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Modular flavour symmetries from the bottom up

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I discuss the application of modular invariance to the flavour problem from a (mostly) bottom-up perspective. In this framework, Yukawa couplings and mass matrices are obtained from modular forms, which are functions of a single complex number: the modulus VEV τ. This VEV can be the only source of symmetry breaking, so no flavons need to be introduced. When τ is close to special values (those preserving residual symmetries), a hierarchical fermion mass spectrum can arise for certain field representations. To illustrate this mechanism, a non-fine-tuned model with hierarchical charged-lepton masses is presented. Some of these apparently ad hoc values of τ turn out to be justified in simple UV-motivated CP-invariant potentials, for which novel CP-breaking minima are found.

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1. Introduction

In recent years, modular flavour symmetries [1] have allowed to open up an interesting avenue within the flavour symmetry approach to the flavour puzzle. The intriguing hierarchies between the masses of fermions of different generations and the contrasting structures of mixing of leptons and quarks continue to defy the theorist's imagination. In an era of increasing precision, the search for models which can relate and explain the patterns of fermion masses, mixing and associated CP violation is warranted.

In the modular approach to flavour, modular symmetry constrains Yukawa couplings and therefore fermion mass matrices to be functions of a single complex scalar field, the modulus τ. In fact, the VEV of τ can be the only source of flavour and CP symmetry breaking [2]. One can therefore, from the outset, avoid the use of flavons and the complicated scalar potentials required to align them, which often call for additional "shaping" symmetries. In this context, flavour structures in the quark and lepton sectors are determined in terms of a limited number of parameters.

In the majority of phenomenologically-viable flavour models based on modular invariance, hierarchies in the charged-lepton and quark masses are obtained by fine-tuning some of the parameters. In practice, this means there is a high sensitivity of observables to model parameters or there are unjustified hierarchies between parameters introduced in the model on an equal footing. Moreover, in practically all of these models, the VEV of the modulus is treated as a free parameter which is determined by confronting model predictions with experimental data. Its value is critical for phenomenological viability and can vary significantly across models. Clearly, determining it from first principles could be used as a powerful selection criterion for flavour models.

This contribution is based on Refs. [3, 4]. In Ref. [3] a formalism is developed that allows to construct models in which fermion (charged-lepton and quark) mass hierarchies follow solely from the properties of the modular forms, avoiding fine-tuning and the need to introduce extra fields.[1] We also investigate the possibility of concurrently obtaining large mixing without fine-tuning in models of lepton flavour. Residual modular symmetries play a crucial role in this analysis. In Ref. [4] it is shown that simple UV-motivated CP-invariant potentials for the modulus have non-fine-tuned CP-breaking minima. Stabilising the modulus at these novel minima allows for natural explanations of fermion mass hierarchies via the mechanism of Ref. [3].

2. Modular symmetries as flavour symmetries

In the supersymmetric modular-invariance approach to flavour, one introduces the modulus chiral superfield τ transforming under the modular group $\Gamma \equiv SL(2, \mathbb{Z})$ via fractional linear transformations. This group is generated by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

obeying $S^2 = R$, $(ST)^3 = R^2 = \mathbb{1}$, and $RT = TR$. For a given element $\gamma \in \Gamma$, one has

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \quad \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{2}$$

[1] For approaches introducing extra (weighted) scalars, see [5, 6].

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while matter superfields ψ_i transform as weighted multiplets [1, 7, 8],

$$\psi_i \to (c\tau + d)^{-k} \rho_{ij}(\gamma)\, \psi_j \,, \tag{3}$$

where ρ is a representation of Γ. We restrict ourselves to integer modular weights k. To use modular symmetry as a flavour symmetry, one fixes a level $N \geq 2$ and assumes that $\rho(\gamma) = \mathbb{1}$ for elements of the principal congruence subgroup, $\Gamma(N) \equiv \{\gamma \in SL(2,\mathbb{Z}),\ \gamma \equiv \mathbb{1} (\text{mod } N)\}$. Hence, ρ is a unitary representation of the finite quotient $\Gamma'_N \equiv \Gamma \,/\, \Gamma(N) \simeq SL(2, \mathbb{Z}_N)$, and an "almost trivial" representation of the full modular group. In the case where matter fields transform trivially under R, ρ is effectively a representation of a smaller finite modular group $\Gamma_N \equiv \Gamma \,/ \left\langle \Gamma(N) \cup \mathbb{Z}_2^R \right\rangle$. For small N, the groups Γ_N and Γ'_N are isomorphic to permutation groups ($\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$) and to their "double covers".

Modular symmetry may then constrain the Yukawa couplings and mass structures of a model in a predictive way. By requiring the invariance of the superpotential under modular transformations, one finds that couplings $Y_{I_1 \dots I_n}(\tau)$ appearing in terms of the type $\psi_{I_1} \dots \psi_{I_n}$ must be special holomorphic functions of τ — they are modular forms of level N — obeying

$$Y_{I_1 \dots I_n}(\tau) \xrightarrow{\gamma} Y_{I_1 \dots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma)\, Y_{I_1 \dots I_n}(\tau) \,. \tag{4}$$

Modular forms carry weights $k_Y = k_{I_1} + \dots + k_{I_n}$ and furnish unitary representations ρ_Y of the finite modular group such that $\rho_Y \otimes \rho_{I_1} \otimes \dots \otimes \rho_{I_n} \supset \mathbf{1}$. Crucially, non-trivial modular forms span finite-dimensional linear spaces. These have relatively low dimensionalities for small values of k and N, leading to a predictive setup in which only a restricted number of τ-dependent Yukawa textures are allowed in the superpotential.

2.1 Residual symmetries

The VEV of τ is restricted to the upper half-plane and plays the role of a spurion, parameterising modular symmetry breaking. There is no value of the modulus VEV preserving the full symmetry group. However, at so-called symmetric points $\tau = \tau_{\text{sym}}$ the modular group is only partially broken, and unbroken generators give rise to residual symmetries.[2] The fundamental domain $\mathcal{D}$ and symmetric points of the modular group are shown in Figure 1. There are only three inequivalent symmetric points [9],

- $\tau_{\text{sym}} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$;
- $\tau_{\text{sym}} = i$, invariant under S, preserving $\mathbb{Z}_4^S$ (note that $S^2 = R$); and
- $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3)$, 'the left cusp', invariant under ST, preserving $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$.

Note that, in the absence of flavons, the value of τ can always be restricted to the fundamental domain $\mathcal{D}$ of the modular group Γ (see e.g. Ref. [10]).

