PROCEEDINGS OF SCIENCE

From von Neumann Algebras to Quantum de Sitter

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In this talk we review recent progress in translating heuristic considerations about quantum properties of de Sitter space into rigorous statements in the framework of Algebraic Quantum Field Theory.

Corfu Summer Institute 2022 "School and Workshops on Elementary Particle Physics and Gravity",
28 August - 1 October, 2022
Corfu, Greece

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PoS(CORFU2022)133

1. Introduction

The aim of this talk is to try to present some recent developments in our understanding of quantum gravity in de Sitter spaces. More specifically, over the years a growing body of heuristic insights has appeared concerning the expected features of quantum gravity on de Sitter space. Rather recently, it has been proposed [1, 2] that some of these insights can be turned into rigorous statements in the framework of algebraic quantum field theory. The various instances of such reformulation are summarized in the following table

Heuristic insight	**Algebraic QFT reformulation**
In QFT, the entanglement entropy of any state for a finite region is infinite.	The algebra of observables associated to a finite region is type III.
In the presence of horizons, the total entropy is finite.	In the presence of horizons, the algebra becomes of type II.
Empty de Sitter has maximum entropy.	The algebra of observables of a static patch of de Sitter is of type II_1.

The structure of the write-up of the talk is the following. We start in section 2 providing a perspective on the holographic principle, and in section 3 we collect some facts about de Sitter space. In section 4 we switch gears and recall some basic notions of entanglement in Quantum Mechanics, and the problems that appear when we try to apply them to Quantum Field Theory. Section 5 is devoted to introduce the bare minimum of von Neumann Algebras, and in section 6 we describe briefly the correspondences that appear in the table above.

2. The holographic principle

In the quest to find a consistent theory of quantum gravity, the holographic principle [3, 4] (see [5] for a review) has turned out to be one of the most remarkable insights. Informally, it asserts that in a theory of quantum gravity the amount of redundant degrees of freedom is such that it must be possible to find an equivalent formulation in terms of non-redundant degrees of freedom living on a hypersuface of one dimension less. Note that there is no guarantee that these non-redundant degrees of freedom have to be described by something familiar, like a local quantum field theory.

The AdS/CFT correspondence [6] provides a very concrete realization of this principle, but it applies to very specific spacetimes. For generic spacetimes, the determination of a holographic dual theory is currently out of reach, and there is no systematic approach in sight. Much of the work has been aimed at providing holographic duals of gravity on maximally symmetric spaces, for the three possible values (positive, zero, negative) of the cosmological constant.

We are currently far from being able to systematically deduce a holographic dual for a given spacetime. Nevertheless, there are some robust features that we can predict about the dual theory, based on the spectrum of black holes in the original spacetime. More specifically, if we grant that the high energy spectrum of the theory is dominated by black holes, we can deduce from the entropy of those black holes the density of states of the putative dual theory. Let's quickly run the argument for Anti de Sitter and Minkowski spaces. Later we will consider the case of de Sitter.

Consider a Schwarzschild black hole of mass M in a D dimensional anti de Sitter space of radius R_{AdS}. Its entropy behaves like [7]

$$S \approx \left(\frac{MR_{AdS}^2}{\ell_D}\right)^{\frac{D-2}{D-1}} \approx E^{\frac{D-2}{D-1}} \tag{1}$$

with ℓ_D the D-dimensional Planck length. On the other hand, for a CFT in d dimensions we have

$$S \approx E^{\frac{d-1}{d}} \tag{2}$$

and we observe that (1) and (2) behave in the same way if we take $d = D - 1$. This simple argument provides a rationale for the celebrated AdS/CFT duality.

If we now run the same argument for Schwarzschild black holes in D dimensional Minkowski space, we arrive at a starkly different conclusion. Their entropy behaves like [7, 8]

$$S \approx (M\ell_D)^{\frac{D-2}{D-3}}$$

and comparing to (2) we learn that no CFT in any dimension has a density of states with that behavior. Thus, it appears that no QFT with UV behavior controlled by a CFT can be dual to gravity in asymptotically Minkowski spaces. This doesn't mean that asympotically Minkowski spaces can't have holographic dual, just that their duals must be more exotic than an ordinary Quantum Field Theory. A particular example is the duality between backgrounds with a linear dilaton and little string theories [9]. Besides these top-down realizations of the holographic principle, there have been various bottom-up proposals. We collect some of those in this table,

	Top-down	Bottom-up
$\Lambda < 0$	AdS/CFT correspondence [6]	O(N) CFT/higher spin gravity in AdS_4 [10].
$\Lambda = 0$	LST/linear dilaton bacgkround [9]	Celestial holography [11]

3. Some facts about de Sitter

de Sitter is the maximally symmetric solution to the vacuum Einstein equations with a positive cosmological constant. The spatial sections of de Sitter are compact, so it is a closed universe. The isometry group of d dimensional de Sitter is O(d,1); this is the conformal group in Euclidean d-1 dimensions. One can introduce many different coordinate systems, that make manifest various properties [12]. The causal structure of the de Sitter is easiest to understand in conformal coordinates

$$ds^2 = \frac{1}{\cos^2 T}\left(-dT^2 + d\Omega_{d-1}^2\right)$$

It has observer dependent cosmological horizons: any given observer can access only a limited region of the full spacetime. In static patch coordinates,

$$ds^2 = -(1-r^2)dt^2 + \frac{1}{1-r^2}dr^2 + r^2 d\Omega_{d-2}^2$$

there is a timelike Killing vector in the static patch (future oriented in the patch, past oriented in the region spacelike separated from the static patch). The symmetry of the static patch is $\mathbb{R}\times SO(D-1)$, this is the subgroup of the full isometry group that gives automorphisms of the static patch.

