture

The Weak Gravity Conjecture (WGC) [25, 26] simplest formulation considers a D-dimensional $U(1)$ gauge theory with a coupling constant g. The WGC requires the existence of at least one state of mass m and charge q which satisfies the inequality:

$$gq \geq \sqrt{\frac{D-3}{D-2}}\kappa_D m, \tag{38}$$

where κ_D is defined as $\kappa_D^2 = 8\pi G_D = \frac{1}{M_{P,D}^{D-2}}$ with $M_{P,D}$ the reduced Planck mass in D dimensions. For these states, this inequality implies that the Newton force is not stronger than the Coulomb force. A particular example of states that saturate this inequality are Kaluza-Klein states, which can be used to investigate more general versions of the WGC where for instance extra scalar, i.e. dilatonic, interactions are present..

Consider compactification from $D + 1$ to D dimension of a free scalar coupled to general relativity. In the lower dimension, it is subject to the gravitational interaction, but also to a scalar (dilatonic) as well as gauge interactions mediated by the component $D + 1, D + 1$ and $D + 1, D$ of the higher dimensional gravitons, respectively.

The tree-level $2 \to 2$ scattering of two KK states $\varphi_n(p_1)\varphi_n(p_2) \to \varphi_n(p_3)\varphi_n(p_4)$:

$$\begin{aligned} i\mathcal{M} =& ig^2 q_n^2 \left(\frac{(p_1+p_3)\cdot(p_2+p_4)}{t} + \frac{(p_1+p_4)\cdot(p_2+p_3)}{u}\right) - 4i\frac{D-1}{D-2}\kappa^2 m_n^4\left(\frac{1}{t}+\frac{1}{u}\right) \\ & - \frac{\kappa^2}{4}\Bigg[\left(p_{1\mu}p_{3\nu} + p_{3\mu}p_{1\nu} - \eta_{\mu\nu}\left(p_1\cdot p_3 - m_n^2\right)\right)\frac{i\mathcal{P}^{\mu\nu\alpha\beta}}{t}\left(p_{2\alpha}p_{4\beta} + p_{4\alpha}p_{2\beta} - \eta_{\alpha\beta}\left(p_2\cdot p_4 - m_n^2\right)\right) \\ & \qquad + (t, p_3, p_4) \leftrightarrow (u, p_4, p_3)\Bigg] \end{aligned} \tag{39}$$

where $\mathcal{P}$ is the spin-2 projector

$$\mathcal{P}^{\alpha\beta\rho\sigma} = \frac{\eta^{\alpha\rho}\eta^{\beta\sigma} + \eta^{\alpha\sigma}\eta^{\beta\rho}}{2} - \frac{\eta^{\alpha\beta}\eta^{\rho\sigma}}{D-2} \tag{40}$$

where the different terms account for contributions from the exchanges of the gauge boson, the dilaton and the graviton, respectively. In the non-relativistic (NR) limit

$$\frac{s-4m_n^2}{m_n^2} \to 0, \qquad \frac{t}{m_n^2} \to 0, \qquad \text{and} \quad \frac{u}{m_n^2} \to 0 \tag{41}$$

which in terms of the mass give

$$i\mathcal{M} \to i\mathcal{M}_{NR} = 4im_n^2 \left[g^2 q_n^2 - \kappa^2 m_n^2 \left(\frac{D-1}{D-2} + \frac{D-3}{D-2} \right) \right] \left(\frac{1}{t} + \frac{1}{u} \right) = 0. \tag{42}$$

The cancellation is the result of a specific relation between the charge and the mass (**??**) that saturates the WGC. In fact, we can ensure the subdominance of gravity by requiring the existence a state with charge q and mass m satisfying the relation

$$g^2 q^2 \geq \left(\frac{\alpha^2}{2} + \frac{D-3}{D-2} \right) \kappa^2 m^2, \tag{43}$$

where α is the strength of the dilatonic coupling due to the gauge coupling of the form $e^{2\sqrt{2}\alpha\kappa\phi}F^2$. The explicit amplitude computation [5] allows to recover the Dilatonic Weak Gravity Conjecture that was derived in [27] (see also [28] for its generalization) from the study of the extremal Einstein-Maxwell-dilaton black hole solutions. In the absence of dilatonic forces, $\alpha = 0$, one retrieves the original WGC condition (38).

The use of amplitudes to investigate the different forms of WGC, in contrast to extremality of black holes, has been useful to formulate a scalar version of the WGC [29, 30] and to state the WGC as a comparison of channels for pair production [31] (valid in four-dimension [5]).

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