PROCEEDINGS OF SCIENCE

Testing universality of gauge theories

Guilherme Catumba[a] **and Alberto Ramos**[a,*]

[a] *IFIC (CSIC-UVEG). Edificio Institutos Investigación, Apt. 22085, E-46071. Valencia, Spain.*

E-mail: alberto.ramos@ific.uv.es, guilherme.telo@ific.uv.es

The continuum limit is one of the main sources of uncertainty in many lattice works. Recent works on the asymptotic form of lattice cutoff effects by Husung et. al. have pointed to complicated functional forms, making difficult in practice the extrapolation to the continuum limit. Simulations at fine lattice spacings remain a challenge due to large autocorrelations, specially if topology freezing is an issue. In this proceedings contributions we test the continuum limit of Yang-Mills using different gauge actions. We discuss how scales derived from the gradient flow should be used for this purpose, and argue that such tests of universality are needed in full QCD in order to get confidence on current Lattice QCD results.

The 39th International Symposium on Lattice Field Theory, LATTICE2022 8th-13th August 2022, Bonn Germany

*Speaker

© Copyright owned by the author(s) under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License (CC BY-NC-ND 4.0).

PoS(LATTICE2022)383

1. Introduction

Lattice QCD provides a non perturbative regulator to the underlying field theory. For any given field theory we can construct many different discretizations. Only after renormalization and the continuum extrapolation all discretizations should give the same predictions for dimensionless observables.

This concept, usually referred as *universality*, is well understood in the framework of Symanzik efective field theory [1]: Any lattice action has a description through an effective continuum theory

$$S_{\rm latt} \overset{a\to 0}{\sim} S_{\rm cont} + a^2 S_2 + \dots \qquad (1)$$

By symmetry arguments the leading term ($S_{\rm cont}$) in this effective description is independent of the chosen discretization.

The Symanzik continuum effective theory is an important tool for lattice field theory: it is the basis of the improvement program for clover Wilson fermions[2], and is behind the arguments of automatic $O(a)$ improvement[3]. Recent works [4–6] have also investigated cutoff effects for different discretizations of Yang Mills and QCD. Their results for a dimensionless function of expectation values

$$\mathcal{P}(a) = \mathcal{P}(0) + a^2 \sum_k \hat{c}_k \mathcal{M}^{\rm RGI}_{\mathcal{P},k} [\alpha(1/a)]^{\gamma_k} \,, \qquad (2)$$

includes the computation of the anomalous dimensions γ_k for Yang-Mills and QCD. The sum in Eq. (2) runs over the possible dimension 6 local operators present in S_2, The interested reader should consult the original works [4–6] for a more detailed explanation. Here we just point that the asymptotic behavior Eq. (2) can be quite complicated, leading to difficult continuum extrapolations. From a practical point of view it is also not clear at which values of the lattice spacing a, this asymptotic behavior starts to set in. Given the poor convergence of lattice bare perturbation theory, one might be worried that in fact all lattice simulations are performed in a region where asymptotic scaling is broken.

These points suggests that a check of universality is more important than ever. We think that in order to claim results with 0.1% precision it is necessary to show that this precision can be reproduced with different discretizations. Using scales derived from the gradient flow [7, 8], we test for universality in the pure gauge theory. We aim at a sub-percent precision[1] by using ranges of lattice spacings usually simulated in large volume simulations (i.e. $0.15\,{\rm fm} \gtrsim a \gtrsim 0.05$ fm).

Although flow scales are ideal in many respect (high statistical precision, and negligible systematics), the description of their cutoff effects in terms of the local Symanzik effective theory is not straightforward [9]. In section 2 we make precise how we plan to test the scaling properties of different actions. Section 3 presents our main preliminary results, still with limited statistics. Finally we conclude in section 4.

[1]Note however that in this proceeding contribution we only have preliminary results with much less precision

PoS(LATTICE2022)383

2. Flow scales as a probe for scaling violations

The gradient flow [7, 10] is nowadays a standard tool in lattice QCD (see [11] for a review). In the continuum the flow equation reads

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t)\,, \tag{3}$$

with initial condition $B_\mu(x,t=0) = A_\mu(x)$, and $G_{\mu\nu}(x,t)$ being the field strength

$$G_{\nu\mu}(x,t) = \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)]\,. \tag{4}$$

