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Reformulation of anomaly inflow on the lattice and construction of lattice chiral gauge theories

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We point out that the integrability condition for lattice chiral determinant of overlap Weyl fermion can be reformulated in parallel with the modern understanding of anomaly inflow based on Dai-Freed theorem and topological classification of global anomalies by bordism invariance. The known relations of the (2n+1)- and (2n+2)-dim domain-wall fermions and (2n)- and (2n+1)-dim overlap fermions, respectively, imply that Dai-Freed theorem and Atiya-Patodi-Singer index theorem. These relations also hold precisely true on the lattice, where the complex phase of (2n+1)-dim overlap fermion determinant defines the η-invariant. This η-invariant becomes “bordism invariant", if the local chiral anomaly density of the (2n+2)-dim overlap fermion is classified as “cohomologically" trivial along with the perturbative condition of gauge anomaly cancellation. Then, the integrability condition is given simply by the fact that the exponentiate of lattice η-invariant square is strictly unity for any admissible (2n+1)-dim gauge-link fields.

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1. Introduction

Since the rediscovery of the Ginsparg-Wilson (GW) relation[1–6], it has become possible to discuss the gauge interaction of Weyl fermions on the lattice in well-defined manners. As for the Abelian lattice chiral gauge theories with anomaly-free set of Weyl fermions, Lüshcer has shown that the gauge-invariant path-integral measure of lattice Weyl fermions can be constructed locally, uniquely and smoothly[7]. Lüshcer has also examined the gauge anomaly in non-Abelian lattice chiral gauge theories and formulated an integrability condition of the chiral determinant of lattice Weyl fermions[8]. This integrability condition was also formulated as the relation between 4-dim. overlap Weyl fermion and 5-dim lattice domain wall fermion[9–14].

Our research aims to reformulate the integrability condition of the chiral determinant of Weyl fermions on the lattice in a more parallel manner with the modern theory of 't Hooft (gauge) anomalies, and derive the necessary and sufficient condition to construct lattice chiral gauge theories. Our discussion parallels the modern theory of anomalies. The current understanding of 't Hooft anomaly implies that the d-dim anomalous theory emerges as a boundary theory of (d+1)-dim SPT theory. This non-trivial relation between a d-dim anomaly and a (d+1)-theory has been known as the anomaly-inflow mechanism[15]. Recent works[16–18] succeeded in reinterpreting the relation based on the Dai-Freed theorem[19] and sophisticating it. Furthermore, the (co)bordism classification of SPT phases enables us to classify anomalies systematically and serves as a powerful tool for examining 't Hooft anomalies in gauge theories.

In this paper, we first consider the 5-dim lattice domain-wall fermion and emerging 4-dim boundary overlap Weyl fermion. Calculating the partition function of the system, we reformulate the anomaly-inflow based on the Dai-Freed theorem on the lattice. Next, observing the 6-dim lattice domain-wall fermion and the boundary 5-dim overlap Dirac fermion, we derive the Atiyah-Patodi-Singer(APS) index theorem on the lattice. Those discussions of 5- and 6-dim domain-wall fermions lead us to define η-invariant on the lattice and its "bordism" invariance the triviality of which indicates the integrability of the lattice Weyl fermion determinant. In conclusion, we present two statements as the necessary and sufficient conditions to construct generic anomaly-free lattice chiral gauge theories.

1.1 Weyl fermions on the lattice and the gauge anomaly

Given a gauge-covariant lattice Dirac operator satisfying the GW relation[1],

$$D\gamma_5 + \gamma_5 D = 2aD\gamma_5 D, \tag{1}$$

Weyl fermion $\psi_-(x), \overline{\psi}_-(x)$ can be defined by constraining Dirac fermion $\psi(x), \overline{\psi}(x)$ with the modified projection operator and the usual one :

$$\psi_-(x) = \hat{P}_-\psi_-(x), \qquad \overline{\psi}_-(x) = \overline{\psi}_-(x)P_+, \tag{2}$$

where

$$\hat{\gamma}_5 = \gamma_5(1-2aD), \quad \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5), \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5). \tag{3}$$