Gibbons and Hawking [13] argued that an observer in de Sitter will measure a non-zero temperature

$$T_{dS} = \frac{1}{2\pi R}$$

and assigned an entropy to the region inside a cosmological horizon

$$S = \frac{A}{4G}$$

where A is the area of the observer dependent cosmological horizon. What is the meaning of this entropy? What is it counting? For black holes, the idea that black hole entropy is von Neumann entropy due to tracing out the interior degrees of freedom is due to R. Sorkin [14], and it is natural to extend it to de Sitter.

3.1 Finite range of entropy

It has been argued [15, 16] that the entropy of any asymptotically de Sitter space is bounded from above by the entropy of empty de Sitter. Here we illustrate this striking property by considering Schwarzschild de Sitter spaces with fixed value of the cosmological constant, and noticing that indeed introducing a black hole reduces the total entropy, and the larger the black hole, the smaller is the entropy of the cosmological horizon plus the black hole horizon.

In the static patch, the Schwarzschild de Sitter metric is (we set $R_{dS} = 1$)

$$ds^2 = -(1 - \frac{2m}{r^{d-3}} - r^2)dt^2 + \frac{1}{1 - \frac{2m}{r^{d-3}} - r^2}dr^2 + r^2 d\Omega^2_{d-2}$$

The horizons are the two real positive roots of $r^{d-1} - r^{d-3} + 2m$. There is a double root when $m^2 = \frac{(d-3)^{d-3}}{(d-1)^{d-1}}$ and it corresponds to the Nariai solution. The double root is $r = \sqrt{\frac{d-3}{d-1}}$. The entropy of the Nariai solution is then $S_N = 2\left(\frac{d-3}{d-1}\right)^{\frac{d-2}{2}} S_{dS}$, which is smaller than S_{dS} for any $d \geq 4$ ($S_N \to \frac{2}{e} S_{dS}$ when $d \to \infty$).

For $d = 4$, the horizons are the solutions of $r^3 - r + 2m = 0$. The discriminant is $4 - 108m^2$, so there is a double root at $m = 1/\sqrt{27}$. The double root is $r = 1/\sqrt{3}$, this is the Nariai solution. For SdS, it decreases until the Nariai value, $S_N = \frac{2}{3} S_{dS}$.

This feature of empty de Sitter having the maximum entropy has prompted the conjecture that the holographic dual of de Sitter ought to involve a finite dimensional Hilbert space [17–20].

4. Entanglement entropy

As advanced in the previous section, one of the suggestions put forward to give meaning to the entropy of de Sitter, is that it corresponds to entanglement entropy. With a view to trying to understand this proposal, in this section we completely shift gears and present the bare minimum concepts to understand entanglement entropy. We start by considering non-relativistic quantum mechanics, and then discuss the complications that arise when generalizing these ideas to quantum field theory. A standard reference is [21].

Consider a Hilbert space that is a tensor product of two Hilbert spaces, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, the generic state can be written as

$$|\Psi\rangle = \sum_{i,j} c_{i,j} |u_i\rangle \otimes |v_j\rangle \tag{3}$$

We will say that $|\Psi\rangle$ is an *entangled* state when there are no vectors $|u\rangle, |v\rangle$ such that $|\Psi\rangle = |u\rangle \otimes |v\rangle$.

By suitable unitary transformations, it is possible to write (3) in a more convenient way. According to the Schmidt decomposition, we can find orthogonal vectors ψ_A^i, ψ_B^j such that

$$|\Psi\rangle = \sum_{i=1}^{\min(d_1,d_2)} \sqrt{p_i}\; |\psi_A^i\rangle \otimes |\psi_B^j\rangle \tag{4}$$

where $\sum_i p_i = 1$ if we normalize $|\Psi\rangle$. Since in general $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$, with this choice, the matrix of coefficients is diagonal + a block of zeroes.

We are going to be particularly interested in the scenario where we only perform measures corresponding to one of the Hilbert spaces above, say $\mathcal{H}_A$. It is then convenient to introduce an object that allows us to restrict ourselves to states in $\mathcal{H}_A$ and operators acting on it. Given a state $|\Psi\rangle$, start by defining the density operator $\rho = |\Psi\rangle\langle\Psi|$. Say we restrict to measures that are just performed in A, so we consider only operators of the form $O = O_A \otimes \mathbb{I}$. In general, there is no $|\psi_A\rangle \in \mathcal{H}_A$ such that

$$\langle\Psi|O|\Psi\rangle = \langle\psi_A|O_A|\psi_A\rangle \;!!$$

However, we can define the reduced density operator

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\, \rho$$

To be concrete, if $|\Psi\rangle$ is written as in (4), then

$$\rho_A = \sum_i p_i |\psi_A^i\rangle\langle\psi_A^i|$$

It follows from $\sum_i p_i = 1$ that $\mathrm{tr}\, \rho_A = 1$. To see the relevance of this operator, let's compute the expected value of an arbitrary operator O_A acting on $\mathcal{H}_A$. In the full Hilbert space it is $O = O_A \otimes \mathbb{I}_B$.

$$\langle\Psi|O|\Psi\rangle = \sum_i p_i \langle\psi_A|O_A|\psi_A\rangle = \mathrm{tr}_{\mathcal{H}_A}\; \rho_A O_A$$

and we have reduced the computation of $\langle\Psi|O|\Psi\rangle$ to quantities defined only in $\mathcal{H}_A$.

