PROCEEDINGS OF SCIENCE

3-loop tadpoles with substructure from 12 elliptic curves

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The generic 2-loop kite integral has 5 internal masses. Its completion by a sixth propagator gives a 3-loop tadpole whose substructure involves 12 elliptic curves. I show how to compute all such kites and their tadpoles, with 200 digit precision achieved in seconds, thanks to the procedure of the arithmetic geometric mean for complete elliptic integrals of the third kind. The number theory of 3-loop tadpoles poses challenges for packages such as `HyperInt` by Erik Panzer and `MZIteratedIntegral` by Kam Cheong Au. In particular, I obtain three surprising empirical reductions to classical polylogarithms of totally massive tadpoles. These have been checked at 600-digit precision.

In memoriam, Gabriel Barton (25 February 1934 to 11 October 2022) and Donald Hill Perkins (15 October 1925 to 30 October 2022), trusted guides and mentors.

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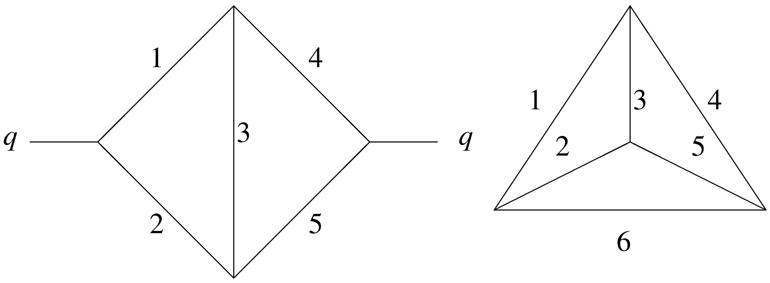
1. Introduction



A 3-loop tetrahedral tadpole is formed by closing a kite with a sixth propagator. In the generic mass case, 6 different kites close to form the same tadpole. Each kite has two elliptic obstructions, coming from intermediate states with three massive particles. At first sight there appears to be scant hope that the tadpole might evaluate to multiple polylogarithms. Yet I shall exhibit three totally massive cases with surprisingly simple empirical reductions to classical polylogarithms.

For the kite integral I extend the methods of [1–3] to the multivariate case, defining the 2-loop scalar kite in 4-dimensional Minkowski space as

$$I(q^2) = -\frac{q^2}{\pi^4}\int \mathrm{d}^4 l \int \mathrm{d}^4 k \prod_{j=1}^{5} \frac{1}{p_j^2 - m_j^2 - \mathrm{i}\epsilon}, \tag{1}$$

$$(p_1,\, p_2,\, p_3,\, p_4,\, p_5) = (l,\, l-q,\, l-k,\, k,\, k-q), \tag{2}$$

with a cut $s \in [s_{\mathrm{L}}, \infty]$ and a branch point s_{L} that is the lowest of the thresholds $\{s_{1,2}, s_{4,5}, s_{2,3,4}, s_{1,3,5}\}$, where $s_{j,k} = (m_j + m_k)^2$ and $s_{i,j,k} = (m_i + m_j + m_k)^2$. To compute the kite it suffices to know the derivative of the discontinuity $I(s + \mathrm{i}\epsilon) - I(s - \mathrm{i}\epsilon) = 2\pi\mathrm{i}\,\sigma(s)$ across the cut, which determines

$$I(q^2) = -\int_{s_{\mathrm{L}}}^{\infty} \mathrm{d}s\, \sigma'(s) \log\left(1 - \frac{q^2}{s}\right). \tag{3}$$

Regularization in $4 - 2\epsilon$ dimensions of the tetrahedral tadpole formed by joining the external vertices of the kite with a propagator $1/(q^2 - m_6^2)$ gives

$$T^{5,4,6}_{1,2,3} = \left(\frac{1}{3\epsilon} + 1\right) 6\zeta_3 + 3\zeta_4 - F^{5,4,6}_{1,2,3} + O(\epsilon), \tag{4}$$

$$F^{5,4,6}_{1,2,3} = \int_{s_{\mathrm{L}}}^{\infty} \mathrm{d}s\, \sigma'(s) \left(\mathrm{Li}_2\left(1 - \frac{m_6^2}{s}\right) + \frac{1}{2}\log^2\left(\frac{\mu^2}{s}\right)\right), \tag{5}$$