The action of the Weyl fermion is written as:

$$S = a^4 \sum \overline{\psi}_-(x) D \psi_-(x). \tag{4}$$

The path-integral measure on the Weyl fermion can be defined with the coefficients of mode expansion of fermions. First, let us introduce chiral basis $\{v_i\}, \{\overline{v}_i\}$ and expand $\psi_-(x), \overline{\psi}_-(x)$ with them:

$$\psi_-(x) = \sum v_i(x)c_i \quad , \quad \hat{P}_- v_i(x) = v_i(x), \tag{5}$$
$$\overline{\psi}_-(x) = \sum \overline{c}_k \overline{v}_k(x) \quad , \quad \overline{v}_k(x)P_+ = \overline{v}_k(x). \tag{6}$$

Then, the path-integral measure can be defined as:

$$\mathcal{D}[\psi]\mathcal{D}[\overline{\psi}] = \prod_j dc_j \prod_k d\bar{c}_k. \tag{7}$$

As a result, the partition function of lattice Weyl fermion can be calculated as follows and takes the form of a chiral determinant:

$$Z = \int \mathcal{D}[\psi_-]\, \mathcal{D}[\bar{\psi}_-]\, \mathrm{e}^{-a^4 \sum_x \bar{\psi}_-(x) D \psi_-(x)} \tag{8}$$
$$= \int \prod_i dc_i \prod_i d\bar{c}_j \mathrm{e}^{-\sum_{ij} \bar{c}_j M_{ji} c_i} = \det M_{ji}, \tag{9}$$

where $M_{ji} = a^4 \sum_x \bar{v}_j D v_i(x)$.

In the above discussion, the constraint condition (5) depends on D through $\hat{P}_-$, and we foist the gauge field dependency of $\psi_-(x)$ on the basis $\{v_i\}$. Therefore, it is not trivial if the choice of the basis $\{v_i\}$ exists so that the partition function is uniquely determined as a function of the gauge field $U(x,\mu)$. In fact, under the unitary transformation Q of the basis,

$$\tilde{v}_i(x) = v_l(x)\left(Q^{-1}\right)_{li}, \quad \tilde{c}_j = \sum_l Q_{jl} c_l, \tag{10}$$

the partition function transforms as

$$\det M_{ji} \longrightarrow \det M_{ji} \det Q. \tag{11}$$

Consequently, the partition function now has a gauge-dependent phase ambiguity, namely, the gauge anomaly.

Overlap Weyl fermions are defined with a gauge-covariant and local solution of GW relation given as follows[2, 3]:

$$D_{\mathrm{ov}} \equiv \frac{1}{2a}\left(1 + X_{\mathrm{w}} \frac{1}{\sqrt{X_{\mathrm{w}}^\dagger X_{\mathrm{w}}}}\right), \qquad X_{\mathrm{w}} = D_{\mathrm{w}} - \frac{m_0}{a} \quad (0 < m_0 < 2), \tag{12}$$

whereD_{w} is a Dirac operator of the massless Wilson fermion:

$$D_{\mathrm{w}} = -\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger. \tag{13}$$

Throughout this paper, let us impose the admissibility condition:

$$\left\|1 - P_{\mu\nu}(x)\right\| < \frac{2}{5d(d-1)}, \tag{14}$$

where $P_{\mu\nu}(x)$ is the plaquette variable and d denotes the dimension. This condition guarantees the locality and the topological structure of the overlap Dirac operator[6].

2. Reformulation of the inegrablity condition in lattice chiral gauge theories

2.1 5-dim DW fermion

We adopt Shamir's definition of the lattice domain-wall fermion[10]. That is, we define it by Wilson-Dirac fermion with the negative mass $-\frac{m_0}{a}$ $(0 < m_0 < 2)$ on the 5-dim lattice with a finite fifth extent $t \in [-N+1, N]$ and impose the Dirichlet(Dir) boundary condition:

$$\psi_-(x,t)|_{t=-N,N+1} = 0, \tag{15}$$

$$\overline{\psi}_-(x,t)\Big|_{t=-N,N+1} = 0, \tag{16}$$

where t denotes the coordinate of 5th axis. This definition naturally introduces the Weyl fermion with the positive and the negative chirality on the $t = -N+1$ boundary and $t = N$ boundary, respectively.