Given a state, we want to not only characterize whether it is entangled or not, but also quantify the amount of entanglement. The most popular measure of entanglement is the von Neumann entropy, defined as

$$S = -\mathrm{Tr}_{\mathcal{H}_A}\; \rho_A \log \rho_A$$

Note that it depends on the full state $|\Psi\rangle$. It satisfies

$$0 \leq S \leq \log(\dim \mathcal{H}_A)$$

It takes the minimum value only if $|\Psi\rangle$ is a product state (no entanglement). So, even if the full $|\Psi\rangle$ is pure, if it is entangled, there is non-zero entanglement entropy. It takes the maximum value when all probabilities are the same. In this case, the density matrix is proportional to the identity.

Modular Hamiltonian. [22] The density matrix ρ is Hermitian and positive semidefinite, so it is diagonalizable with eigenvalues real and semipositive. It then makes sense to define the modular Hamiltonian

$$\rho = e^{-H} \qquad \rho = \frac{e^{-H}}{\text{tr } e^{-H}}$$

for some Hermitian operator H. In general H is not local (it can't be written as a local expression of the fields). Let's emphasize that the modular Hamiltonian H is not the Hamiltonian of the QM/QFT. This allows us to think of entanglement entropy as a thermal entropy of a system with H and $\beta = 1$.

Thermofield double Consider a thermal system with Hamiltonian H at temperature β, the density matrix is

$$\rho = \frac{e^{-\beta H}}{\text{Tr } e^{-\beta H}}$$

Given a pure state $|\Psi\rangle$ of a large system, we saw how to obtain the reduced density operator ρ of a subsystem. Conversely, given ρ, it is possible to "purify" it by constructing a pure $|\Psi\rangle$ of a larger system. This purification process is hugely non-unique. In the particular case that ρ corresponds to a thermal state, there is a particular choice of purification that is very convenient. It is known as the thermofield double and it is defined as follows. Start by doubling the system and introduce

$$|\Psi\rangle_{TFD} = \frac{1}{\sqrt{Z}} \sum_i e^{-\frac{\beta E_i}{2}} |i\rangle_A \otimes |i\rangle_B$$

It is immediate to check that indeed for any $O = O_A \otimes \mathbb{I}$,

$$\langle\Psi|O|\Psi\rangle = \frac{1}{Z} \sum_i e^{-\beta E_i} \langle i|O_A|i\rangle = \text{tr}_{\mathcal{H}_A}\ \rho O_A$$

As an important application, the thermofield double provides a way to describe statistic mechanics in the large volume limit [23]. The thermofield double is a Physics realization of the Gelfand-Naimark-Segal (GNS) construction.

4.1 Entanglement entropy in QFT

The straightforward application of the concepts introduced above in Quantum Field Theories runs into trouble, as we now quickly recall. See [24–26] for reviews of entanglement entropy in QFT.

Let's illustrate the main issues with one of the first computations of entanglement entropy in QFT [27], see also [28]. Consider a scalar field in $\mathbb{R}^{3,1}$. Discretize $\mathbb{R}^3$ by a lattice.

$$H = \frac{1}{2} G^{MN} P_M P_N + \frac{1}{2} V_{MN} q^M q^N$$

V is symmetric and positive definite, so there is W such that $W^2 = V$. Let's focus on the ground state

$$\Psi_0(q^A) \propto e^{-\frac{1}{2} W_{AB} q^A q^B}$$

and from it we defined the density operator (if we think of Ψ_0 like a vector, ρ is like a matrix)

$$\rho(q^A, q'^B) \propto e^{-\frac{1}{2}W_{AB}(q^A q^B + q'^A q'^B)}$$

Now, consider a region Ω in $\mathbb{R}^3$. While [27] refer to this region as the interior of a black hole, nothing in their actual computation relies on this interpretation. If we assume that we can't access the oscillators in the interior of this region, it makes sense to consider the reduced density matrix for the complement of Ω (Greek indices label oscillators inside Ω),

$$\rho_{\text{red}} = \int \prod_\alpha dq^\alpha \ \langle q^a, q^\alpha | \rho | q'^b, q^\alpha \rangle$$

After doing the integral, one can attempt to compute the corresponding von Neumann entropy $S(\rho_{\text{red}})$, but the answer is divergent. Let's understand the origin of this divergence. S is dimensionless, so for a scale invariant QFT, S can't depend on any scale of the region. The answer is thus 0 or ∞, and the authors of [27] discard the possibility that the answer is 0. For a massive scalar, the entropy of a region with characteristic scale R can depend on mR. If S was now finite, we would expect that it tends to zero as $R \to 0$. But this limit is equivalent to $m \to 0$, which as argued above should give an infinite entanglement entropy. It must then be the case that S is still infinite in the massive case. We learn that the divergence is not removed by a mass term, so it is an ultraviolet divergence.

We can regulate the computation by introducing a short-distance cut-off ϵ near the boundary of the region. This eliminates the contribution from very high frequency modes that are likely the cause of the divergence. Then [27]

$$S = c\frac{A}{\epsilon^2} + \dots \tag{5}$$

Tantalizingly, the resulting entanglement entropy is proportional to the area A of the boundary of the region. However, we have to face the fact that the entanglement entropy is divergent. This leading UV divergence is rather universal, in the sense that it is independent of the state. Intuitively, this is a reflection that in QFT, at short distance, all states look like the vacuum. This universality of the divergence points to a serious difficulty in assuming that when we divide spacetime in two regions A,B, then $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If that were the case, one would have product states $\psi_A \otimes \psi_B$, and those have no entanglement.