where μ is the scale of dimensional regularization. This is more convenient than a 5-dimensional integral from Schwinger parametrization. With $\mu = m_6 = 1$, the finite part is given by

$$F^{5,4,6}_{1,2,3} = \int_0^{\infty} \mathrm{d}x_1 \ldots \int_0^{\infty} \mathrm{d}x_5\, \frac{1}{U^2} \log\left(1 + \sum_{k=1}^{5} x_k m_k^2\right) \tag{6}$$

after setting $x_6 = 1$ in the Symanzik polynomial of the tetrahedron,

$$\begin{aligned} U = {} & x_3(x_1x_2 + x_4x_5) + x_6(x_1x_4 + x_2x_5) + x_3x_6(x_1 + x_2 + x_4 + x_5) \\ & + x_2x_4(x_1 + x_3 + x_5 + x_6) + x_1x_5(x_2 + x_3 + x_4 + x_6). \end{aligned} \tag{7}$$

The binary case, with masses equal to zero or unity, was considered in [4–7].

2. The derivative of the discontinuity of the kite

In the absence of anomalous thresholds, the non-elliptic contribution from 2-particle cuts is

$$\sigma'_{\rm N}(s) = \Theta(s - s_{1,2})\sigma'_{1,2}(s) + \Theta(s - s_{4,5})\sigma'_{4,5}(s) \tag{8}$$

where Θ is the Heaviside step function. I denote the square root of the Källén function by

$$\Delta(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)} \tag{9}$$

with convenient abbreviations $\Delta_{j,k}(s) = \Delta(s, m_j^2, m_k^2)$ and $\Delta_{i,j,k} = \Delta_{j,k}(m_i^2)$. Then

$$D_{j,k}(s) = \frac{r}{s - (m_j - m_k)^2} \log\left(\frac{1 + r}{1 - r}\right), \quad r = \left(\frac{s - (m_j - m_k)^2}{s - (m_j + m_k)^2}\right)^{1/2} \tag{10}$$

provides the logarithms in

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) = \Re\left((s + \alpha)D_{4,5}(s) + L_{4,5} + \sum_{i=0,+,-} C_i \frac{D_{4,5}(s) - D_{4,5}(s_i)}{s - s_i}\right) \tag{11}$$

with constants

$$C_0 = -(m_1^2 - m_2^2)(m_4^2 - m_5^2), \quad C_\pm = \alpha s_\pm + \beta, \quad L_{4,5} = \log\left(\frac{m_4 m_5}{m_3^2}\right), \tag{12}$$

$$\alpha = \frac{(m_1^2 - m_4^2)(m_2^2 - m_5^2)}{m_3^2} - m_3^2, \quad \beta = \frac{(m_1^2 m_5^2 - m_2^2 m_4^2)(m_1^2 - m_2^2 - m_4^2 + m_5^2)}{m_3^2}, \tag{13}$$

$$s_0 = 0, \quad s_\pm = \frac{m_1^2 + m_2^2 - 2m_3^2 + m_4^2 + m_5^2 - \alpha}{2} \pm \frac{\Delta_{1,3,4}\Delta_{2,3,5}}{2m_3^2} \tag{14}$$

where $s_\pm$ locate leading Landau singularities of triangles that form the kite. These may lead to anomalous thresholds [8], to be considered later.

The elliptic contribution comes from 3-particle intermediate states, giving

$$\sigma'_{\rm E}(s) = \Theta(s - s_{2,3,4})\sigma'_{2,3,4}(s) + \Theta(s - s_{1,3,5})\sigma'_{1,3,5}(s). \tag{15}$$