In order to define chiral fermions, we need to assign different 4-dim link fields on two boundaries. Therefore, let $U_0(x;\mu)$ and $U_1(x;\mu)$ be the 4-dim gauge fields such that couple to ψ_+ and ψ_-, respectively:

$$U(x,t;\mu)|_{t=-N+1} = U_0(x;\mu), \tag{17}$$

$$U(x,t;\mu)|_{t=N} = U_1(x;\mu). \tag{18}$$

In addition, we must define the 5-dim link fields consistent with the boundary link fields. Thus we take the 5-dim path c to smoothly interpolate $U_0(x,\mu)$ and $U_1(x,\mu)$ along the 5th axis. Note that it has to satisfy the 5-dim version of the admissibility condition.For a practical purpose, we suppose that 5-dim link fields vary in a finite interval along the path c, namely the interval $t \in [-\Delta+1, \Delta]$ with a fixed Δ, sufficiently small compared to N.

Further, we can also define the domain-wall fermion with anti-periodic(AP) boundary condition. To do so, we first combine two 5-dim lattice spaces and take $t \in [-N+1, 3N]$ and interpolate 5-dim link fields with the path $\tilde{c} = c_1 c_2^{-1}$.

2.2 5-dim DW fermion and the Dai-Freed theorem

In this section, let us consider the 5-dim domain-wall fermion with Dir. boundary condition and reformulate the anomaly inflow on the lattice based on the Dai-Freed theorem. Let $\mathbb{Y}|_{\mathrm{Dir/AP}}$ be the 5-dim lattice space with Dir/AP boundary condition and $\mathbb{X}_{0,1}$ denotes the two boundaries of $\mathbb{Y}|_{\mathrm{Dir}}$.

The partition function of the domain-wall fermion and emerging boundary Weyl fermion can be calculated as follows[13]:

$$\lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir}}^{c}}{\left|\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{AP}}^{c\cdot c^{-1}}\right|^{1/2}} = \det\left(\overline{v} D_{\mathrm{ov}} v^1\right) \det\left(\overline{v} D_{\mathrm{ov}} v^0\right)^* \frac{\det\left(v^{1\dagger} \prod_{t\in\tilde{c}} T_t v^0\right)}{\left|\det\left(v^{1\dagger} \prod_{t\in\tilde{c}} T_t v^0\right)\right|}. \tag{19}$$

where T in the 3rd term is a transfer matrix and defined as $T_t = \frac{1-aH_t/2}{1+aH_t/2}$. Here are some remarks about the equation(19). First, at the LHS, we calculate the partition function of the 5-dim domain-wall fermion with the Dir boundary condition, divided by the same with the AP boundary condition.

It is because we would like to focus on the contribution of the boundary Weyl fermions. As a virtue of this procedure, the bulk term, the 3rd term in the RHS, becomes only a pure phase. We denote this phase with $i\,2\pi\eta_{\mathrm{DF}}(\,\mathbb{Y}|_{\mathrm{Dir}}\,)$:

$$\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right) := \frac{\det\left(v^{1\dagger}\prod_{t\in\tilde{c}}T_t v^0\right)}{\left|\det\left(v^{1\dagger}\prod_{t\in\tilde{c}}T_t v^0\right)\right|}. \tag{20}$$

Second, the 1st and 2nd terms specify the boundary contributions, namely the effective action of 4-dim Weyl fermions. As mentioned in section1.1, it takes the form of the chiral determinant and breaks the gauge symmetry through the chiral bases $\{v_i^1\}, \{v_i^0\}$. We denote these effective actions with $\Gamma(\mathbb{X}_1\cup\overline{\mathbb{X}}_0)$. Consequently, the equation (19) can be written in a simpler form:

$$\lim_{N\to\infty}\frac{\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir}}^{c}}{\left|\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{AP}}^{c\cdot c^{-1}}\right|^{1/2}} = \exp\left(\Gamma(\mathbb{X}_1\cup\overline{\mathbb{X}}_0)\right)\exp\left(i2\pi\eta_{\mathrm{DF}}(\,\mathbb{Y}|_{\mathrm{Dir}}\,)\right). \tag{21}$$

A closer look at this equation shows that each term in RHS is anomalous, but by combining them, we get LHS, which is anomaly-free. Actually, the boundary effective action and the bulk term have the same $\{v_i\}$ dependency and give the exact same U(1) bundle. This relation reformulates the anomaly inflow on the lattice based on the Dai-Freed theorem.