Let's briefly mention another example. Consider Rindler space, *i.e.* the causal development of the half-space $x^1 > 0$. Then [29] the modular Hamiltonian is just the boost generator in the x^1-direction, [26]

$$\rho \propto e^{-K} \qquad K = \int_{x^1>|t|} d^{d-1}x \, x^1 T_{00}(x)$$

If the state is the vacuum, the modular operator is $\Delta = e^{-2\pi K}$, where K is a Lorentz boost. Since K has continuous spectrum all the real numbers, the spectrum of Δ is all the positive numbers. This is Unruh : the thermal description shows various degrees of entanglement (not maximal entanglement).

4.2 Entanglement entropy and gravity

We have found that entanglement entropy in QFT presents a universal UV divergence, eq. (5). There are no counterterms within QFT itself that can absorb this divergence. One possible criticism to the previous calculation is that modes with arbitrarily high frequency play a determinant role, but we are neglecting their backreaction on the background geometry. It is conceivable that in a calculation that takes into account that backreation, the resulting entanglement entropy is finite. We will now recall the argument [30] (see also [31]) that in the presence of horizons, this divergence can be absorbed in Newton's constant.

In the presence of a horizon, the total entropy receives two contributions, one from the horizon and one from the matter fields

$$S_{\text{gen}} = \frac{A}{4G} + S_{\text{out}} \tag{6}$$

The key point of [30] is that both terms in the RHS of (6) are divergent by themselves, but their sum is finite. We have already seen that the entanglement entropy of the matter fields outside the horizon yields a divergent result, eq. (5). An explicit computation shows that Newton's constant, that appears in the first term of the RHS of (6) gets renormalized, and the precise correction to the bare constant depends on the matter fields in the required way for these two divergences to cancel each others. This calculation has been generalized to various matter fields, see [32] for a review. While most of these computations are presented for black hole horizons, they have been also carried out for a de Sitter cosmological horizon [33].

Let's briefly mention another approach to characterize the entanglement entropy of de Sitter space. Within the context of the AdS/CFT duality, the Ryu-Takayanagi [34, 35] formula gives a way to compute holographically the entanglement entropy of a region in the boundary. The UV divergence that appears in the QFT computations is now realized as an IR divergence in the geometric computation. While we lack a firmly established dual for de Sitter space, one can apply the Ryu-Takayanagi proposal also to de Sitter, expecting that it captures the entanglement entropy of the putative dual [36]. Strikingly, the entanglement entropy of a region is now finite. This suggests that the dual is not a full quantum field theory, perhaps a truncated CFT.

One can compute holographically more than just the von Neumann entropy. In [36] they computed all Renyi entropies, found them to vanish, and from this they concluded that the density matrix of the dual of de Sitter is maximally mixed. However, see [37] for an argument that the are corrections to the density matrix such that it is not proportional to the identity.

5. von Neumann algebras

We have discussed entanglement entropy in QFT and its universal UV divergence. We have also recalled some indications that a putative holographic dual of de Sitter space might have entanglement entropy not corresponding to a full QFT. In the remains of this talk, we want to explore the possibility that the Algebraic formulation of QFT is particularly well suited to understand these results. Algebraic QFT attempts to formulate QFT in terms of the operators that one can define on a bounded region of spacetime, and their relations [22]. Recalling the early days of Quantum Mechanics, when Schrödinger formulated it in terms of wavefunctions, and Heisenberg in terms of commutation relations among operators, informally one can think of Algebraic QFT as Heisenberg vision for QM on steroids.

In particular, these operators form a particular kind of algebras, von Neumann algebras, and they will be the focus of our attention. In this section we introduce some very basic notions about von Neumann algebras, and in the next section we discuss their relevance in Physics.

Recall that an algebra is a vector space A over a field K with a bilinear product

$$A \times A \to A$$

so there are three operations, sum, product by scalar (=element of K) and product.

We are ultimately interested in von Neumann algebras, which are a particular type of C^* Banach algebras. To understand what this means, let's recall some definitions. A Banach algebra is an associative algebra that is also a Banach space, *i.e.* a complete normed space[1]. A C^*- algebra is a Banach algebra over $\mathbb{C}$ with an involution $* : A \to A$, which sends $x \in A$ to $x^* \in A$, so it has the properties of complex conjugation/taking the adjoint.

To define a von Neumann algebra, we need to be aware that given a sequence of operators $T_n, n \in \mathbb{N}$, there are inequivalent ways to define that the limit of T_n as $n \to \infty$ is the operator T. These inequivalent ways are called topologies. The norm $|| \ ||$ in the Banach algebra does define a topology, but this *norm topology* is too restrictive for our purposes. Let's introduce the weak topology, using language that assumes that the von Neumann algebra acts on vectors of a Hilbert space. We say that a_n converges to a in the weak topology if for all $|\psi\rangle, |\xi\rangle \in \mathcal{H}$, $\lim_{n\to\infty}\langle\psi|a_n|\xi\rangle = \langle\psi|a|\xi\rangle$. The weak operator topology (WOT) is weaker than the norm topology. That means that there are sequences of operators that don't have a limit using the norm topology, but they do using the WOT. For instance, let $\mathcal{H}$ be an ∞-dimensional separable Hilbert space, and $K(\mathcal{H})$ the algebra of its compact operators. $K(\mathcal{H})$ is a C^* algebra that contains all finite-rank projector operators. The WOT limit of these operators is the identity, but the identity is not a compact operator, so it is included in the von Neumann algebra, but not in the C^* algebra.