Gauge invariant composite operators at positive flow time are automatically renormalized [12]. In particular the action density in units of the flow time t

$$t^2\langle E(x,t)\rangle\,, \qquad E(x,t) = \mathrm{tr}\{G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\}\,. \tag{5}$$

is dimensionless but depends on the scale $\mu = 1/\sqrt{8t}$. This makes it an ideal candidate to define a renormalized coupling or as a scale setting tool. In particular t-like [7] scales are defined by the condition

$$t^2\langle E(t)\rangle\Big|_{t=t_c} = c\,. \tag{6}$$

Different values of c define different scales t_c. In particular the choice $c = 0.3$ leads to the definition of the scale t_0[7].

w-like scales [13] use instead the logarithmic derivative of $t^2\langle E(t)\rangle$

$$t\frac{\mathrm{d}}{\mathrm{d}t}t^2\langle E(t)\rangle\Big|_{t=w_c^2} = c\,, \tag{7}$$

with the usual scale w_0 being defined by using again $c = 0.3$.

In this work we will make use of two different t-like scales (t_0 and t_1) and two different w-like scales (w_A and w_B). They are defined by:

$$t^2\langle E(t)\rangle = \begin{cases} 0.300 & (t = t_0) \\ 0.500 & (t = t_1) \end{cases}\,, \tag{8}$$

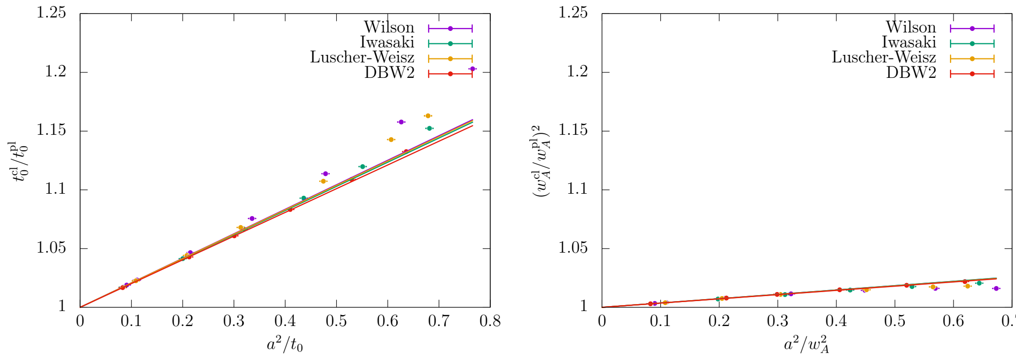
$$t\frac{\mathrm{d}}{\mathrm{d}t}t^2\langle E(t)\rangle = \begin{cases} 0.285 & (t = w_A^2) \\ 0.550 & (t = w_B^2) \end{cases}\,. \tag{9}$$

The apparently "weird" choices for w_A, w_B are chosen so that, in the pure gauge theory, $t_0 \approx w_A^2$ and $t_1 \approx w_B^2$.

2.1 Checking for scaling violations using flow quantities

In order to motivate how to check for scaling violations using flow scales, we will first show how this should not be done. There are several choices to solve the flow equation Eq. (3) and compute the energy density Eq. (5) on the lattice. They differ by cutoff effects. Typical choices for solving the flow equation are the Wilson flow [7] or the Zeuthen flow [9], while typical choices for

PoS(LATTICE2022)383



(a) The ratio $t_0^{\rm cl}/t_0^{\rm pl}$. **(b)** The ratio $(w_A^{\rm cl}/w_A^{\rm pl})^2$.

Figure 1: Ratios of plaquette/clover discretizations show similar scaling violations when computed with different gauge actions. On the other hand t-like scales show larger cutoff effects than w-like scales. See text for more details.

evaluating the action density use the clover or the plaquette definition of $G_{\mu\nu}$ (see [11] for a more detailed discussion). All these different choices differ just by cutoff effects.

One might think that measuring a flow scale (i.e. t_0) with different discretizations, for example using the clover ($t_0^{\rm cl}$) and plaquette ($t_0^{\rm pl}$) discretizations for $E(t)$, is an ideal probe for scaling violations. For example, the ratio $t_0^{\rm cl}/t_0^{\rm pl}$ is on one hand statistically very precise (numerator and demoninator are very correlated), and on the other hand the continuum limit is known

$$\lim_{a\to 0} \frac{t_0^{\rm cl}}{t_0^{\rm pl}} = 1\,, \tag{10}$$

since numerator/denominator are just different discretizations of the same observable.