Let us emphasize that the amount $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$ indicates the bulk dependency of the gauge configuration, precisely speaking, the dependence of the interpolation path c. Based on the above discussion, we can always cancel the gauge anomaly of the 4-dim lattice Weyl fermion with the 5-dim object $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$; thus, we no longer have the problem with the anomaly. Instead, it now comes down to the new situation of 5-dim dependency.

2.3 Lattice η-invariant and integrability condition

In this section, we discuss the relation between the amount $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$ and the lattice η-invariant and formulate the integrability condition of lattice Weyl fermion with the lattice η invariant. First, we define the lattice η-invariant with the phase of the determinant of the 5-dim overlap Dirac operator, following a precedented work[14]:

$$\exp\left(i2\pi\eta(\,\mathbb{Y}|_{\mathrm{Dir/AP}}\,)\right) := \lim_{N\to\infty}\left[\frac{\det D_{\mathrm{ov}}^{(5)}\Big|_{\mathrm{Dir/AP}}}{\left|\det D_{\mathrm{ov}}^{(5)}\Big|_{\mathrm{Dir/AP}}\right|}\right]^2 \tag{22}$$

$$= \lim_{N\to\infty}\frac{\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir/AP}}}{\left|\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir/AP}}\right|}. \tag{23}$$

This definition coincides with the usual definition of the η invariant under the classical continuum limit. Then, by definition, the equation (21) reads

$$\mathrm{e}^{i2\pi\eta(\,\mathbb{Y}|_{\mathrm{Dir}})} = \mathrm{e}^{i2\pi\eta_{\mathrm{DF}}(\,\mathbb{Y}|_{\mathrm{Dir}})}\mathrm{e}^{i\,\mathrm{Im}\,\Gamma(\mathbb{X}_1\cup\overline{\mathbb{X}}_0)} \tag{24}$$

and, with the equation (20), this in turn implies

$$\frac{\mathrm{e}^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_1}_{\mathrm{Dir}}\right)}}{\mathrm{e}^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_2}_{\mathrm{Dir}}\right)}} = \mathrm{e}^{i2\pi\eta\left(\mathbb{Y}|^{c_1 c_2^{-1}}_{\mathrm{AP}}\right)}. \tag{25}$$

Now, let us come back to our problem of the 5-dim dependency of the partition function of the 4-dim lattice Weyl fermion. Reminding that $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$ is the bulk contribution of the partition function and the ratio $\mathrm{e}^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_1}_{\mathrm{Dir}}\right)}/\mathrm{e}^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_2}_{\mathrm{Dir}}\right)}$ measures the bulk dependency of the interpolation paths. Accordingly, the integrability condition is now formulated by the following statement:

$$\text{“}e^{i2\pi\eta(\mathbb{Y}|_{\mathrm{AP}})} = 1 \text{ for arbitrary gauge configurations satisfying the admissibility condition.''} \tag{26}$$

However, it is practically impossible to compute $e^{i2\pi\eta(\mathbb{Y}|_{\mathrm{AP}})}$ for any gauge configurations. Then it is essential to introduce the “bordism” invariance on the lattice, as we will see in later, to reduce the computation.

2.4 6-dim DW fermions and APS index theorem

In this section, we formulate the APS index theorem on the lattice, considering the 6-dim latticedomain-wall fermion and boundary 5-dim overlap Dirac fermion[20]. Let $\mathbb{Z}|_{\mathrm{Dir/AP}}$ be the 5-dim lattice space with Dir/AP boundary condition and $\mathbb{Y}_{0,1}$ denotes the two boundaries of $\mathbb{Z}|_{\mathrm{Dir}}$. We use s to indicate the coordinate of 6th axis.