We are ready to define von Neumann algebras. A von Neumann algebra is a C^* algebra closed under the weak operator topology.

As the simplest example, let $\mathcal{H}$ be a Hilbert space and O a (linear) operator $O : \mathcal{H} \to \mathcal{H}$. We define the norm of the operator as follows

$$||O|| = \sup_{x\neq 0} \frac{||Ox||}{||x||}$$

If an operator has finite norm, we say it is a bounded operator. The set of bounded operators of a Hilbert space, $B(\mathcal{H})$ is a von Neumann algebra.

To further proceed, we need various definitions. The center of an algebra A is

$$Z(A) = \{x \in A| \ \ xy = yx \ \ \forall y \in A\}$$

An algebra such that the center is just $\mathbb{C}$ (*i.e.* the identity matrix multiplied by a complex number), is called a factor (a factor can be thought of as the analog of simple Lie algebra).

[1]I.e., it has a norm $|| \ || : A \to \mathbb{R}$, such that with the distance induced by this norm, every Cauchy sequence is convergent.

Let $A \subset B(\mathcal{H})$. The commutant A' of A in $B(\mathcal{H})$ is

$$A' = \{x \in B(\mathcal{H})| \ \ xy = yx \ \ \forall y \in A\}$$

The commutant is always a von Neumann algebra, even if the original algebra is not.

The double commutant theorem by von Neumann shows that if A is a C^* algebra and contains the unit operator, the three following conditions are equivalent:

1. A=A"
2. A is closed in the strong operator topology (SOT).
3. A is closed in the WOT.

If we define the algebra without reference to a Hilbert space, we need to define states without introducing the Hilbert space. A state ω is a linear functional that assigns a complex number to any operator, $\omega : A \to \mathbb{C}$. For comparison, when we have a Hilbert space available, we can think of states as mapping an operator to its expected value in that state: $a \to \langle \Psi | a | \Psi \rangle$. We require that $\omega(a^\dagger a) \geq 0$ and $\omega(\mathbb{I}) = 1$.

Trace A trace τ is a linear functional $\tau : A \to \mathbb{C}$ such that $\tau(a^\dagger a) \geq 0$ and $\tau(ab) = \tau(ba)$.

Tracial state. A tracial state is a trace that is also a state, $\tau(\mathbb{I}) = 1$.

Example: Consider the algebra of $M_{n\times n}(\mathbb{C})$. The ordinary trace satisfies $Tr(MN) = Tr(NM)$ and $Tr(M^\dagger M) = \sum_{ij} |m_{ij}|^2 \geq 0$ so it is a trace. A tracial state is $\tau = \frac{1}{n}$ Tr.

Projector. A projector is an operator P that satisfies $P^2 = P$ and $P^\dagger = P$. The spectral theorem allows to write any self-adjoint operator as a sum of projectors. Projectors play an important role in von Neumann algebras, because in a sense they generate it. Indeed, define $\mathcal{P}(A) = \{p \in A | p^2 = p = p^\dagger\}$. Then $A = \mathcal{P}(\mathcal{A})''$. The same is not true for generic C^* algebras [38].

Crossed product [1] Suppose a von Neumann algebra A acts on a Hilbert space $\mathcal{H}$, and let T be a self-adjoint operator on H that generates a group of automorphisms of A. This means

$$e^{iTs}\, a\, e^{-iTs} \in A \qquad \forall a \in A, \ \ s \in \mathbb{R}$$

We can construct a larger algebra, the crossed product $A \times \mathbb{R}$. Introduce a new real variable X, and let $L^2(\mathbb{R})$ be the space of square-integrable functions on X. Then $H \otimes L^2(\mathbb{R})$ is a larger Hilbert space and the crossed product algebra acts on $H \otimes L^2(\mathbb{R})$, generated by operators of the form $ae^{isT} \otimes e^{isX}$

5.1 Classification of von Neumann algebras

Next, we want to discuss the classification of certain von Neumann algebras [38, 39]. The full classification of von Neumann algebras appears to be out of reach, but we are only interested in the classification of very particular von Neumann algebras: hyperfinite factors. First we will quote the classification of factors, and then that of hyperfinite factors.

A factor is an algebra with trivial center. It is the analog of a simple Lie algebra. It turns out (theorem 4.34 in [38]) that factors can be classified by the action of the trace on the projectors of the algebra. To connect with things that are more familiar, in finite dimensions, a projector P can be diagonalized to a matrix with 1s and 0s in the diagonal. The trace of P is just the number of 1s, and more invariantly, it is the dimension of the space it projects onto, so it is an integer between 0 and dim $\mathcal{H}$.

Theorem [38]: if M is a factor, then the action of the trace on the projectors of M is

- {0,1,2,. . . ,n}. M is type I_n
- $\mathbb{N} \cup \infty$. M is type I_∞
- [0,1]. M is type II_1
- $[0, \infty]$. M is type II_∞
- $\{0,\infty\}$. M is type III.

Recall that for finite dimensions, the trace acting on a projector gives the dimension of the space it projects onto. Thus, for type II, this trace is a generalization (to $\mathbb{R}$!) of the notion of dimension. It is then clear that this is not quite the notion of trace that we are familiar with in Physics. On the other hand, note that for type III algebras there are no non-trivial values of the trace; one can say that these algebras "have no trace".

