PROCEEDINGS OF SCIENCE

The cyclic symmetries in the representations of unitary discrete subgroups

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Dark matter may be stable because of a conserved $\mathbb{Z}_N$ (cyclic) symmetry. Usually N is assumed to be 2, but it may also be larger than 2.
This $\mathbb{Z}_N$ is usually assumed to be in a direct product with some other symmetry group. The full symmetry group of the theory is then $G = \mathbb{Z}_N \times G'$. We suggest another possibility.
Many discrete subgroups of $U(D)$, for any $D \geq 2$, have a non-trivial center $\mathbb{Z}_N$, even if they are not the direct product of that $\mathbb{Z}_N$ with some other group. When that happens, the irreducible representations ('irreps') of the group may either represent all the elements of that $\mathbb{Z}_N$ by the unit matrix, or else they may represent that $\mathbb{Z}_N$ faithfully. If ordinary matter is placed in a representation where $\mathbb{Z}_N$ is represented by 1, and dark matter is placed in irreps that represent $\mathbb{Z}_N$ faithfully, then dark matter is stabilized by that $\mathbb{Z}_N$.
We have scanned all the discrete groups in the `SmallGroups` library with order ≤ 2000 that are not the direct product of a cyclic group with some other group. We have determined their centers and whether they are subgroups of one or more groups $SU(D)$ or $U(D)$. We have found that very many groups, especially subgroups of $U(D)$ but not of $SU(D)$, have non-trivial centers $\mathbb{Z}_N$, mostly with N of the form $2^p \times 3^q$ but also with other values of N.

8th Symposium on Prospects in the Physics of Discrete Symmetries (DISCRETE 2022)
7-11 November, 2022
Baden-Baden, Germany

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†D.J. thanks the Lithuanian Academy of Sciences for support through the project DaFi2021.

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1. Introduction

The lightest Dark Matter (DM) particles must be stable (*i.e.* unable to decay), on cosmological timescales. The stability is determined by an unbroken cyclic $\mathbb{Z}_N$ symmetry which is non-trivial (*i.e.* it has $N \geq 2$), such that standard matter is invariant under $\mathbb{Z}_N$ while DM is not. The $\mathbb{Z}_N$ charge different from 1 of the lightest DM particle prevents it from decaying to standard matter, which has $\mathbb{Z}_N$ charge 1.

In many models this stability is provided by $N = 2$. But, some authors have considered possibilities $N > 2$ such DM stabilized by a $\mathbb{Z}_3$ symmetry [1, 2]. Larger cyclic groups have been used to stabilize DM, like $\mathbb{Z}_4$, $\mathbb{Z}_5$ and $\mathbb{Z}_6$ [3, 4, 5] or a general $\mathbb{Z}_N$ [6, 7].

The cyclic $\mathbb{Z}_N$ symmetry that stabilizes DM may be the center of a larger internal-symmetry group G. The simplest possibility consists in G being a discrete group of order O that is isomorphic to the direct product $\mathbb{Z}_N \times G'$, where G' is a group of order O/N. In that case, all the irreducible representations of G consist of the product of an irrep of $\mathbb{Z}_N$ and an irrep of G'. The particles of the Standard Model (SM) must be placed in the trivial representation of $\mathbb{Z}_N$ while DM particles are placed in non-trivial representations of $\mathbb{Z}_N$.

But there exist also discrete groups G that cannot be written as the direct product of a cyclic group and a smaller group but may have a non-trivial $\mathbb{Z}_N$ center. In that case an irrep of G may represent $\mathbb{Z}_N$ non-trivially (*viz.* when some elements of $\mathbb{Z}_N$ are represented by a phase f with $f \neq 1$ but $f^N = 1$). If $\mathbb{Z}_N$ remains unbroken when G is broken, and if there are particles with $\mathbb{Z}_N$ charge different from 1, then those particles play the role of DM, while the particles with $\mathbb{Z}_N$ value 1 are standard matter. This mechanism had already been suggested before, *viz.* in Ref. [8] and studied in recent works by Refs. [9] and [10].