Figure 1 shows the ratios $t_0^{\rm cl}/t_0^{\rm pl}$, $(w_A^{\rm cl}/w_A^{\rm pl})^2$ computed for the Wilson [14], Iwasaki [15], tree-level improved Luscher-Weisz [16] and DBW2 [17] gauge actions. A naive interpretation of these figures might lead to the following **wrong conclusions**

1. All four considered gauge actions show similar scaling violations.

2. w-like scales have significantly smaller scaling violations.

In order to understand why this conclusions are worng, it is crucial to understand the Symanzik expansion for flow quantities.

2.2 The Symanzik effective theory for flow scales

Flow observables are non-local from the 4d perspective, avoiding the naive use of the Symanzik effective theory Eq. (1). The Symanzik effective theory for flow scales [9] uses the 5d local formulation of the gradient flow [12] (see figure 2). One adds a fifth dimension to the usual 4d space time with the flow time t as coordinate. The theory lives only in the region of space $t \geq 0$. At $t = 0$ the action is the usual 4d action, but in the bulk ($t > 0$) the action consists on a Lagrange

PoS(LATTICE2022)383

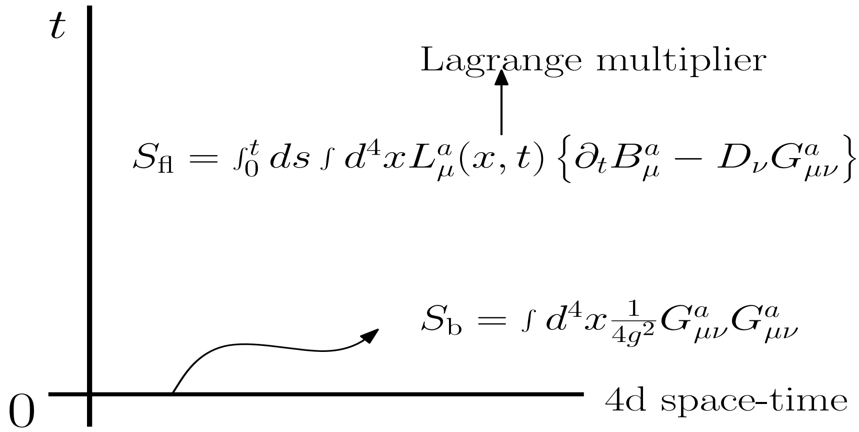


Figure 2: The analysis of flow observables using the Symanzik effective field theory description has to be done using the local 5d formulation of the flow [9].

multiplier term that enforces the flow equation. The total action of the theory consists in the sum of the two pieces

$$S^{5\rm d} = S_{\rm fl} + S_{\rm b} \,. \tag{11}$$

It is to this total local action that the Symanzik expansion can be applied. On a typical lattice computation one is using a 5d action in a rather indirect way: the boundary ($S_{\rm b}$) part is given by the gauge action simulated (i.e. Wilson/Iwasaki/...), while $S_{\rm fl}$ is implicitly defined when one solves the flow equation. For example, by using he Wilson flow versus the Zeuthen flow one is effectively using a different lattice discretization of $S_{\rm fl}$. The Symanzik expansion should be derived using the 5d action

$$S^{5\rm d}_{\rm latt} \overset{a\to 0}{\sim} S^{5\rm d}_{\rm cont} + a^2 S_{2,\rm b} + a^2 S_{2,\rm fl} + \dots \,. \tag{12}$$

Note the following points:

- $S_{2,\rm b}$ represents the usual term that appears in the Symanzik expansion Eq. (1). It enters in the cutoff effects of all quantities (i.e. when we determine the proton mass, or the muon $g-2$).
- $S_{2,\rm fl}$ only enter in the scaling violations of flow quantities. It comes from the discretization effects of the lattice flow equation.

From this expression and the classical expansion of a lattice discretization of the action density

$$E_{\rm latt}(t) \overset{a\to 0}{\sim} E(t) + a^2 E_2(t) \,, \tag{13}$$

one obtains the Symanzik expansion for flow scales. For the case of t_0 (similar expressions can be derived for w-like scales) we have

$$\begin{aligned} t_0^{\rm pl} &\overset{a\to 0}{\sim} t_0 - \frac{a^2}{D_0}\left\{t_0^2\langle E(t_0)S_{2,\rm b}\rangle + t_0^2\langle E(t_0)S_{2,\rm fl}\rangle + t_0^2\langle E_2^{\rm pl}(t_0)\rangle\right\} \\ t_0^{\rm cl} &\overset{a\to 0}{\sim} t_0 - \frac{a^2}{D_0}\left\{t_0^2\langle E(t_0)S_{2,\rm b}\rangle + t_0^2\langle E(t_0)S_{2,\rm fl}\rangle + t_0^2\langle E_2^{\rm cl}(t_0)\rangle\right\} \end{aligned}$$