It contains complete elliptic integrals of the third kind of the form

$$P(n, k) = \frac{\Pi(n, k)}{\Pi(0, k)}, \quad \Pi(n, k) = \int_0^{\pi/2} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} \tag{16}$$

with $\Pi(0, k) = (\pi/2)/\text{AGM}\,(1, \sqrt{1 - k^2})$ determined by the arithmetic-geometric mean of Gauss [9]. With $s = w^2$, an integration over the phase space of particles 2, 3 and 4 determines

$$k^2 = 1 - \frac{16 m_2 m_3 m_4 w}{W}, \quad W = (w_+^2 - m_+^2)(w_-^2 - m_-^2) \tag{17}$$

with $w_\pm = w \pm m_2$ and $m_\pm = m_3 \pm m_4$. Then I obtain

$$\sigma'_{2,3,4}(w^2) = \frac{4\pi m_3 m_4}{\text{AGM}\,(\sqrt{16 m_2 m_3 m_4 w}, \sqrt{W})} \Re\left(\sum_{i=+,-} E_i \frac{P(n_i, k) - P(n_1, k)}{t_i - t_1}\right) \tag{18}$$

with coefficients and arguments given, as compactly as possible, by

$$E_\pm = \frac{m_2^2 - m_3^2 + m_5^2}{2m_5^2} \pm \left(\frac{m_4^2 - m_5^2 - w^2}{2m_5^2}\right) \frac{\Delta_{2,3,5}}{\Delta_{4,5}(w^2)}, \tag{19}$$

$$t_\pm = \frac{\gamma \pm \Delta_{2,3,5}\Delta_{4,5}(w^2)}{2m_5^2}, \quad t_1 = m_1^2, \quad n_i = \frac{(w_-^2 - m_+^2)(t_i - m_-^2)}{(w_-^2 - m_-^2)(t_i - m_+^2)}, \tag{20}$$

$$\gamma = (m_2^2 + m_3^2 + m_4^2 - m_5^2 + w^2)m_5^2 + (m_2^2 - m_3^2)(m_4^2 - w^2). \tag{21}$$

An AGM procedure speedily evaluates $P(n,k) = \Pi(n,k)/\Pi(0,k)$ to high precision, as follows.

1. Initialize $[a,\ b,\ p,\ q] = [1,\ \sqrt{1-k^2},\ \sqrt{1-n},\ n/(2-2n)]$. Set $f = 1 + q$.
2. Set $m = ab$ and $r = p^2 + m$. Replace $[a,\ b,\ p,\ q]$ by the new values in the vector $[(a+b)/2,\ \sqrt{m},\ r/(2p),\ (r-2m)q/(2r)]$. Add q to f.
3. If $|q/f|$ is sufficiently small, return $P(n,k) = f$, else go to step 2.

This converges very quickly, for $n \notin [1,\infty]$. On the cut with $n \geq 1$, replace n by $n' = k^2/n < 1$, to obtain the principal value $\Re P(n,k) = 1 - P(n',k)$.

2.1 Criterion for an anomalous contribution

Suppose that $s_{4,5} \geq s_{1,2}$. Then

$$\sigma'(s) = \sigma'_{\rm N}(s) + \sigma'_{\rm E}(s) + C_{\rm A}\frac{\Theta(s-s_{4,5})}{\Delta_{4,5}(s)}\,\Re\left(\frac{2\pi {\rm i}\Delta_{4,5}(s_-)}{s-s_-}\right) \tag{22}$$

with $C_{\rm A} \neq 0$ if and only if $(m_1+m_2)(m_3^2+m_1m_2) < m_1m_5^2 + m_2m_4^2$ and at least one of $\Delta_{1,3,4}$ and $\Delta_{2,3,5}$ is imaginary, in which case $C_{\rm A} = \pm1$ is the sign of $\Im\Delta_{4,5}(s_-)$, with s_- given in (14).

This value of $C_{\rm A}$ is required by the elliptic contribution at high energy. With $L_k = m_k^2\log(s/m_k^2)$, the large-s behaviour

$$s^2\sigma'(s) = 2L_3 + \sum_{k=1,2,4,5}(L_k + m_k^2) + O\left(\frac{\log(s)}{s}\right) \tag{23}$$

invariably holds. The elliptic contribution $\sigma'_{\rm E}$ in (22) is oblivious to the anomalous threshold problem. Its high-energy behaviour determines the value $C_{\rm A} \in \{0, 1, -1\}$, ensuring (23).

3. Tadpoles and number theory

The rescaling $m_k \to \kappa\, m_k$ gives $F \to F + 12\zeta_3\log(\kappa)$ for the finite part F. To standardize, I set $\mu = \max(\{m_k\}) = 1$.