In a CP- and modular-invariant theory [2, 10], an additional $\mathbb{Z}_2^{\text{CP}}$ symmetry is preserved for $\operatorname{Re}\tau = 0$ or for τ on the border of $\mathcal{D}$, while is broken at generic values of τ. All three symmetric values listed above preserve the CP symmetry.

[2] The R generator is unbroken for any value of τ, so that a $\mathbb{Z}_2^R$ symmetry is always preserved.

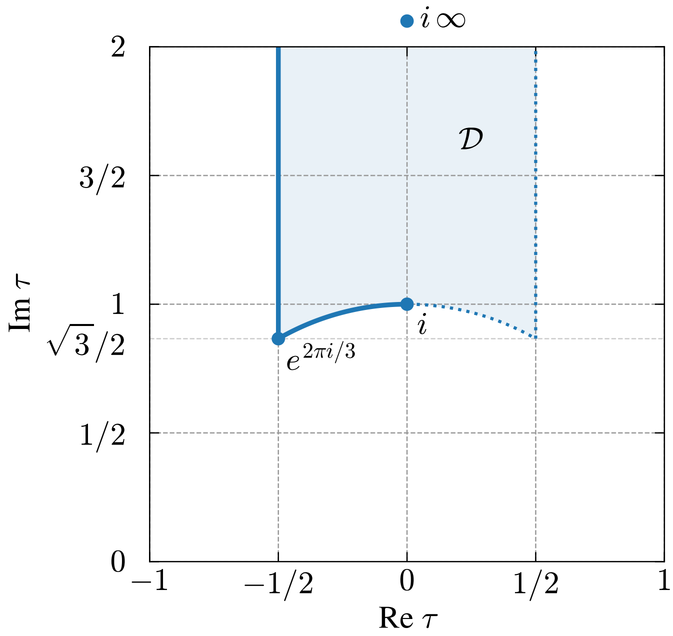

Figure 1: The fundamental domain $\mathcal{D}$ of the modular group Γ and its three symmetric points (from [10]).

3. Mass hierarchies without fine-tuning

3.1 Mass matrices close to symmetric points

At a symmetric point, flavour textures can be severely constrained by the residual symmetry group, which may enforce the presence of multiple zero entries in the mass matrices. As τ moves away from its symmetric value, these entries will generically become non-zero. The magnitudes of such (residual-)symmetry-breaking entries are controlled by the size of the departure ϵ of τ from τ_{sym} and by the field transformation properties under the residual symmetry group, which may depend on modular weights [3].

Consider a modular-invariant bilinear $\psi_i^c\, M(\tau)_{ij}\, \psi_j$, where the superfields ψ and ψ^c transform under the modular group as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma)\, \psi\,, \qquad \psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma)\, \psi^c\,, \tag{5}$$

so that each $M(\tau)_{ij}$ is a modular form of level N and weight $K \equiv k + k^c$. Modular invariance requires $M(\tau)$ to transform as

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger\,. \tag{6}$$

Taking τ to be close to the symmetric point, and setting γ to the residual symmetry generator, one can use this transformation rule to constrain the form of the mass matrix. Consider, for instance, the T-diagonal representation basis for group generators, in which $\rho^{(c)}(T) = \text{diag}(\rho_i^{(c)})$, and take τ 'close' to $\tau_{\text{sym}} = i\infty$, i.e. large enough $\text{Im}\,\tau$. By setting $\gamma = T$ in eq. (6), one finds

$$M_{ij}(T\tau) = \left(\rho_i^c \rho_j\right)^* M_{ij}(\tau)\,. \tag{7}$$

It is convenient to treat the M_{ij} as a function of $q \equiv \exp(2\pi i\tau/N)$, so that $|\epsilon| \equiv |q| = e^{-2\pi \operatorname{Im}\tau/N}$ parameterises the deviation of τ from the symmetric point. Note that the entries $M_{ij}(q)$ depend analytically on q and that $q \xrightarrow{T} \zeta q$, with $\zeta \equiv \exp(2\pi i/N)$. Expanding (7) in powers of q, one finds

$$\zeta^n M_{ij}^{(n)}(0) = (\rho_i^c \rho_j)^* M_{ij}^{(n)}(0)\,, \tag{8}$$

where $M_{ij}^{(n)}$ denotes the n-th derivative of M_{ij} with respect to q. It follows that $M_{ij}^{(n)}(0)$ can only be non-zero for values of n such that $(\rho_i^c \rho_j)^* = \zeta^n$. Also, in the symmetric limit $q \to 0$ the entry $M_{ij} = M_{ij}^{(0)}$ is only allowed to be non-zero if $\rho_i^c \rho_j = 1$. More generally, if $(\rho_i^c \rho_j)^* = \zeta^l$ with $0 \le l < N$,

$$M_{ij}(q) = a_0\, q^l + a_1\, q^{N+l} + a_2\, q^{2N+l} + \ldots \tag{9}$$

in the vicinity of the symmetric point. It crucially follows that the entry M_{ij} is expected to be $O(|\epsilon|^l)$ whenever $\operatorname{Im}\tau$ is large. The power l only depends on how the representations of ψ and ψ^c decompose under the residual symmetry group $\mathbb{Z}_N^T$. A similar analysis can be carried out for $\tau_{\text{sym}} = i$ with $|\epsilon| = |(\tau - i)/(\tau + i)|$, and for $\tau_{\text{sym}} = \omega$ with $|\epsilon| = |u|$ and $u \equiv (\tau - \omega)/(\tau - \omega^2)|$ [3].

The important takeaway message is that as τ departs from a symmetric value τ_{sym} — with ϵ parameterising the deviation — the zero entries of fermion mass matrices become $O(|\epsilon|^l)$. The exponents l are extracted from products of factors which correspond to representations of the residual symmetry group (see Ref. [3] for further detail).