PoS(LATTICE2022)383

where $D_0 = \mathrm{d}/\mathrm{d}t|_{t=t_0}\, t^2\langle E(t)\rangle$. It is now clear that the ratio

$$\frac{t_0^{\rm pl}}{t_0^{\rm cl}} \overset{a\to 0}{\sim} 1 - \frac{a^2}{D_0}\left\{t_0^2\langle E_2^{\rm pl}(t_0)\rangle - t_0^2\langle E_2^{\rm cl}(t_0)\rangle\right\} \tag{14}$$

only shows "trivial" classical cutoff effects. One does not learn anything about the potential a^2 scaling violations in any other quantity: to order a^2 this ratio does not depend on the choice of gauge action, since both $S_{2,\rm b}$ drops in the ratio. This explains the plots in Fig. 1.

3. Testing for universality with improved observables

It is clear that if we want to test universality or say something about the scaling properties of some action, we have to get rid of the scaling violations that depend on the discretization of the flow equation/observable. Fortunately it is possible to completely eliminate these sources of cutoff effects [9]. The key idea is that the theory at $t > 0$ is basically classical, and therefore classical a^2 improvement removes all a^2 cutoff effects [9]. In particular, if one uses the Zeuthen flow.

$$a^2\frac{d}{dt}V_\mu(x,t) = -g_0^2\left(1+\frac{a^2}{12}D_\mu D_\mu^*\right)\frac{\delta S^{\rm LW}[V]}{\delta V_\mu(x,t)}V_\mu(x,t)\,, \quad \big(V_\mu(x,0)=U_\mu(x)\big) \tag{15}$$

then $S_{2,\rm fl} = 0$. Also the clasically improved observable $E^{\rm imp}(t) = \frac{4}{3}E^{\rm pl}(t) - \frac{1}{3}E^{\rm cl}(t)$ has $E_2^{\rm imp} = 0$. With these choices of discretizations, the scaling violations on a flow scale read

$$t_0^{\rm imp} \overset{a\to 0}{\sim} t_0 - \frac{a^2}{D_0}\, t_0^2\langle E(t_0)S_{2,\rm b}\rangle\,, \tag{16}$$

i.e. only cutoff effects originating from the action chosen for the simulation $S_{2,\rm b}$ enter in the scaling violations. This improvement puts flow scales on a similar footing to other spectral quantities, that have as only source of cutoff effects the choice of lattice action.

3.1 A comment on the shift in the initial condition

In principle flow quantities generate an additional a^2 counterterm. This is related with a shift in the initial condition: in Eq. (15) one could use as initial condition

$$V_\mu(t,x)\Big|_{t=0} = \exp\{c_b(g_0^2)g_0^2\partial_{x,\mu}S_g[U]\}U_\mu(x) \tag{17}$$

where the parameter $c_b(g_0^2)$ has to be tuned in order to remove the extra counterterm [2]. Note however that these cutoff effects can be understood as a classical cutoff effect. Since the flow equation is a first order ODE in t, the shift at $t = 0$, can be understood as a shift at a later time $t_s > 0$. In particular, for flow scales one is interested in the quantity