For specific von Neumann algebras, this classification can be further refined. A hyperfinite von Neumann algebra is an algebra that appears as the limit of matrix algebras, as we allow them to grow in size. For Lie algebras, Ado's theorem ensures that all Lie algebras are isomorphic to a matrix algebra, so this is a big difference with von Neumann algebras.

Classification of hyperfinite factors [section 4.12 of [38]] Let M be a hyperfinite factor

- If M is type I, then $M \simeq B(\mathcal{H})$ for some $\mathcal{H}$. with dim $\mathcal{H}$=d for type I_d. d can be finite or infinite
- There is a unique hyperfinite type II_1 factor. Call it R.
- There is a unique hyperfinite type II_∞ factor, $R \otimes B(\ell^2)$
- There is a new invariant for type III factors. In paricular, there is a unique hyperfinite III_1 factor.

In the case of type II_∞, since the trace for type I_∞ is not defined for every element, the same happens for II_∞.

5.2 Explicit construction of hyperfinite factors

We now sketch explicit constructions of hyperfinite II_1 and III_1 factors. These factors are unique up to isomorphism, but there are many ways to construct them.

Hyperfinite type II_1 factor A first way to construct it is to consider a nested set of embeddings of $M_{2^n}(\mathbb{C}) \subset M_{2^{n+1}}(\mathbb{C})$. [40, 41], normalizing each time the ordinary trace. This seems to be a case of "direct limit". The resulting algebra is C^*, but not a von Neumann algebra. Now, applying the GNS construction to that C^* algebra with respect to the trace, one obtains a von Neumann algebra, that is the hyperfinite II_1 factor.

As a second construction, consider two systems A and B of N qubits each, and maximally entangled pairwise (N Bell pairs). This is like the thermofield double at $\beta = 0$ (or H=0).

$$|\Psi\rangle = \frac{1}{2^{N/2}} \bigotimes_{n=1}^{N} \sum |i\rangle_{A,n} \otimes |i\rangle_{B,n}$$

Now consider operators a that act upon $k \leq N$ qubits of the A system. We can define $F(a) = \langle\Psi|a|\Psi\rangle$. Crucially, maximal entanglement of each pair implies that $F(ab) = F(ba)$. To see this, note that at finite N, $\rho = \frac{1}{2^N}\mathbb{I}$. Let's illustrate this with operators acting on the first A qubit,

$$F(ab) - F(ba) = \langle\uparrow|[a,b]|\uparrow\rangle + \langle\downarrow|[a,b]|\downarrow\rangle = \text{tr}\,[a,b] = 0$$

Now take the limit of $N \to \infty$, allowing operators that act on "not too many" qubits. This doesn't quite yield a von Neumann algebra, one has to complete the resulting algebra by adding operators that appear in the weak limit. Importantly, the F defined above is well defined in the large N limit.

Recall that for a single qubit the projectors are

$$P_\uparrow = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_\downarrow = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

It is easy to convince oneself that by having projectors for some qubits, we get any rational value $\in [0,1]$ for the trace. Then the closure provides all values in $[0,1]$.

Hyperfinite type III_1 factor Let's repeat the same construction as before, but now each pair of the N qubits is non-maximally entangled.

$$|\Psi\rangle = \frac{1}{Z} \otimes_{n=1}^{N} \left(|\uparrow\uparrow\rangle_n + e^{-\frac{\beta E}{2}} |\downarrow\downarrow\rangle_n \right)$$

Again, let's consider operators that act only of the first $k \leq N$ R qubits, and define again

$$F(a) = \langle\Psi|a|\Psi\rangle$$

but now in general $F(ab) \neq F(ba)$, so it can't be interpreted as a trace. To see this, it is enough to consider operators acting just on the first qubit.

$$F(ab) - F(ba) = \langle\uparrow|[a,b]|\uparrow\rangle + e^{-\frac{\beta E}{2}} \langle\downarrow|[a,b]|\downarrow\rangle$$

The crucial difference is that now we can't construct a function satisfying $F(ab) = F(ba)$. This illustrates the generic feature that type III von Neumann algebras don't have a trace. The fact that type II algebras have a trace while type III don't will be of the utmost importance for us. The reason is that if we have a trace $\tau : A \to \mathbb{C}$, then *any* linear functional $F : A \to \mathbb{C}$ is of the form $F(a) = \tau(a\rho)$ for some $\rho \in A$. This will allow us to define a notion of density matrix for type II algebras, even if they don't act irreducibly on any Hilbert space. To understand the claim, recall that in ordinary (finite dim) linear algebra, if M is the space of $n \times n$ matrices, the trace defines an isomorphism

$$\text{tr}\ : M \to M^* \tag{7}$$

$$m \to \text{tr}\,(m\quad) \tag{8}$$

Notice that for type II_1, 1 is a valid density matrix. Once we have a density matrix, we can define a von Neumann entropy for type II algebras. For type III we can't do the same, there is no notion of density matrix.

To conclude, let's present a scorecard for the different algebras we have encountered: type I algebras are algebras of bounded operators $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Type II and III are not; this is important: if the algebra of operators defined on a region in spacetime is of type II or type III, there is no Hilbert space associated to that region, only the full Hilbert space is defined. Moreover, type II algebras have a notion of trace, which allows us to define density matrices and entropy. Type III algebras don't have such a notion of trace, and consequently no notion of density matrix either.