The partition function of the 6-dim domain-wall fermion with the Dir boundary condition can be evaluated in the same manner as the section 2.2, and we get:

$$\lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(6)}\Big|^{c}_{\mathrm{Dir}}}{\left|\det X_{\mathrm{w}}^{(6)}\Big|^{c^{-1}}_{\mathrm{AP}}\right|^{1/2}} = \det\left(\bar{v} D_{\mathrm{ov}} v^1\right) \det\left(\bar{v} D_{\mathrm{ov}} v^0\right)^* \frac{\det\left(v^{1\dagger} \prod_{t\in\bar{c}} T_t v^0\right)}{\left|\det\left(v^{1\dagger} \prod_{t\in\bar{c}} T_t v^0\right)\right|}. \tag{27}$$

In the 6-dim case, we can choose the basis $\{v_i\}$ as follows:

$$v_i(x) \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} X^{(5)} \frac{1}{\sqrt{X^{(5)\dagger} X^{(5)}}} \\ 1 \end{pmatrix} \phi_i(x), \tag{28}$$

where

$$\left(X^{(5)\dagger} X^{(5)}\right) \phi_i(x) = (\lambda_i)^2 \phi_i(x), \quad \sum \phi_i(x)\phi_i(y)^\dagger = I_{4\times4}\delta_{x,y}. \tag{29}$$

As a result, unlike the 5-dim case, we can factorize the boundary and the bulk terms so that neither is independent on the basis $\{v_i\}$, meaning anomaly-free. This choice of chiral basis corresponds to taking the APS boundary condition. Therefore, based on our choice of the basis, we can rewrite the equation (27) as follows:

$$\lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(6)}\Big|^{c}_{\mathrm{Dir}}}{\left|\det X_{\mathrm{w}}^{(6)}\Big|^{cc^{-1}}_{\mathrm{AP}}\right|^{1/2}} \;=\; \det D^{(5)}_{\mathrm{ov}}\Big|_{\mathrm{Y}_1} \left\{\det D^{(5)}_{\mathrm{ov}}\Big|_{\mathrm{Y}_0}\right\}^* \mathrm{e}^{i\pi P(\mathbb{Z}|^{c})}, \tag{30}$$

where we define the amount P so that the 3rd term denotes the bulk contribution:

$$e^{i\pi P\left(\mathbb{Z}|^{c}\right)} = \frac{\det \mathcal{T}}{\left|\det \mathcal{T}\right|}, \tag{31}$$

$$\mathcal{T} = \frac{1}{2}\left(\frac{1}{\sqrt{X^\dagger X}}X^\dagger \quad 1\right)\Bigg|_{\mathbb{Y}_1} \prod_{t\in\tilde{c}} T_t^{(5)} \left(\begin{array}{c} X\frac{1}{\sqrt{X^\dagger X}} \\ 1 \end{array}\right)\Bigg|_{\mathbb{Y}_0}. \tag{32}$$

This choice of chiral basis corresponds to taking the APS boundary condition. The amount $e^{i\pi P(\mathbb{Z}|^{c})}$ indicates the interpolation dependency of the 6-dim gauge configuration, and it can be expressed with the phase of the domain-wall fermion with the AP boundary condition:

$$e^{i\pi P(\mathbb{Z}|^{c_1})}e^{-i\pi P(\mathbb{Z}|^{c_2})} = \lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(6)}\Big|_{\mathrm{AP}}^{c_1 c_2^{-1}}}{\left|\det X_{\mathrm{w}}^{(6)}\Big|_{\mathrm{AP}}^{c_1\cdot c_2^{-1}}\right|} \tag{33}$$

$$:= \lim_{N\to\infty} e^{i\pi Q^{(6)}\left(\mathbb{Z}|^{c_1\cdot c_2^{-1}}\right)}. \tag{34}$$