$$t^2\langle E^{\rm imp}(t + c_b(g_0^2)a^2)\rangle\,, \tag{18}$$

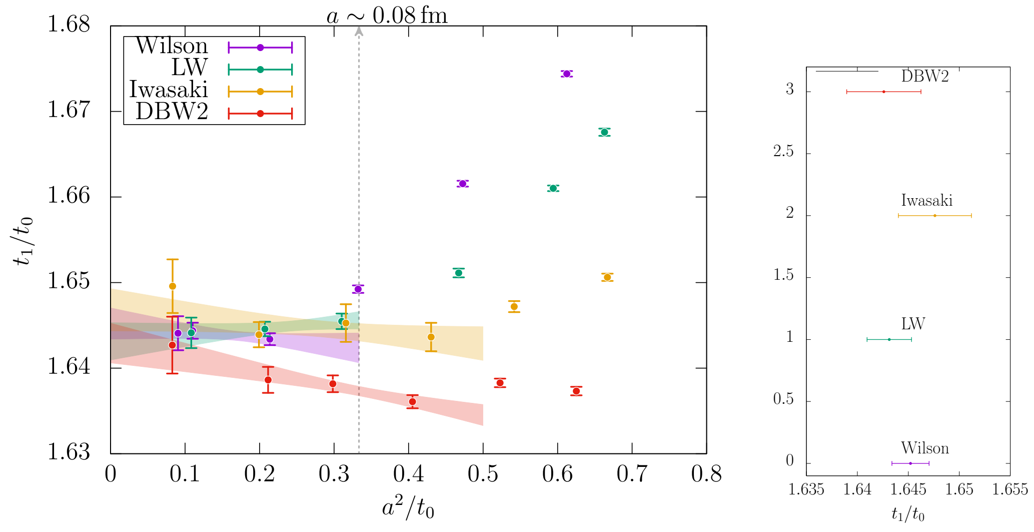
that has an effect in t_0

$$t_0(c_b) \overset{a\to 0}{\sim} \quad t_0 - \frac{a^2}{D_0}\left\{t_0^2\langle E(t_0)S_{2,b}\rangle + c_b t_0^2\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}E(t)\right\}\,, \tag{19}$$

i.e. this effect is a pure classical a^2 effect. Different values of c_b produce an a^2 effect, with corrections being a^4 but no logarithmic corrections.

[2]This shift in the initial condition is similar to the τ-shift introduced in [18].

PoS(LATTICE2022)383



(a) Continuum extrapolation. **(b)** Continuum values.

Figure 3: The ratio t_1/t_0 computed with different gauge actions. We see good agreement in the continuum values with about a 1% precision.

3.2 Results

All in all, these observations suggests using ratios of improved flow quantities measured at different flow times as probes for scaling violation. For example using

$$\frac{t_1^{\rm imp}}{t_0^{\rm imp}} \stackrel{t\to 0}{\sim} \frac{t_1}{t_0} - a^2 \left\{ \frac{t_1^2}{D_1} \langle E(t_1) S_{2,\rm b}\rangle - \frac{t_0^2}{D_0}\langle E(t_0) S_{2,\rm b}\rangle \right\} \tag{20}$$

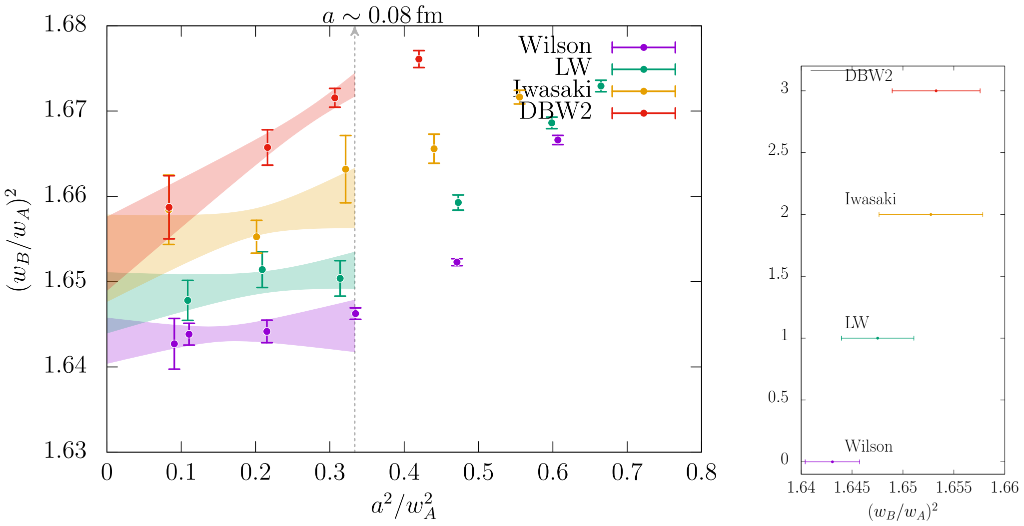
we see that the scaling violations enter only with $S_{2,\rm b}$.

Figure 3 and 4 show that this ratios display a reasonable agreement after the continuum limit is taken. Of course the precision is still poor, and more work is required.