I define a tetrahedral tadpole to be perfect if and only if the Källén function vanishes at each of its 4 vertices, thereby avoiding all resolutions of square roots. Promoting the subscripts and superscripts of F to arguments that denote the 6 masses, I define the two-parameter perfect tadpoles

$$\widehat{F}(x,y) = F_{(x,y,1)}^{(1-y,1-x,|x-y|)} = \widehat{F}(y,x) = \widehat{F}(1-x,1-y) \tag{24}$$

with symmetries restricting distinct cases to $x \geq y \geq 1-x \geq 0$ and hence $x \in [\frac{1}{2}, 1]$. In QED, I identified tetralogarithms in two perfect binary tadpoles, obtaining [6]

$$\widehat{F}(1,0) = F^{(1,1,0)}_{(1,1,0)} = 17\zeta_4 + 16U_{3,1}, \quad \widehat{F}(1,1) = F^{(0,0,0)}_{(1,1,1)} = 12\zeta_4, \tag{25}$$

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \tfrac{1}{2}\zeta_4 + \tfrac{1}{2}\zeta_2 \log^2(2) - \tfrac{1}{12}\log^4(2) - 2\,\mathrm{Li}_4(\tfrac{1}{2}). \tag{26}$$

Now consider the elliptic route to evaluating $\widehat{F}(\frac{1}{2}, \frac{1}{2})$. With $(m_3, m_6) = (1, 0)$ and $m_1 = m_2 = m_4 = m_5 = \frac{1}{2}$, I obtained

$$\widehat{F}(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2}\int_1^\infty \mathrm{d}s\,\left(\widehat{\sigma}'_{\mathrm{N}}(s) + \widehat{\sigma}'_{\mathrm{E}}(s)\right)\log^2(s), \tag{27}$$

$$w^2\widehat{\sigma}'_{\mathrm{N}}(w^2) = \Theta(w-1)\left(2\log\left(\frac{r+1}{r-1}\right) - 4r\log(2)\right),\ r = \frac{w}{\sqrt{w^2-1}}, \tag{28}$$

$$w^2\widehat{\sigma}'_{\mathrm{E}}(w^2) = \frac{4\pi(1-P(n,k))\Theta(w-2)}{\mathrm{AGM}(2\sqrt{w}, (w-1)\sqrt{w^2+2w})},\ n = \frac{w^2-2w}{(w-1)^2},\ \frac{k^2}{n} = \frac{(w+1)^2}{w^2+2w} \tag{29}$$

and readily discovered a new reduction of a perfect tadpole to tetralogarithms

$$\widehat{F}(\tfrac{1}{2}, \tfrac{1}{2}) = 30\zeta_3\log(2) - 16\zeta_4 - 32U_{3,1}. \tag{30}$$

3.1 Relations between tadpoles

In addition to the two-parameter family $\widehat{F}(x, y)$, there is a one-parameter family $\widehat{G}(x) = F^{(x,1-x,1)}_{(x,1-x,1)}$ of perfect tadpoles, with $x \in [0, \frac{1}{2}]$ and $\widehat{G}(0) = 17\zeta_4 + 16U_{3,1}$.

I used the efficient AGM of Gauss to obtain 200 digits of

$$\widehat{G}(\tfrac{1}{2}) = -\int_1^\infty \mathrm{d}s\,\left(\widehat{\sigma}'_{\mathrm{N}}(s) + \widehat{\sigma}'_{\mathrm{E}}(s)\right)\mathrm{Li}_2(1-s) \tag{31}$$

to which all routes are elliptic. This revealed the intriguing empirical relation

$$2\widehat{F}(\tfrac{1}{2}, \tfrac{1}{2}) + 2\widehat{F}(1, \tfrac{1}{2}) + \widehat{G}(\tfrac{1}{2}) = 42\zeta_4 + 24\zeta_3\log(2). \tag{32}$$