Matter fields ψ furnish 'weighted' representations $(\mathbf{r}, k)$ of the finite modular group Γ'_N. Whenever a residual symmetry is preserved by τ, fields decompose into unitary representations of the residual symmetry group. Modulo a possible $\mathbb{Z}_2^R$ factor, these are the cyclic groups $\mathbb{Z}_N^T$, $\mathbb{Z}_2^S$, and $\mathbb{Z}_3^{ST}$ (cf. section 2.1). To illustrate the decomposition of representations at symmetric points, take as an example a $(\mathbf{3}, k)$ triplet ψ of S'_4. It transforms under the unbroken $\gamma = ST$ at $\tau = \omega$ as

$$\psi_i \xrightarrow{ST} (-\omega - 1)^{-k}\, \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j = \omega^k \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j\,. \tag{10}$$

One can check that the eigenvalues of $\rho_{\mathbf{3}}(ST)$ are 1, ω and ω^2, and so in a suitable (ST-diagonal) basis the transformation rule explicitly reads

$$\psi \xrightarrow{ST} \omega^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \psi = \begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{k+1} & 0 \\ 0 & 0 & \omega^{k+2} \end{pmatrix} \psi\,, \tag{11}$$

which means that ψ decomposes as $\psi \rightsquigarrow \mathbf{1}_k \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+2}$ under the residual $\mathbb{Z}_3^{ST}$. One can similarly find the residual symmetry representations for any other 'weighted' multiplet. The decompositions of the weighted representations of Γ'_N ($N \le 5$) under the three residual symmetry groups have been collected in Appendix A of Ref. [3].

It can be shown that for $\tau \simeq i\infty$ the phase factors ρ_i correspond to the $\mathbb{Z}_N^T$ irreps into which ψ decomposes and that the product $(\rho_i^c \rho_j)^*$ indeed matches some power ζ^l with $0 \le l < N$, as tacitly assumed above. For $\tau \simeq \omega$ the relevant products match ω^l with $l = 0, 1, 2$. Finally, due to the fact that $M(\tau)_{ij}$ is R-even, the fields ψ_i^c and ψ_j need to carry the same R-parity. It then follows that for $\tau \simeq i$ the relevant phase factor products are restricted to ± 1. In practice, this means that the degree of suppression of mass matrix elements is given by $|\epsilon|^l$ where l can take the values $l = 0, 1, \ldots, N-1$ if $\operatorname{Im}\tau$ is large; $l = 0, 1, 2$ if $\tau \simeq \tau_{\text{sym}} = \omega$; or $l = 0, 1$ if $\tau \simeq \tau_{\text{sym}} = i$. Our results are summarised in Table 1.

τ_{sym}	Residual sym. group (ρ_{sym})	$\lvert\epsilon\rvert$	Possible powers $\lvert\epsilon\rvert^l$	How to find leading $\lvert\epsilon\rvert^l$
i	$\mathbb{Z}_2$ (S)	$\left\lvert\frac{\tau-i}{\tau+i}\right\rvert$	$l=0,1$	$\tilde{\rho}_i^c\tilde{\rho}_j \overset{?}{=} (-1)^l$
ω	$\mathbb{Z}_3$ (ST)	$\lvert u\rvert \equiv \left\lvert\frac{\tau-\omega}{\tau-\omega^2}\right\rvert$	$l=0,1,2$	$\tilde{\rho}_i^c\tilde{\rho}_j \overset{?}{=} \omega^l$
$i\infty$	$\mathbb{Z}_N$ (T)	$\lvert q\rvert$	$l=0,1,\ldots,N$	$(\rho_i^c\rho_j)^* \overset{?}{=} \zeta^l$

Table 1: Summary of possible powers of $|\epsilon|$ appearing in the mass matrix, in an appropriate ρ_{sym}-diagonal basis, and how to determine them for each (i,j). Here, $\overset{(\sim)}{\rho_i}$ and $\overset{(\sim)}{\rho}{}_j^c$ are the decompositions of ψ and ψ^c under the residual symmetry group. In practice, one may disregard $\mathbb{Z}_2^R$ factors here (cf. Appendix A of [3]).

3.2 Hierarchical structures

The results found so far allow us to construct hierarchical mass matrices in the vicinity of a symmetric point. As an example, consider a model at level $N=5$ with large $\operatorname{Im}\tau$ and matter fields $\psi \sim (\mathbf{3},k)$ and $\psi^c \sim (\mathbf{3}',k^c)$. One has the decompositions $\psi \rightsquigarrow 1_0\oplus\mathbf{1}_1\oplus\mathbf{1}_4$ and $\psi^c \rightsquigarrow 1_0\oplus\mathbf{1}_2\oplus\mathbf{1}_3$ under the residual group at the symmetric point $\tau_{\text{sym}}=i\infty$. One can then identify $\rho_i=\operatorname{diag}(1,\zeta,\zeta^4)$ and $\rho_i^c=\operatorname{diag}(1,\zeta^2,\zeta^3)$, with $\zeta=\exp(2\pi i/5)$, and derive the power structure

$$M(\tau(\epsilon)) \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix}, \quad \text{with } \epsilon = e^{-2\pi \operatorname{Im}\tau/5}, \tag{12}$$

which corresponds to a hierarchical $(1,\epsilon,\epsilon^4)$ spectrum. .

Note that $K=k+k^c$ must be large enough that sufficient modular forms contribute to $M(\tau)$. For instance, for $K=2$ the superpotential may turn out to include a unique contribution:

$$W \supset \sum_s \alpha_s \left(Y_{\mathbf{5}}^{(5,2)}(\tau)\psi^c\psi\right)_{\mathbf{1},s} \quad\Rightarrow\quad M(\tau)=\alpha\begin{pmatrix} \sqrt{3}Y_1 & Y_5 & Y_2 \\ Y_4 & -\sqrt{2}Y_3 & -\sqrt{2}Y_5 \\ Y_3 & -\sqrt{2}Y_2 & -\sqrt{2}Y_4 \end{pmatrix}_{Y_{\mathbf{5}}^{(5,2)}}, \tag{13}$$

where the α_s are coupling constants, the sum is taken over all possible singlets s and $Y_{\mathbf{r}(,\mu)}^{(N,K)}$ denotes the modular form multiplet of level N, weight K and irrep $\mathbf{r}$, with μ possibly labelling linearly independent multiplets of the same type (Y_i are the corresponding components). At leading order in $\epsilon=|q|$, one has $(Y_1,Y_2,Y_3,Y_4,Y_5)\simeq\left(-1/\sqrt{6},\, q,\, 3q^2,\, 4q^3,\, 7q^4\right)$ up to normalisation and the power structure indeed matches that of eq. (12). However, one can check that the determinant of M vanishes identically for any τ and the spectrum is $\sim(1,\epsilon,0)$, with one massless fermion. This issue is solved at weight $K=4$. Then, the multiplets $Y_{\mathbf{4}}^{(5,4)}$, $Y_{\mathbf{5},1}^{(5,4)}$, and $Y_{\mathbf{5},2}^{(5,4)}$ are available and the spectrum is indeed of the type $(1,\epsilon,\epsilon^4)$ without a massless fermion. While the ϵ power-counting in eq. (13) may resemble that of a Froggatt-Nielsen mechanism [11], our framework is unrelated and can be regarded as an improvement. Instead of having an unknown $O(1)$ coefficient for each mass matrix entry, entries depend only on τ and a limited number of superpotential parameters.