We can specifically exhibit the amount of $Q^{(6)}\left(\mathbb{Z}|^{c_1\cdot c_2^{-1}}\right)$ as a lattice sum of a local topological field $q^{(6)}(z)$:

$$Q^{(6)} = \sum_{y,s\in c_1 c_2^{-1}} q^{(6)}(z), \tag{35}$$

$$q^{(6)}(z) := -\frac{1}{2}\,\mathrm{tr}\left\{\frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}\Bigg|_{\mathrm{AP}}\right\}(z,z), \qquad H_{\mathrm{w}} = \gamma_7 X_{\mathrm{w}}^{(6)}, \tag{36}$$

where z denotes the 6-dim coordinate. Although $Q^{(6)}\left(\mathbb{Z}|^{c_1\cdot c_2^{-1}}\right)$ originally depended on a loop $c_1 c_2^{-1}$, we can use the locality of $q^{(6)}(z)$ to decouple the contributions from c_1 and c_2^{-1}. Thus we can write:

$$P\left(\mathbb{Z}|^{c}\right) = \lim_{N\to\infty}\sum_{y,s\in c} q^{(6)}(z). \tag{37}$$

Now, let us focus on the phase of the equation (30) this time. First, for the phase of the LHS, one can define an index I on the lattice as follows:

$$I\left(\mathbb{Z}|_{\mathrm{Dir}}\right) = -\sum_{z}\frac{1}{2}\,\mathrm{tr}\left\{\frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}\Bigg|_{\mathrm{Dir}}\right\}(z,z). \tag{38}$$

It has been proved that this index coincides with the APS index in the continuum theory[21–23]; hence we get the lattice APS index from the phase of the LHS.

Next, it follows from the definition of the lattice η invariant (22) that the phase of the boundary terms in the RHS, 1st and 2nd terms, give the lattice η invariants. Besides, the bulk term in the LHS has already been just a phase. Therefore, from the phase of the equation (30), we get:

$$I\left(\mathbb{Z}|_{\mathrm{Dir}}\right) = P\left(\mathbb{Z}|^{c}\right) + \eta\left(\mathbb{Y}_1|_{\mathrm{AP}}\right) - \eta\left(\mathbb{Y}_0|_{\mathrm{AP}}\right). \tag{39}$$

This result exactly reproduces the APS index theorem on the lattice[1].

[1] Strictly speaking, it only holds with modulo 2π, but it has been shown that the leftover term vanishes when perturbative anomaly cancels out, and thus, we can simplify the equation. [14]

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2.5 The cohomological problem and "Bordism" invariance of lattice eta invariant

Now, let us look closely at the amount $P(\mathbb{Z}|^c)$. As mentioned above, $P(\mathbb{Z}|^c)$ can be expressed with the lattice sum of a lattice topological field q, which thus satisfies the relation:

$$\sum_{y,s\in c} \delta_\eta q^{(6)}(z) = 0. \tag{40}$$

Such a local topological field $q(z)$ is expected to be classified cohomologically[8], as well as the case in the continuum theory. Thereby assuming this cohomological problem, we can rewrite $q(z)$ with a lattice Chern class $c(z)$ and local gauge-invariant currents k as follows:

$$q^{(6)}(z) = c_3(z) + \partial^*_\mu k_\mu(z). \tag{41}$$

Additionally, let us suppose the perturbative anomaly cancellation condition $\Sigma_R \operatorname{Tr}_R \left[T^a \left\{T^b, T^c\right\}\right] = 0$. As it means $c(z) = 0$, $q(z)$ becomes cohomologically trivial:

$$q^{(6)}(z) = \partial^*_\mu k_\mu(z). \tag{42}$$

under the condition. Now $q(z)$ is a mere total derivative term, and hence P, the lattice sum of $q(z)$, can be represented with only the boundary contributions:

$$P\left(\mathbb{Z}|^c\right) = \sum_y k_s(z)|_{\mathbb{Y}_1} - \sum_y k_s(z)|_{\mathbb{Y}_0} \tag{43}$$

This description leads us to define a new amount, adding the boundary contribution from $P(\mathbb{Z}|^c)$ to η, which is originally a boundary object. Namely, we define a new amount as follows:

$$\check{\eta}\left(\mathbb{Y}_{1,0}\big|_{\mathrm{AP}}\right) = \eta\left(\mathbb{Y}_{1,0}\big|_{\mathrm{AP}}\right) + \sum_y k_s(y,s)|_{\mathbb{Y}_{1,0}}. \tag{44}$$