A non-elliptic route to $\widehat{F}(1, \frac{1}{2})$ led to multiple polylogarithms in an alphabet of forms, $\mathrm{d}x/(x-a_i)$, with $a_i \in \{0,\ 1,\ -1,\ -2,\ -\frac{1}{2}\}$. With help from Steven Charlton, I found the surprising evaluation

$$F^{(\frac{1}{2},\frac{1}{2},1)}_{(\frac{1}{2},\frac{1}{2},1)} = \widehat{G}(\tfrac{1}{2}) = 6\left(2\zeta_4 - 3\mathrm{Li}_4(\tfrac{1}{4})\right) + 8\left(2\zeta_3 - 3\mathrm{Li}_3(\tfrac{1}{4})\right)L - 12\,\mathrm{Li}_2(\tfrac{1}{4})L^2 - 4L^4 \tag{33}$$

with $L = \log(2)$ and classical polylogs giving 10000 digits in less than a second. Relation (32) evaluates an integral of a trilog against complete elliptic integrals of the first and second kinds:

$$4\int_0^1 \mathrm{d}y\left(\frac{1}{y} - 1\right)T(y)Z(y) = \widehat{G}(\tfrac{1}{2}) + 16\,\zeta_4 + 32\,U_{3,1} - 30\,\zeta_3\log(2)\,, \tag{34}$$

$$T(y) = \mathrm{Li}_3(u) - \tfrac{1}{2}\,\mathrm{Li}_2(u)\log(u)\,, \quad u = \frac{y}{(1+y)^2}\,, \tag{35}$$

$$Z(y) = \frac{y(1+y)K(k) + E(k)}{(1+y+y^2)\sqrt{1+y}}\,, \quad k^2 = 1 - y^3\,, \tag{36}$$

$$K(k) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}\mathrm{d}\theta, \quad E(k) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{1/2}\mathrm{d}\theta\,. \tag{37}$$

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I remark that the elliptically obstructed massless 10-point double-box integral of [10, 11] involves the integral of trilogarithms against the reciprocal of the square root of a quartic.

Binary tadpoles, with $m_k \in \{0,1\}$, evaluate to multiple polylogarithms in an alphabet containing sixth roots of unity, with $\lambda = (1+\sqrt{-3})/2$ appearing if three massive edges meet at a vertex, where $\Delta_{i,j,k} = \sqrt{-3}$. For example, with 5 unit edges

$$F^{(1,1,1)}_{(1,1,0)} = \frac{109}{6}\left(\frac{\pi}{3}\right)^4 + 16\Re\left(\frac{\mathrm{Li}_2^2(\lambda)}{6} + \sum_{m>n>0}\frac{\lambda^{3m+2n}}{m^3 n}\right). \tag{38}$$

There are linear relations between binary tadpoles [6],

$$3F^{(1,1,1)}_{(0,0,0)} = F^{(0,0,0)}_{(1,1,1)} + 2F^{(1,0,0)}_{(1,1,0)}, \tag{39}$$

$$3F^{(0,0,0)}_{(1,1,0)} = F^{(0,0,0)}_{(1,0,0)} + 2F^{(0,0,0)}_{(1,1,1)}, \tag{40}$$

$$F^{(1,1,1)}_{(1,1,1)} + F^{(1,0,0)}_{(1,0,0)} = F^{(1,1,0)}_{(1,1,0)} + F^{(1,1,1)}_{(0,0,0)}, \tag{41}$$

the last of which surprisingly reduces a totally massive imperfect case to polylogs, with

$$F^{(1,1,1)}_{(1,1,1)} = 16\zeta_4 + 8U_{3,1} + 4\mathrm{Cl}_2^2(\pi/3) \tag{42}$$

containing the square of the Clausen value $\mathrm{Cl}_2(\pi/3) = \Im\mathrm{Li}_2(\lambda)$.

3.2 Three more reductions of tadpoles to polylogs

In further evidence of the simplicity of perfect tadpoles, I reduced $\widehat{F}(1,\frac{2}{3})$, $\widehat{F}(\frac{2}{3},\frac{2}{3})$ and $\widehat{F}(\frac{3}{4},\frac{3}{4})$ to weight-4 products of $\{\log(2), \log(3), \zeta_2, \mathrm{Li}_2(\frac{1}{4}), \zeta_3, \mathrm{Li}_3(\frac{1}{9}), \mathrm{Li}_3(\frac{1}{4})\}$, together with $\mathrm{Li}_4(x)$ for $x \in \{\frac{1}{9},\frac{1}{4},\frac{1}{3},\frac{1}{2}\}$ and the depth-2 multiple polylogarithm $\mathrm{Li}_{2,2}(\frac{1}{4},1) = \sum_{m>n>0}(2^m mn)^{-2}$.