We are interested in identifying all possible 3×3 hierarchical mass matrices arising from the described mechanism for $N\leq5$. We scan over representations $\mathbf{r}$ and $\mathbf{r}^c$, rejecting spectra with

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$
			$\tau \simeq i\infty$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$ with $\mathbf{1}_a \neq (\mathbf{1}_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes [\mathbf{2} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}']$, $\mathbf{3}' \otimes [\mathbf{2} \oplus \mathbf{1}'$, or $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}']$, $\hat{\mathbf{3}}' \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}']$, $\hat{\mathbf{3}} \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

Table 2: Hierarchical mass patterns which can be realised in the vicinity of symmetric points. Subscripts run over irreps of a certain dimension, and $\mathbf{1}'''_a = \mathbf{1}_a$ for $N = 3$, while $\mathbf{1}''_a = \mathbf{1}_a$ for $N = 4$.

massless fermions. Note that, in the reducible case, we assume a common weight (and a common $\rho(R)$) to be shared across the decomposition. For $\tau \simeq i$ the hierarchical pattern cannot be produced solely as a consequence of the smallness of ϵ, since mass matrix entries are either $O(1)$ or $O(\epsilon)$. The full results of the scan are given in Appendix B of Ref. [3]. It is only possible to obtain *hierarchical* spectra for a small list of representation pairs, the most promising of which are collected in Table 2. We have excluded from this summary table reducible representations made up of three copies of the same singlet, as in those cases the number of superpotential parameters is unappealingly high.

4. Charged-lepton masses and large lepton mixing without fine-tuning

4.1 Viable PMNS matrix in the symmetric limit

Inspired by the above results, we have searched and built viable and predictive S'_4 and A'_5 lepton flavour models, marked in blue in Table 2, see section 3.4 of [3]. In these models, the slightly-broken residual symmetry allows to successfully produce hierarchical charged-lepton masses without tuning the corresponding couplings. However, tuning is still present in the neutrino sector, as residual symmetries constrain the PMNS matrix, forcing some of its entries to be zero. It is known that only a limited number of flavour symmetry representation choices for lepton fields L and E^c may give rise to a viable PMNS matrix in the symmetric limit [12]. Viability in our case means that either none of its entries vanishes, or only the (13) entry vanishes as $\epsilon \to 0$. A modular-symmetric model of lepton flavour with hierarchical charged-lepton masses may be free of fine-tuning if it satisfies any of the properties [3]:

1. $L \rightsquigarrow 1 \oplus 1 \oplus 1$, $E^c \rightsquigarrow 1 \oplus r$, where 1 is some real singlet and r is some (possibly reducible) representation such that $r \not\supset 1$;

2. $L \rightsquigarrow \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}^*$, $E^c \rightsquigarrow \mathbf{1}^* \oplus r$, where $\mathbf{1}$ is some complex singlet, $\mathbf{1}^*$ is its conjugate, and r is some (possibly reducible) representation such that $r \not\supset \mathbf{1}, \mathbf{1}^*$.

3. all charged-lepton masses vanish in the symmetric limit;

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}_{E^c} \otimes \mathbf{r}_L$	Property
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$	1 or 4
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$	2
			$\tau \simeq i\infty$	$[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'] \otimes [\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}']$, $[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}''] \otimes [\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'']$	2
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

Table 3: Hierarchical charged-lepton mass patterns which may be realised in the vicinity of symmetric points without fine-tuned mixing (PMNS close to the observed one in the symmetric limit).

4. all light neutrino masses vanish in the symmetric limit.

Applying this filter to the promising hierarchical cases of Table 2, one is left with the representation pairs of Table 3. Note there is no surviving possibility for $A_5^{(\prime)}$.

4.2 Scan of predictive S'_4 models with $\tau \simeq \omega$

Finally, we consider the most structured surviving cases within Table 3. These arise for S'_4, $\tau \simeq \omega$ and E^c and L being a triplet and the direct sum of three singlets, respectively. The expected charged-lepton spectrum is $(1, \epsilon, \epsilon^2)$. We have performed a systematic scan restricting ourselves to promising models involving the minimal number of effective parameters (9, including $\mathrm{Re}\,\tau$ and $\mathrm{Im}\,\tau$). Right-handed neutrino fields N^c are present since Weinberg dimension-5 operator models require more parameters. Aiming at minimal and predictive models, we impose a generalised CP symmetry enforcing the reality of coupling constants [2]. Out of 48 models, we have identified the only one which is viable and not fine-tuned, and is consistent with the 2σ range for the Dirac CPV phase, predicting $\delta \simeq \pi$. For this model, $L = L_1 \oplus L_2 \oplus L_3$ with $L_1, L_2 \sim (\hat{\mathbf{1}}, 2)$, $L_3 \sim (\hat{\mathbf{1}}', 2)$, $E^c \sim (\hat{\mathbf{3}}, 4)$ and $N^c \sim (\mathbf{3}', 1)$. The corresponding superpotential reads:

$$\begin{aligned} W = &\Big[\alpha_1 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_1\right)_{\mathbf{1}} + \alpha_2 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_1\right)_{\mathbf{1}} + \alpha_3 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_2\right)_{\mathbf{1}} + \alpha_4 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_2\right)_{\mathbf{1}} + \alpha_5 \left(Y^{(4,6)}_{\mathbf{3}} E^c L_3\right)_{\mathbf{1}}\Big] H_d \\ &+ \Big[g_1 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_1\right)_{\mathbf{1}} + g_2 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_2\right)_{\mathbf{1}} + g_3 \left(Y^{(4,3)}_{\hat{\mathbf{3}}'} N^c L_3\right)_{\mathbf{1}}\Big] H_u + \Lambda \left(Y^{(4,2)}_{\mathbf{2}} (N^c)^2\right)_{\mathbf{1}} . \end{aligned} \tag{14}$$

Since L_1 and L_2 are indistinguishable, one can set $\alpha_2 = 0$ without loss of generality.