As a result, the following identity follows from the APS index theorem (39):

$$\exp(\pi i I\,(\mathbb{Z}|_{\mathrm{Dir}})) = \exp(\pi i\check{\eta}\,(\mathbb{Y}_1|_{\mathrm{AP}}))\exp(-\pi i\check{\eta}\,(\mathbb{Y}_0|_{\mathrm{AP}})), \tag{45}$$

$$1 = \exp(2\pi i I\,(\mathbb{Z}|_{\mathrm{Dir}})) = \exp(2\pi i\check{\eta}\,(\mathbb{Y}_1|_{\mathrm{AP}}))\exp(-2\pi i\check{\eta}\,(\mathbb{Y}_0|_{\mathrm{AP}})). \tag{46}$$

The last relation tells us that $\exp(2\pi i\check{\eta}\,(\mathbb{Y}_1|_{\mathrm{AP}}))$ and $\exp(2\pi i\check{\eta}\,(\mathbb{Y}_0|_{\mathrm{AP}}))$ give the same amount if there exists a 6-dim lattice space such that $\partial\mathbb{Z} = \overline{\mathbb{Y}}_0 \sqcup \mathbb{Y}_1$. This property is the lattice analog for the notion of bordism invariance.

2.6 The triviality of $\exp(2\pi i\check{\eta}\,(\mathbb{Y}_0|_{\mathrm{AP}}))$ and the integrability condition

As mentioned in sec2.3, the integrability condition has come down to the statement "$e^{i2\pi\eta(\mathbb{Y}|_{\mathrm{AP}})} = 1$ for arbitrary gauge configurations satisfying the admissibility condition". Now we can amend it using $\check{\eta}$:

$$\text{“}e^{i2\pi\check{\eta}(\mathbb{Y}|_{\mathrm{AP}})} = 1 \text{ for arbitrary gauge configurations satisfying the admissibility condition.”} \tag{47}$$

Due to the powerful property of the 'bordism' invariance of $e^{i2\pi\check{\eta}(\mathbb{Y}|_{\mathrm{AP}})}$, we only need to calculate if with a finite number of 'bordism' equivalent gauge configurations. In conclusion, the original issue of the anomaly cancellation condition, namely the integrability condition of the 4-dim lattice Weyl fermion, eventuated the calculation task of the amount $e^{i2\pi\check{\eta}(\mathbb{Y}|_{\mathrm{AP}})}$. After confirming that $e^{i2\pi\eta(\mathbb{Y}|_{\mathrm{AP}})} = 1$, we can define the partition function of the 4-dim Weyl fermion as follows:

$$\mathrm{e}^{\Gamma(\mathbb{X}_1)} = \det\left(\bar{v} D_{\mathrm{ov}} v^1\right) \mathrm{e}^{i2\pi\check{\eta}_{\mathrm{DF}}\left(\mathbb{Y}|^c_{\mathrm{Dir}}\right)}. \tag{48}$$

Our discussion above guarantees that this partition function is uniquely determined up to a constant (gauge-independent) phase.

3. Summary

With the above discussion, we have formulated the construction of generic lattice chiral gauge theories. As a first step towards this formulation, we first focused on the 5-dim lattice domain-wall fermion. We defined an amount $\check{\eta}_{\mathrm{DF}}\left(\mathbb{Y}|^c_{\mathrm{Dir}}\right)$, which has been shown to coincide with the section of the determinant line bundle of the 4-dim boundary lattice chiral fermions. This mechanism is a lattice analog of the anomaly inflow based on the Dai-Freed theorem.
Next, we focused on the 6-dim lattice domain-wall fermion and produced the APS index theorem on the lattice. In addition, we introduced an amount $e^{i2\pi\check{\eta}(\mathbb{Y}|_{\mathrm{AP}})}$, assuming the cohomological problem. In conclusion, the necessary and sufficient conditions to construct lattice chiral gauge theories can be summarized in two statements:

1. (The cohomological problem) local lattice topological fields become cohomologically trivial under the perturbative anomaly cancellation condition.

2. (The integrability condition) $e^{i2\pi\check{\eta}(\mathbb{Y}|_{\mathrm{AP}})} = 1$ for representative gauge configuration of "bordism" equivalence

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