3.3 Stringent tests for kites and tadpoles

1. Elliptic terms do not depend on the order of phase-space integrations [12].

2. The derivative of the discontinuity of a kite satisfies the sum rule

$$\int_{s_{\mathrm{L}}}^{\infty} \mathrm{d}s\, \sigma'(s)\log\left(\frac{s}{s_{\mathrm{L}}}\right) = 6\zeta_3. \tag{43}$$

3. The high-energy behaviour (23) of $\sigma'(s)$ holds irrespective of anomalous thresholds.

4. Benchmarks for kites, given by Stefan Bauberger and Manfred Böhm [13] to 6 decimal digits and by Stephen Martin [14], to 8 decimal digits, are confirmed and then extended to 100 digits in less than a second.

5. The same tadpole is obtained by integrating over 6 distinct kites.

6. Binary tadpoles with $m_k \in \{0,1\}$ agree with my previous reductions to poloylogs of sixths roots of unity.

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3.4 Number theory

So far, one might guess that a tadpole with rational masses evaluates to multiple tetralogarithms in an alphabet whose number field is no larger than the compositum $Q(\Delta_{1,3,4}, \Delta_{2,3,5}, \Delta_{1,2,6}, \Delta_{4,5,6})$ of the quadratic number fields associated by Gunnar Källén to the vertices of the tetrahedron. Yet that is not the case. The imperfect binary tadpole $F^{(1,0,0)}_{(1,1,0)}$ involves $\mathrm{Cl}_2^2(\pi/3)$, but the Källén field is rational. Faced with this rather limited, yet potent, evidence, I arrive at three suggestions, each too weak to be dignified as a well-tested conjecture.

1. Tetrahedral tadpoles with rational masses reduce to multiple tetralogarithms whose alphabet lies in an algebraic number field.
2. If the tadpole is perfect, the alphabet is rational.
3. If the tadpole is imperfect, the alphabetic field may include the Källén field.

3.5 Experimentum crucis

Most remarkably, I found an empirical relation between the totally massive imperfect tadpole $F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$ with Källén field $Q(\sqrt{-3})$ and the perfect tadpole $\widehat{G}(\frac{1}{2})$ in (33), namely

$$F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} = 3\zeta_3\log(2) - 4U_{3,1} + 10\zeta_4 + 10\mathrm{Cl}_2^2(\pi/3) - \tfrac{1}{2}\widehat{G}(\tfrac{1}{2}). \tag{44}$$

It took less than a minute to validate (44) at 600-digit precision. It implies that

$$4\int_2^\infty \frac{\mathrm{d}w}{w}\left(\mathrm{Li}_2\left(1-\frac{1}{w^2}\right)-\zeta_2\right)Y(w) = \zeta_4 - 4U_{3,1} + 7\zeta_3\log(2), \tag{45}$$

$$Y(w) = \frac{\Pi(0,k) - \Pi(n,k) - 6\,\Pi(\widehat{n},k)}{(w-1)\sqrt{w^2+2w}}, \tag{46}$$

$$k^2 = 1 - \frac{4}{(w-1)^2(w+2)}, \quad n = 1 - \frac{1}{(w-1)^2}, \quad \widehat{n} = 1 - \frac{2}{w(w-1)}. \tag{47}$$

4. Comments and summary

1. Elliptic substructure of 2-loop kites and 3-loop tadpoles is not a problem. The time taken to evaluate a complete elliptic integral, of whatever kind, is commensurate with the time for a logarithm and less than the time for a dilogarithm. Thanks to Gauss, elliptic integrals should be embraced, not feared.
2. Anomalous terms are not problematic. They submit to Gauss, at high energy.
3. The number theory of tadpoles is subtle. As found in (33,42,44), totally massive tadpoles may be polylogarithmic. Yet every route to their evaluation has an elliptic obstruction. Notwithstanding earnest efforts by Yajun Zhou and myself, proofs of these three surprising reductions currently elude us.
4. With at least one massless edge, non-elliptic routes permit proofs, ex post facto, of reductions discovered empirically, using `Pari/GP`. For the proofs, we relied on procedures implemented in `Maple` by Erik Panzer [15] and in `Mathematica` by Kam Cheong Au [16].

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