At leading order in a small parameter $|\epsilon|$, with $\epsilon \equiv 1 - \frac{1+\sqrt{3}}{1-i} \frac{\varepsilon}{\theta}$ and $|\epsilon| \simeq 2.8 \left|\frac{\tau-\omega}{\tau-\omega^2}\right|$ in the context of this section,[3] the charged-lepton mass matrix reads

$$M_e^\dagger \simeq -\frac{3(\sqrt{3}-1)^6}{\sqrt{13}} v_d \alpha_1 \theta^{12} \begin{pmatrix} 1 & \tilde{\alpha}_3 + \frac{\sqrt{13}}{2}\tilde{\alpha}_4 & \frac{i\sqrt{39}}{2}\tilde{\alpha}_5 \\ \sqrt{3}\,\epsilon & \sqrt{3}\left(\tilde{\alpha}_3 - \frac{\sqrt{13}}{2}\tilde{\alpha}_4\right)\epsilon & \frac{i\sqrt{13}}{2}\tilde{\alpha}_5\,\epsilon \\ \frac{5}{2}\,\epsilon^2 & \frac{1}{4}\left(10\tilde{\alpha}_3 + \sqrt{13}\tilde{\alpha}_4\right)\epsilon^2 & -\frac{5i\sqrt{13}}{4\sqrt{3}}\tilde{\alpha}_5\,\epsilon^2 \end{pmatrix}, \tag{15}$$

[3] This local definition is motivated by the fact that $\varepsilon/\theta = (1-i)/(1+\sqrt{3})$ at $\tau = \omega$, with ε, θ defined in Ref. [10].

while the charged-lepton mass ratios follow the expected ϵ-pattern and are given by

$$\frac{m_e}{m_\mu} \simeq 2\frac{|\tilde{\alpha}_4\tilde{\alpha}_5|\sqrt{4+\left(2\tilde{\alpha}_3+\sqrt{13}\tilde{\alpha}_4\right)^2+39\tilde{\alpha}_5^2}}{3\tilde{\alpha}_4^2+\left[1+\left(\tilde{\alpha}_3-\sqrt{13}\tilde{\alpha}_4\right)^2\right]\tilde{\alpha}_5^2}\,|\epsilon|\,, \quad \frac{m_\mu}{m_\tau} \simeq 4\sqrt{13}\frac{\sqrt{3\tilde{\alpha}_4^2+\left[1+\left(\tilde{\alpha}_3-\sqrt{13}\tilde{\alpha}_4\right)^2\right]\tilde{\alpha}_5^2}}{4+\left(2\tilde{\alpha}_3+\sqrt{13}\tilde{\alpha}_4\right)^2+39\tilde{\alpha}_5^2}\,|\epsilon|\,, \tag{16}$$

with $\tilde{\alpha}_i \equiv \alpha_i/\alpha_1$. Up to an overall normalisation $\mathcal{K}$, the light neutrino mass matrix is given by

$$M_\nu \simeq \mathcal{K}\,\epsilon \begin{pmatrix} 0 & 0 & \tilde{g}_3 \\ 0 & 0 & \tilde{g}_2\tilde{g}_3 \\ \tilde{g}_3 & \tilde{g}_2\tilde{g}_3 & 2i\sqrt{\frac{2}{3}}\tilde{g}_3^2 \end{pmatrix} \tag{17}$$

at leading order in $|\epsilon|$, where $\tilde{g}_i \equiv g_i/g_1$. The smallness of $|\epsilon|$ does not constrain the M_ν contribution to mixing, which depends only on the g_i, and large mixing angles are allowed. Note that there is a massless neutrino even though N^c is a triplet. The fit of the model yields ($N\sigma \simeq 0.563$):

$$\begin{aligned}
&\frac{m_e}{m_\mu} = 0.00475^{+0.00061}_{-0.00052}\,, \quad \frac{m_\mu}{m_\tau} = 0.0556^{+0.0136}_{-0.0116}\,, \quad \Sigma m_\nu = 0.0588^{+0.0002}_{-0.0002}\text{ eV}\,,\\
&\delta m^2 = 7.38^{+0.35}_{-0.44}\times 10^{-5}\text{ eV}^2\,, \quad |\Delta m^2| = 2.48^{+0.05}_{-0.04}\times 10^{-3}\text{ eV}^2\,, \quad r = 0.0298^{+0.00196}_{-0.0023}\,,\\
&\sin^2\theta_{12} = 0.304^{+0.039}_{-0.036}\,, \quad \sin^2\theta_{13} = 0.0221^{+0.0019}_{-0.002}\,, \quad \sin^2\theta_{23} = 0.539^{+0.0522}_{-0.099}\,,\\
&m_{\beta\beta} = 0.00144^{+0.00035}_{-0.00033}\text{ eV}\,, \quad \frac{\delta}{\pi} = 1\pm\mathcal{O}(10^{-6})\,, \quad \frac{\alpha}{\pi} = 1\pm\mathcal{O}(10^{-5})\,.
\end{aligned} \tag{18}$$

The viable region in the τ plane corresponds to a neutrino spectrum with NO and is located very close to $\tau_{\text{sym}} = \omega$, as can be seen from Figure 2. The annular form of the region is explained by the fact that the phase of $(\tau-\omega)$ has no effect on the observables, as it enters only through ϵ and its effects are suppressed by the smallness of $|\epsilon|$. Therefore, in the regime $\tau \simeq \omega$ this model is effectively described by 8 rather than 9 parameters:

$$\begin{aligned}
&|\epsilon(\tau)| = 0.0186^{+0.0028}_{-0.0023}\,, \quad \tilde{\alpha}_3 = 2.45^{+0.44}_{-0.42}\,, \quad \tilde{\alpha}_4 = -2.37^{+0.36}_{-0.30}\,, \quad \tilde{\alpha}_5 = 1.01^{+0.06}_{-0.06}\,,\\
&\tilde{g}_2 = 1.5^{+0.15}_{-0.14}\,, \quad \tilde{g}_3 = 2.22^{+0.17}_{-0.15}\,, \quad v_d\,\alpha_1 = 4.61^{+1.32}_{-1.33}\text{ GeV}\,, \quad \frac{v_u^2\,g_1}{\Lambda} = 0.268^{+0.057}_{-0.063}\text{ eV}\,.
\end{aligned} \tag{19}$$

5. Modulus stabilisation

For the model of the previous section, data requires τ to have a value near the cusp $\tau_{\text{sym}} = \omega$. The viable region for τ is a small ring around the cusp (see Fig. 2) of radius $|u| \simeq 0.007$, with $u \equiv (\tau-\omega)/(\tau-\omega^2)$ and has a best-fit point at

$$\tau \simeq -0.496 + 0.877\,i\,. \tag{20}$$

The question to address is whether such data-driven values of τ are ad hoc or can be naturally justified by a dynamical principle, from a top-down perspective.[4] In what follows, we analyse known and relatively simple supergravity-motivated modular- and CP-invariant potentials for the modulus τ. Their q- and u-expansions have been used to show the existence of new global CP-breaking minima.