6. von Neumann algebras in Physics

We are finally ready to explain how some of the heuristic insights about quantum gravity in de Sitter space find a precise formulation in algebraic quantum field theory, and more particularly in the language of von Neumann algebras. Some references on the role of von Neumann algebras in Physics are [22, 25, 39, 42, 43].

The first question we need to address is what is the von Neumann algebra of operators in ordinary Quantum Field Theory. Consider a Quantum Field Theory defined on a manifold M, and an open subset $U \subset M$. For free field theories, Araki [44] argued that the von Neumann algebra is of type III. According to [42, 45] this result extends to interacting theories and so it is quite universal: for all relativistic QFTs the algebra of observables defined on an open and bounded region is universal, it is the unique hyperfinite type III_1 factor. Witten [25, 43] provides a very heuristic argument for this universality: consider any state in a QFT and an entangling surface. Locally we can approximate the entangling surface by two opposite Rindler wedges in Minkowski space, and at short distances all states behave like the vacuum. If the state is the vacuum, the modular operator is $\Delta = e^{-2\pi K}$, where K is a Lorentz boost. Since K has continuous spectrum all the real numbers, the spectrum of Δ is all the positive numbers. This is just the Unruh effect: the thermal description shows various degrees of entanglement (not maximal entanglement), pointing towards type III_1.

The identification of the algebra of operators in an open region as type III algebras gives a formal explanation to the universal divergence encountered when trying to compute the entanglement entropy, eq. (5). There is no notion of trace or density matrix for these algebras, and that is the formal reason behind the failure to obtain a finite entanglement entropy for any state.

The universality of the von Neumann algebra for all QFTs, regardless of the matter content, interactions and spacetime dimension raises an obvious question: how are different QFTs defined if they all share the same algebra of observables for an open region? According to [26, 42, 45] the answer is that the specific properties of particular QFTs are encoded in the nesting of these algebras. For comparison, recall that in QM, the harmonic oscillator and the hydrogen atom have the same Hilbert space, and the same algebra of bounded operators.

Recently, type III algebras have appeared in the context of the AdS/CFT duality, in two novel ways [46, 47]. A first appearance concerns the algebra of single trace operators in the large N limit of $\mathcal{N} = 4$ SYM, at $T > T_H$ defined over the entire boundary spacetime. It is conjectured to be an emergent type III_1 algebra. It is dual to the bulk algebra of the R region in a Schwarzschild AdS black hole. According to Witten [1], this emergent type III_1 algebra, in the strict large N limit, has a non-trivial center, so it is not a factor.

A second appearance concerns the algebra of operators defined on a local region of the boundary spacetime at zero temperature (single copy of the CFT). This algebra at finite N is expected to be of type III_1 by the arguments mentioned above. The claim is that in the large N limit, there is a different, emergent, type III_1 algebra.

6.1 Change in the type of algebra due to gravity

The novel physical insight we want to discuss is the following: even at very weak coupling, in the presence of a horizon gravity realizes the enlargement of the von Neumann algebra by crossed product, thus turning the type III algebras of the previous section into type II algebras [1, 2]. This qualitative change in the algebra of operators has rather important implications, since as we discussed in section 5, one can define density matrices for type II algebras, something that it is not possible for type III. This qualitative change has been discussed for black hole horizons [1] and for de Sitter [2]. In the case of black hole horizons the resulting algebra is of type II_∞, while for de Sitter is of type II_1. In either case, this change from type III to type II algebra explains why the notion of entropy is better defined in the presence of gravity. Moreover, in the de Sitter case, the fact that the algebra is of type II_1 has a more detailed implication; recall that the type II_1 algebra has a state of maximum entropy, namely the maximally mixed state with matrix density equal to 1. This provides a precise formulation for the insight that empty de Sitter has maximal entropy.

Let's start by going over the argument for the change in the algebra in the presence of a black hole. In [1] Witten follows [46, 47] and considers the thermofield double of $\mathcal{N} = 4$ SU(N) SYM at temperatures above the Hawking-Page temperature. Starting in the strict large N limit, we restrict our attention to subtracted single trace operators. Consider first operators that *don't* commute. They form two algebras that we denote by $A_{R,0}, A_{L,0}$. By the arguments above, they are type III_1. They are factors, since they have a trivial center. The 0 subindex is to remind us that so far we are only considering operators that don't commute. The Hamiltonians on each side of the double, H_L, H_R, are not in $A_{R,0}, A_{L,0}$. On the other hand

$$U \equiv \frac{1}{N}\left(H_R - \langle H_R\rangle\right)$$

does survive the large N limit. In the strict large N limit, U commutes with every operator in $A_{R,0}$. We might think that we need to introduce a similar operator for H_L, but in the strict large N limit, it is argued in [1] that they are the same. Now introduce the following algebras

$$A_R \equiv A_{R,0} \otimes A_U \qquad A_L \equiv A_{L,0} \otimes A_U$$

A_R, A_L are type III, but they are not factors, in both cases their center is A_U.

So far, the discussion was in the strict large N limit. Now we consider $1/N$ corrections, *i.e.* small G_N effects. So far, we used U for a boundary operator that has a gravity dual. From now on, U refers to the gravity dual. The most important 1/N correction is that now

$$\frac{1}{N}H'_R = U + \frac{1}{\beta N}\hat{h}$$

where $\hat{h} = \beta(H'_R - H'_L)$. This implies that the algebras $A_{R,L}$ turn into crossed products

$$A_R = A_{R,0} \times | A_{U+\hat{h}/\beta N}$$

The argument in [1] is at order $1/N$. There is no guarantee this structure holds at finite N, in fact we expect that at finite N, A_R becomes type I. The status is the following, U belongs to A_L, so it commutes with A_R. It is not in the center of A_R. On the other hand $U + \hat{h}/\beta N$ is an element of A_R, but it is not central. A_R has become a factor. The discussion has been presented in a way that doesn't make manifest the symmetry between A_R and A_L, but it is argued in [1] that nonetheless the symmetry is there.