4. Conclusions

- The asymptotic scaling of different gauge actions [4, 5] show complicated functional forms in the approach to the continuum limit. A strong test of universality is needed in order to gain confidence in lattice QCD results.
- The lack of systematic effects in the determination of flow scales make them ideal candidates to test for scaling. We have all used different discretizations for the observable (clover/plaquette/...), flow (Wilson/Symanzik) or shift in the initial condition to test our continuum extrapolations. Now it is time to stop doing it. The Symanzik expansion shows that these tests can be very misleading: finding an agreement in the continuum extrapolation is relatively easy since we are only probing the classical a^2 effects present at $t > 0$.
- We propose to use ratios of **improved** flow scales to test universality and the scaling of different actions. It has to be noted that it is possible that this ratio suffer from accidental

PoS(LATTICE2022)383



(a) Continuum extrapolation. **(b)** Continuum values.

Figure 4: The ratio $(w_A/w_B)^2$ computed with different gauge actions. We see reasonable agreement in the continuum values with about a 2% precision.

cancelations (see Eq. (20)). Still an agreement at the sub-percent level using this approach would be very comforting.

We have shown preliminary results for ratios of flow scales using different gauge actions (Wilson, Iwasaki, tree-level Symanzik improved and DBW2). Our precision is still low, but sub-percent results will follow. We believe that such tests should be performed for different lattice QCD actions.

Aknowledgements

The authors acknowledge financial support from the Generalitat Valenciana (genT program CIDEGENT/2019/040), and the Ministerio de Ciencia e Innovacion (PID2020-113644GB-I00).

We acknowledges the computer resources at Artemisa, funded by the European Union ERDF and Comunitat Valenciana as well as the technical support provided by the Instituto de Fisica Corpuscular, IFIC (CSIC-UV).

References

[1] K. Symanzik *NATO Sci. Ser. B* **59** (1980) 313–330.

[2] M. Lüscher, S. Sint, R. Sommer, and P. Weisz *Nucl. Phys.* **B478** (1996) 365–400, arXiv:hep-lat/9605038 [hep-lat].

[3] **ALPHA** Collaboration, R. Frezzotti, S. Sint, and P. Weisz *JHEP* **07** (2001) 048, arXiv:hep-lat/0104014.

PoS(LATTICE2022)383

[4] N. Husung, P. Marquard, and R. Sommer *Phys. Lett. B* **829** (2022) 137069, arXiv:2111.02347 [hep-lat].

[5] N. Husung, P. Marquard, and R. Sommer *Eur. Phys. J. C* **80** no. 3, (2020) 200, arXiv:1912.08498 [hep-lat].

[6] N. Husung arXiv:2206.03536 [hep-lat].

[7] M. Lüscher *JHEP* **08** (2010) 071, arXiv:1006.4518 [hep-lat]. [Erratum: JHEP 03, 092 (2014)].

[8] S. Borsanyi, S. Dürr, Z. Fodor, C. Hoelbling, S. D. Katz, *et al.* *JHEP* **1209** (2012) 010, arXiv:1203.4469 [hep-lat].

[9] A. Ramos and S. Sint *Eur. Phys. J. C* **76** no. 1, (2016) 15, arXiv:1508.05552 [hep-lat].

[10] R. Narayanan and H. Neuberger *JHEP* **0603** (2006) 064, arXiv:hep-th/0601210 [hep-th].

[11] A. Ramos *PoS* **LATTICE2014** (2015) 017, arXiv:1506.00118 [hep-lat].

[12] M. Lüscher and P. Weisz *JHEP* **1102** (2011) 051, arXiv:1101.0963 [hep-th].

[13] S. Borsanyi, S. Durr, Z. Fodor, C. Hoelbling, S. D. Katz, *et al.* *JHEP* **1209** (2012) 010, arXiv:1203.4469 [hep-lat].

[14] K. G. Wilson, "Quarks and Strings on a Lattice," in *New Phenomena in Subnuclear Physics: Proceedings, International School of Subnuclear Physics, Erice, Sicily, Jul 11-Aug 1 1975. Part A*, p. 99. 1975. [,0069(1975)].

[15] S. Itoh, Y. Iwasaki, Y. Oyanagi, and T. Yoshie *Phys. Lett. B* **148** (1984) 153–156.

[16] M. Lüscher and P. Weisz *Phys.Lett.* **B158** (1985) 250.

[17] **QCD-TARO** Collaboration, P. de Forcrand *et al.* *Nucl. Phys. B Proc. Suppl.* **53** (1997) 938–941, arXiv:hep-lat/9608094.

[18] A. Cheng, A. Hasenfratz, Y. Liu, G. Petropoulos, and D. Schaich *JHEP* **1405** (2014) 137, arXiv:1404.0984 [hep-lat].

PoS(LATTICE2022)383