In this work we make a systematic survey of the centers of all the discrete groups G of order up to 2000 that cannot be written as the direct product of a cyclic group and another group and that have some faithful irreducible representation ('firrep').[1] We identify the center $\mathbb{Z}_N$ of each of those groups, and also the dimensions D of their firreps. Also we construct various tables with the integers O, N, and D. We find that very many discrete groups, especially those that are not subgroups of any continuous group $SU(D)$, have centers $\mathbb{Z}_N$ with $N \geq 2$, and N is sometimes quite large. Comprehensive listings of the groups that we have studied are available online at `https://github.com/jurciukonis/GAP-group-search` [11].

2. Group search

The defining representation of $SU(D)$ consists of the $D \times D$ unitary matrices with determinant 1. In this representation, the center of $SU(D)$ is formed by the D diagonal matrices

$$\Delta \times \mathbb{1}_D,\ \Delta^2 \times \mathbb{1}_D,\ \Delta^3 \times \mathbb{1}_D,\ \ldots,\ \Delta^D \times \mathbb{1}_D = \mathbb{1}_D, \tag{2.1}$$

where $\Delta = \exp(2i\pi/D)$. Thus, the center of $SU(D)$ is a $\mathbb{Z}_D$ group. More generally, if m is an integer that divides D and $\mu = \exp(2i\pi/m)$, then there is a cyclic symmetry $\mathbb{Z}_m$ given, in the

[1] We do not survey groups of order either 512, 1024, or 1536, because there are millions or billions of groups of those orders.

defining representation of $SU(D)$, by

$$\mathbb{Z}_m = \left\{\mu \times \mathbb{1}_D,\ \mu^2 \times \mathbb{1}_D,\ \mu^3 \times \mathbb{1}_D,\ \ldots,\ \mu^m \times \mathbb{1}_D = \mathbb{1}_D\right\}. \tag{2.2}$$

One or more discrete subgroups of $SU(D)$ may then have $\mathbb{Z}_m$ as their center. Thus, discrete subgroups of $SU(D)$ that are not subgroups of any $U(D')$ with $D' < D$ may have very few centers. For particular cases of $SU(D)$ please see Ref. [10].

On the other hand, discrete subgroups of $U(D)$ do not bear the constraint that the determinants of the matrices in their defining representations should be 1. As a consequence, if

$$\mathbb{Z}_t = \left\{\theta \times \mathbb{1}_D,\ \theta^2 \times \mathbb{1}_D,\ \theta^3 \times \mathbb{1}_D,\ \ldots,\ \theta^t \times \mathbb{1}_D = \mathbb{1}_D\right\}, \tag{2.3}$$

where $\theta = \exp(2i\pi/t)$, is the center of a discrete subgroup of $U(D)$, then there appears to be *a priori* no restriction on t. An example of $U(2)$ would be the discrete group $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$ having order 16 and `SmallGroups` identifier $[16, 6]$ as illustrated in Ref. [10].

Motivated by this observation that discrete subgroups of $U(D)$ may in general have diverse centers, in our work we have surveyed many discrete groups in order to find out their centers and also which groups $U(D)$ they are subgroups of. Therefore we have made a scan over all the discrete groups of order $O \leq 2000$ in the `SmallGroups` library[2], using `GAP`[3] except the groups of order either 512, 1024, or 1536. Then we have discarded all the groups that are isomorphic to the direct product of a smaller (*i.e.* of lower order) group and a cyclic group. We also have used `GAP` to find out all the irreps of each remaining group, and then to ascertain whether those irreps are faithful or not. We have discarded all the groups that do not have any firrep. We have thus obtained 87,349 non-isomorphic groups, that are all listed in our tables available at [11]. We have computed the determinants of the matrices of each firrep in order to find out whether all those determinants are 1 or not. We have also looked for all the matrices in the firreps that are proportional to the unit matrix and we have checked that those matrices form a group $\mathbb{Z}_N$ for some integer N (which for some groups is just 1). We have also examined the question whether each D-dimensional firrep is equivalent to a representation through matrices of $SU(D)$.[4]