For a black hole, this comes about because there are two canonically conjugate modes that appear to behave differently: one is a mode that describes a fluctuation of the mass of the black hole. The other, its conjugate, is a relative time shift between observations in the left and right regions. Call these variables X,P. If $X \in \mathbb{R}$, there is a new algebra $B(L^2(\mathbb{R}))$. It turns out to be the the modular automorphism group of the ordinary QFT III_1 algebra, so the result is a II_∞ algebra.

This has been considered for asymptotically AdS and asymptotically flat black holes. Consider the maximally extended solution. X is βH_L the ADM energy measured on the left, while H+X is realized by H_R, the ADM measured on the right. Since the entropy of fields outside the BH can be arbitrarily large, there is no upper bound on the total entropy (BH+out).

Now consider de Sitter space. Crucially, we are dealing with a closed universe, so isometries must be treated as constraints [48, 49]. This is something we are familiar with in gauge theories; for gauge theories on compact space, even at vanishing gauge one must impose the Gauss constraint, and that has important effects on the spectrum, *e.g.* [50]. In the case of de Sitter, if we try to just restrict to an algebra invariant under the constraints, there is nothing left! [2]. The way out proposed in [2] is to introduce an observer. Formally this is implemented by enlarging the Hamiltonian, $H \to \hat{H} = H + H_{obs}$ with energy q bounded from below $q \geq 0$. This has the effect of enlarging the algebra

$$A_1 = A_0 \otimes B(L^2(\mathbb{R}_+))$$

We impose again the Hamiltonian constraint, but now for the enlarged Hamiltonian $\hat{H}$, thus keeping only $A_1^{\hat{H}}$, the $\hat{H}$ invariant part of A_1, *i.e.* the subalgebra that commutes with $\hat{H}$. To construct A_1, first ignore the $q \geq 0$ constraint. Then, operators of the form

$$\hat{a} = e^{ipH} a e^{-ipH}$$

where $p = -id/dq$, commute with $H + q$. Indeed

$$[e^{ipH} a e^{-ipH}, H] = e^{ipH}[a, H] e^{-ipH} \qquad [e^{ipH} a e^{-ipH}, q] = -e^{ipH}[a, H] e^{-ipH}$$

q is also invariant, trivially $[q, H + q] = 0$. According to Takesaki duality, those operators are the only ones in the algebra. It then it follows from results of Connes and Takesaki that the algebra is of type II_∞. Finally, to take into account the constraint $q \geq 0$, consider the projection $\theta(q)$ and the algebra $\hat{A} = \theta(q) A_{cr} \theta(q)$. The result a type II_1 algebra with trivial center.

The tracial state is identified with empty de Sitter, thus giving a rigorous formulation of the insight that empty de Sitter has maximal entropy.

7. Parting comments

To wrap up, let me mention some questions and ideas that arose while preparing this talk.

First, we have been referring to von Neumann algebras as algebras of observables, but these algebras contain operators that are not self-adjoint. What can we say about the reduction to self-adjoints? Apparently, algebras of self-adjoint operators are known as Jordan algebras. They are not associative, so they are harder to work with. Also, any operator O is the sum of two self-adjoint operators

$$O = \frac{1}{2}\left(O + O^{\dagger}\right) + \frac{i}{2}\left(O - O^{\dagger}\right)/i$$

so perhaps the reason we deal with von Neumann algebras is that we can always write any operator as a sum of two self-adjoint operators.

When identifying the algebra of observables of the static patch of de Sitter space with a type II_1 algebra, the dimension of de Sitter space never entered the discussion. This is most likely a generalized version of the universality of algebras observed in the absence of gravity. It would be interesting to explicitly construct nets of the II_1 factor that correspond to de Sitter in various dimensions.

The arguments leading to the identification of the algebra of observables of the static patch of de Sitter space with a type II_1 algebra is rather indirect [2]: start with a type III algebra; consider the presence of an observer that via a crossed product turns it into type a II_∞, and finally take into account that the energy of the observer is bounded from below to reduce the algebra to type II_1. On the other hand, in the Mathematics literature there are reasonably explicit constructions of the hyperfinite II_1 factor, particularly in terms of Clifford algebras [40, 41, 51]. It might be worth use this explicit constructions to come up with a toy model of holography for de Sittter.

An obvious objection to the proposal of [2] is that the entropy of a type II_1 algebra is not bounded from below, while the entropy of de Sitter is bounded from below by the entropy of the Nariai solution. Perhaps the way out is that the derivation of type II_1 is made for very small G_N, so it doesn't capture black holes in de Sitter.

It will be important to identify the algebras of observables for other spacetimes. An interesting next case to consider is a Nariai space, *i.e.* the largest black hole that fits in de Sitter. The geometry has two horizons, but they have the same temperature, so it's looks easier than the generic Schwarzschild de Sitter solution.

Looking ahead, the arguments reviewed in this talk suggest that we might be on the road to prove rigorously that in the presence of a horizon

$$S = S_{hor} + S_{mat}$$

is UV finite. But what about entanglement entropy of QFT in a compact region without horizon? Does gravity at finite G_N also render it UV finite?

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