[4]For other attempts to determine the value of τ on the basis of dynamical considerations see, e.g., [13–16].

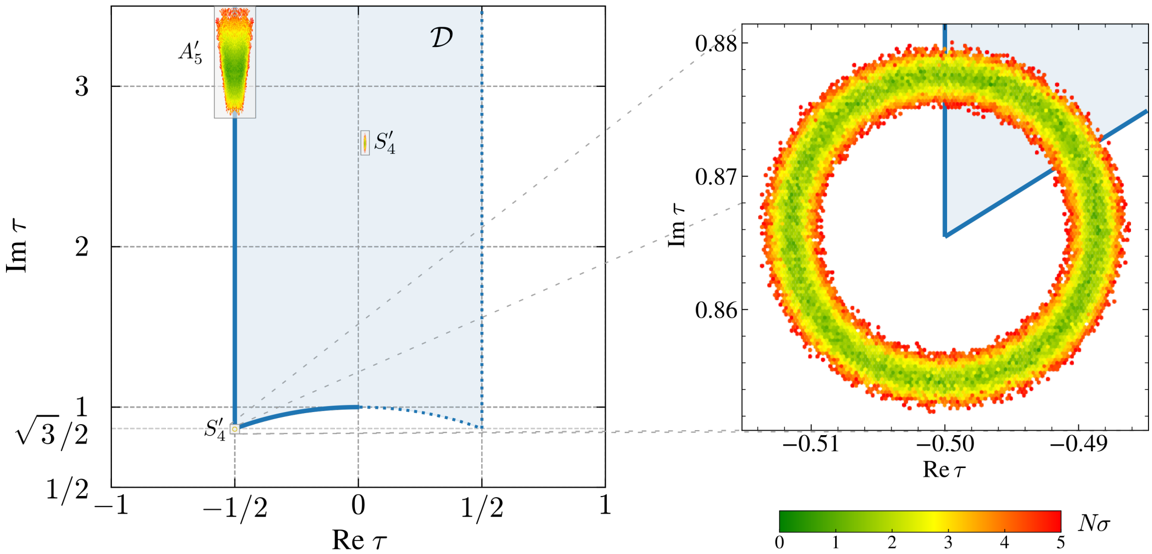

Figure 2: Allowed regions in the τ plane for the viable S'_4 and A'_5 lepton flavour models of section 3.4 of [3] and for the S'_4 model discussed here (left). The region corresponding to the latter is magnified (right).

5.1 Scalar potential

We focus on a known class of modular-invariant potentials which are functions of τ alone and independent of the level N [17, 18]. These simplified models are nevertheless explicit examples of $\mathcal{N} = 1$ supergravity models.[5] We consider the Kähler potential

$$K(\tau, \overline{\tau}) = -\Lambda_K^2 \log(2 \operatorname{Im} \tau)\,, \tag{21}$$

where Λ_K is a scale (mass dimension one). The relevant action depends on the Kähler-invariant

$$G(\tau, \overline{\tau}) \;=\; \kappa^2 K(\tau, \overline{\tau}) + \log\left|\kappa^3 W(\tau)\right|^2\,, \tag{22}$$

where $\kappa^2 = 8\pi/M_P^2$, M_P being the Planck mass. Modular invariance of G implies that the superpotential W carries modular weight $-\mathfrak{n}$, where $\mathfrak{n} = \kappa^2\Lambda_K^2$. We consider integer values of $\mathfrak{n}$, in line with [17, 18]. The superpotential can then be parameterised in terms of the Dedekind η function and a modular-invariant function H, as

$$W(\tau) \;=\; \Lambda_W^3\, \frac{H(\tau)}{\eta(\tau)^{2\mathfrak{n}}}\,, \tag{23}$$

where Λ_W is a mass scale so that $H(\tau)$ is dimensionless. The most general H without singularities in the fundamental domain can be cast in the form [17]:

$$H(\tau) \;=\; (j(\tau) - 1728)^{m/2}\, j(\tau)^{n/3}\, \mathcal{P}\left(j(\tau)\right)\,, \tag{24}$$

making use of the Klein j function. Here, m and n are non-negative integers and $\mathcal{P}$ is a polynomial in $j(\tau)$. The scalar potential follows from the explicit forms of $K(\tau, \overline{\tau})$ and $W(\tau)$ given above,

$$V(\tau, \overline{\tau}) = \frac{\Lambda_V^4}{(2 \operatorname{Im} \tau)^{\mathfrak{n}} |\eta(\tau)|^{4\mathfrak{n}}} \left[\left| iH'(\tau) + \frac{\mathfrak{n}}{2\pi} H(\tau) \hat{G}_2(\tau, \overline{\tau}) \right|^2 \frac{(2 \operatorname{Im} \tau)^2}{\mathfrak{n}} - 3|H(\tau)|^2 \right]\,, \tag{25}$$

[5] A fully realistic string compactification is expected to involve other moduli, as well as gauge and matter fields [18].

where we have defined $\Lambda_V = \left(\kappa^2 \Lambda_W^6\right)^{1/4}$ and $\hat{G}_2$ is the non-holomorphic Eisenstein function of weight 2, given by $\hat{G}_2(\tau,\overline{\tau}) = G_2(\tau) - \pi/\mathrm{Im}\,\tau$ G_2 is its holomorphic counterpart, which can be related to the Dedekind function via

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi} G_2(\tau)\,. \tag{26}$$

It is not difficult to show that the potential $V(\tau,\bar{\tau})$ is modular-invariant.

We are interested in the simple cases investigated in Refs. [17, 18], for which $\mathfrak{n} = 3$ corresponds to the number of compactified complex dimensions and the simplest choice $\mathcal{P}(j) = 1$ is taken. The scalar potential then reads:

$$V(\tau,\overline{\tau}) = \frac{\Lambda_V^4}{8(\mathrm{Im}\,\tau)^3|\eta|^{12}}\left[\frac{4}{3}\left|iH' + \frac{3}{2\pi}H\hat{G}_2\right|^2 (\mathrm{Im}\,\tau)^2 - 3|H|^2\right]\,, \tag{27}$$

and is CP-symmetric for $\mathcal{P}(j) = 1$, i.e., invariant under a reflection with respect to the imaginary axis, $\tau \to -\overline{\tau}$. We consider the form of $H(\tau)$ given in Eq. (24) for different values of m and n.