There are relatively few groups that have firreps with different dimensions. For instance: A_5 has firreps of dimensions three, four, and five for which all generators have determinant 1. The discrete group $\mathrm{GL}(2,3)$, with `SmallGroups` identifier $[48, 29]$, has two 2-dimensional generators with determinant 1 while the third one has determinant -1; hence, we classify $\mathrm{GL}(2,3)$ as a subgroup of $U(2)$, but it is not a subgroup of $SU(2)$. On the other hand, there is another faithful

[2]This library [12] contains all the finite groups of order less than 2001, but for order 1024—because there are about 4.9×10^{10} of groups of order 1024. `SmallGroups` also contains some groups for some specific orders larger than 2000. In `SmallGroups` the groups are ordered by their orders; for each order, the complete list of nonisomorphic groups is given. Each discrete group is labeled $[O, n]$ by `SmallGroups`, where O is the order of the group and $n \in \mathbb{N}$ is an integer that distinguishes among the non-isomorphic groups of the same order.

[3]`GAP` [13] is a system for computational discrete algebra that provides a programming language and includes many functions that implement various algebraic algorithms. With `GAP` it is possible to study groups and their representations, to find the subgroups of larger groups, and so on.

[4]All the irreps of discrete groups are equivalent to representations through unitary matrices, and therefore we know that all the generators that `GAP` provides to us are equivalent to unitary generators. In order to know whether the generators belong to $SU(D)$ we just compute their determinants.

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irrep of GL(2, 3), through 4×4 unitary matrices, all of them with determinant 1. Therefore, we classify GL(2, 3) as a subgroup of both $U(2)$ and $SU(4)$, but GL(2, 3) earns these two classifications through different irreps. We have found just 2787 such discrete groups, out of the total 87,349 groups that we have surveyed; they are collected in table Intersections at [11].

On the other hand, the group $\Sigma(36 \times 3)$, that has `SmallGroups` identifier [108, 15], has irreps of dimensions 1, 3, and 4, but the 1- and 4-dimensional irreps are unfaithful—all the firreps have dimension 3. This group is not included in table Intersections but it appears in table U(D)&SU(D) at [11], because some 3-dimensional irreps has determinant 1 for all generators, while other 3-dimensional irreps do not have determinant 1 for all generators.

It is worth to mention also that the scan over the `SmallGroups` library to find the firreps of all possible dimensions constituted a computationally very expensive task. Some groups of orders 1728 and 1920 require quite a few CPU hours to find the irreps. Orders 768, 1280, and 1792 have more than one million non-isomorphic groups of each order and therefore require many CPU hours to scan over all of them.

3. Conclusions

In this paper we have pointed out that the stability of Dark Matter may be determined by a non-trivial $\mathbb{Z}_N$ center of the larger internal symmetry group G, while G is not a direct product $\mathbb{Z}_N \times G'$. Also we summarize the search of these centers of discrete groups that have faithful irreducible representations that was performed in Ref. [10]. In our survey we have found groups with non-trivial $\mathbb{Z}_N$ centers with the corresponding properties:

- $\mathbb{Z}_N$ for $N \leq 162$.
- $N = 2^p \times 3^q$ for <u>all</u> the integers p and q such that $N \leq 162$.
- $N = 2^p \times 5$ for $0 \leq p \leq 3$.
- $N = 7$, $N = 11$, $N = 14$, $N = 15$, and $N = 25$.
- The number N always divides the order O of the group. The integer O/N always has at least two prime factors: $O/N = 4$, 6, 8, 9, 10, 12, 14, and so on.

Also the relations between the integer N characterizing the center $\mathbb{Z}_N$ of each group and the identifiers of the group series defined in Ref. [14] were found and given in Ref. [10].

Listings of obtained 87,349 non-isomorphic groups that we have studied are available online at `https://github.com/jurciukonis/GAP-group-search`.

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