5.2 q- and u- expansions

To analyse the potential of Eq. (27), we express the functions j (and therefore H, H') and $\hat{G}_2$ in terms of the Dedekind η and its derivatives. It then suffices to know the q-expansion of η, i.e., its expansion in powers of $q = e^{2\pi i\tau}$, up to a certain order (note the different definition of q). This expansion has the well-known form

$$\eta = q^{1/24}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}} = q^{1/24}\left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + O(q^{22})\right) \tag{28}$$

and converges rapidly within the fundamental domain, where $|q| \le e^{-\sqrt{3}\pi} \simeq 0.004$.

To guarantee a robust numerical analysis near the cusp, we also develop an expansion of the potential in terms of the parameter u. Using $\tau = \omega^2(\omega^2 - u)/(1-u)$, one can write η as a function of u. It further proves useful to define $\tilde{\eta}(u) = (1-u)^{-1/2}\eta(u)$, since symmetry dictates $\tilde{\eta}$ to be a power series in u^3. We obtain

$$\tilde{\eta}(u) \simeq e^{-i\pi/24}\left(0.800579 - 0.573569u^3 - 0.780766u^6 - 0.150007u^9\right) + O(u^{12})\,. \tag{29}$$

5.3 Global minima for various m, n

We now look into the global minima of the potential V of Eq. (27). In [17], it was conjectured that all extrema of V would lie at CP-conserving values of τ. Therein, the cases $(m,n) = (0,0), (1,1), (0,3)$ were explicitly examined. While we have verified these particular results, we have further found that the potentials $V_{m,n}$ with $n = 0$ but $m > 0$ do allow for CP-breaking global minima, located in the vicinity of the cusps. Our results are summarised in Fig. 3. The minima fall into several classes depending on values of m and n:

(0, 0) is a single minimum at $\tau \simeq 1.2i$ on the imaginary axis, corresponding to the case $m = n = 0$;

(0, *n*) is a single minimum at the symmetric point $\tau = i$ attained when $m = 0$, $n \neq 0$;

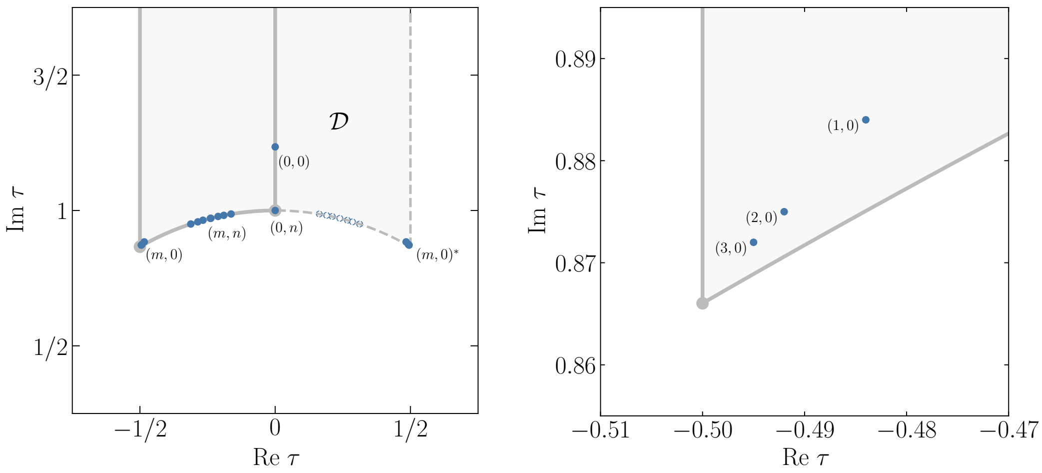

Figure 3: Global minima of the potentials $V(\tau,\bar{\tau})$, Eq. (27), see text for details. Note that points on the right half of the unit arc, which are CP-conjugates of the (m,n) minima, are excluded as they lie outside the fundamental domain. The right panel shows the series $(m,0)$ in the vicinity of the left cusp in more detail.

$(m,0)$ and $(m,0)^*$ are a pair of degenerate minima for each $m \neq 0$ and $n = 0$: $(m,0)$ is located in the vicinity of the left cusp, approaching it as m increases, while $(m,0)^*$ is its CP-conjugate;

(m,n) is a series of CP-conserving minima on the unit arc, corresponding to $m \neq 0$, $n \neq 0$; these minima shift towards $\tau = \omega$ ($\tau = i$) along the arc as m (n) grows.

The $(m,0)^{(*)}$ minima slightly depart from the left (right) cusp symmetric point and the boundary, and may explain both CP violation and hierarchical mass patterns in an economical way.[6]

Since the $(m,0)^*$ minima are trivially related to the $(m,0)$ minima via CP reflection, in what follows we concentrate on the latter series only, i.e., we study the behaviour of $V_{m,0}$ in the vicinity of the left cusp. $V_{m,0}$ expands in powers of $|u|$, with coefficients possibly depending on the phase of u. Denoting this phase as ϕ, i.e., $u = |u|e^{i\phi}$, $\phi \in [-\pi/3, 0]$, we find:

$$V_{m,0} = \Lambda_V^4 \, \frac{1728^m}{\sqrt{3}\,\tilde{\eta}_0^{12}} \left\{-1 - 2\,|u|^2 + \left(A_m^2 - 3\right)|u|^4\right\} + O(|u|^6)\,, \tag{30}$$

where $A_m \simeq 68.78\,m + 4.30$ and $\tilde{\eta}_0^{12} \simeq 0.800579$. Apart from the overall scale, the potential $V_{m,0}$ in Eq. (30) depends on only one parameter — m, which takes positive integer values. One can check that the quartic term coefficient $(A_m^2 - 3)$ is positive for any $m \geq 1$, so up to $O(|u|^6)$ the potential has the well-known Mexican-hat profile. This clearly indicates that the cusp $\tau = \omega \leftrightarrow |u| = 0$ is not the minimum. Instead, it is a local maximum. The true minimum is attained at

$$|u|_{\min} \simeq (A_m^2 - 3)^{-1/2} \simeq A_m^{-1} = \frac{0.0145}{m + 0.0625}\,. \tag{31}$$

This equation yields a series $|\epsilon| \simeq 2.8|u|_{\min} \simeq 0.0383, 0.0197, 0.0133, \ldots$ for $m = 1, 2, 3, \ldots$. Quite remarkably, by choosing $m = 2 \leftrightarrow |\epsilon| \simeq 0.0197$ one finds a value of $|\epsilon|$ within the allowed

[6]Moreover, we have verified numerically for several values of $m \neq 0$ ($n = 0$) that the quantity $h^T = \exp(G/2)G^T$, where the superscript denotes derivation with respect to $T = -i\tau$, has a non-zero VEV at the new found minima, indicating that, for these potentials, SUSY is broken spontaneously by the modulus F-term VEV.

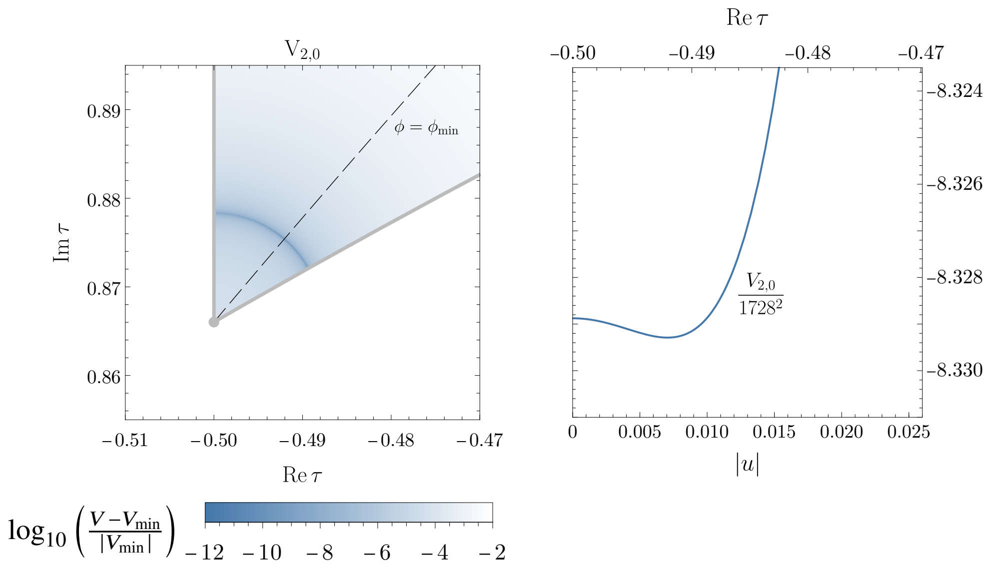

Figure 4: Potential $V_{2,0}(\tau,\bar{\tau})$ in the vicinity of the cusp (left panel) and its 1-dimensional projection onto the curve $\phi = \phi_{\min}$ (right panel), in units of Λ_V^4.

range of the model of the previous section. In short, the potential $V_{m,0}$ completes the non-fine-tuned model of Ref. [3] by providing a dynamical origin of the smallness of the deviation of τ from its symmetric value.

So far we have not discussed the phase of u at the minimum, $\phi_{\min}$. It may seem from Eq. (30) that the potential is independent of ϕ, thus having a flat direction. However, expanding $V_{m,0}$ to higher orders in $|u|$ reveals a mild dependence on ϕ. Up to an overall factor, we have

$$\begin{aligned} V_{m,0} \propto -1 - 2\,|u|^2 + \left(A_m^2 - 3\right)|u|^4 + \left(-4 + 2A_m^2 + B_m^2 \cos 6\phi\right)|u|^6 \\ + 2A_m B_m^2 \cos 3\phi\,|u|^7 + \left(-5 + 3A_m^2 + 2B_m^2 \cos 6\phi\right)|u|^8 + O(|u|^9)\,, \end{aligned} \tag{32}$$

where $B_m^2 \simeq 4730.60\,m^2 - 2069.73\,m + 33.26$. The ϕ-dependent contribution to the potential is dominated by

$$B_m^2 \cos 6\phi\,|u|^6 + 2A_m B_m^2 \cos 3\phi\,|u|^7 \propto \cos 6\phi + 2A_m|u| \cos 3\phi \simeq \cos 6\phi + 2\cos 3\phi \tag{33}$$

at $|u| = |u|_{\min}$. This expression is minimised at $\phi_{\min} \simeq -2\pi/9 = -40°$, independently of m, in excellent agreement with the minima obtained numerically. In the case $m = 2$, one gets

$$u_{\min} \simeq \frac{0.0145}{2 + 0.0625}\,e^{-2\pi i/9} \leftrightarrow \tau_{\min} \simeq -0.492 + 0.875i\,. \tag{34}$$

This shows that the minimum not only deviates from the *symmetric point*, which may be responsible for mass hierarchies, but also from the *boundary* of the fundamental domain, providing an origin of CP breaking. Indeed, $\phi = 0$ corresponds to the left vertical boundary, while $\phi = -\pi/3$ corresponds to the arc, so that the minimum lies in between. This can be seen in Fig. 4, which shows the potential $V_{2,0}$ in the vicinity of the cusp.

6. Summary

In modular-invariant theories of flavour, hierarchical fermion masses may arise solely due to the proximity of the modulus to a point of residual symmetry $\tau_{\rm sym} = i, \omega$ or $i\infty$. In particular, if ϵ parameterises the deviation of τ from $\tau_{\rm sym}$ with $|\epsilon| \ll 1$, the degree of suppression of mass matrix elements is given by $|\epsilon|^l$ where l can take the values $l = 0, 1, ..., N-1$ if $\mathrm{Im}\,\tau$ is large; $l = 0, 1, 2$ if $\tau \simeq \tau_{\rm sym} = \omega$; or $l = 0, 1$ if $\tau \simeq \tau_{\rm sym} = i$. Here, N is the level of the finite modular group $\Gamma_N^{(\prime)}$. As summarised in Table 1, the specific value of l depends only on how the representations of the fermion fields entering the mass term bilinear decompose under the residual symmetry group.

Furthermore, we have found that it is only possible to obtain *hierarchical* spectra for a small list of representation pairs, assuming a common weight across reducible representations (see Table 2). Having scanned these models, we found two viable ones based on S_4' and A_5', both in the 'vicinity' of $\tau_{\rm sym} = i\infty$, in which charged-lepton hierarchies arise naturally as a consequence of the described mechanism. However, a certain degree of fine-tuning is still required due to the need for large corrections to the symmetric-limit PMNS matrix. One may avoid it if the model satisfies one of four conditions. Accordingly, we have constructed and presented a viable model based on S_4' modular symmetry with $\tau \simeq \omega$, which is free of fine-tuning in both the charged-lepton and neutrino sectors. The charged-lepton mass pattern is predicted to be $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^2)$ with $\epsilon \simeq 0.02$.

Finally, we have shown that values of τ with such small deviations from ω naturally arise in simple SUGRA-motivated potentials for the modulus. These potentials yield pairs of degenerate CP-conjugate global minima which break the CP symmetry spontaneously. Despite the simple framework, these results hint at an interesting connection between bottom-up and top-down pieces of the (modular) flavour puzzle.

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