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# Loop-string-hadron formulation of an SU(3) gauge theory with dynamical quarks

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The Hamiltonian formalism for lattice gauge theories has experienced a resurgence of interest in recent years due to its relevance for quantum simulation, a major goal of which is the solution of sign problems in QCD. The particular formulation of the Hamiltonian formalism is itself an important design decision, where factors to consider include (non)locality of the degrees of freedom, (non)Abelian constraints, and computational costs associated with simulating the Hamiltonian. This work represents a key step toward understanding the costs and benefits associated with the loop-string-hadron (LSH) formulation of lattice gauge theories by generalizing the original SU(2) construction to SU(3) (in 1+1 D). We show that the SU(3) LSH construction is indeed a straightforward generalization of its SU(2) counterpart with all salient theoretical features left intact—particularly the conversion of SU(3) Clebsch-Gordan coefficients into explicit functions of LSH number operators. The validity of the LSH approach is underscored by demonstrating numerical agreement with the better-known purely-fermionic formulation of the theory (with open boundary conditions).

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## 1. Introduction

Understanding the properties and interactions of hadrons is crucial for a wide range of physical phenomenon. The strong force that governs these interactions is described by the theory of quantum chromodynamics (QCD) which is a non-Abelian theory with symmetry group SU(3). This is a strongly interacting theory at low energies that requires non-perturbative methods like lattice QCD to calculate its low energy predictions. Although being very successful [1, 2], the method of lattice QCD suffers from a 'sign problem' in certain situations, like nonzero chemical potential, topological terms in the Lagrangian, and far-from-equilibrium and real-time dynamics, since it uses Monte Carlo method to evaluate QCD path integrals with imaginary time.

The Hamiltonian formalism, on the other hand, performs the real time evolution of states and calculates observables by measuring operator expectation values in a time evolved state, and thus, does not obviously suffer from a sign problem. However, it suffers from the exponential growth of Hilbert space with system size which can be tackled by using quantum simulators that also feature exponentially large Hilbert spaces, and are naturally expressed in the Hamiltonian framework. The paradigm of quantum simulation for particle physics problems is rapidly growing [3], and in the case of non-Abelian gauge theories, a number of Hamiltonian formulations for a given model have been proposed including the original Kogut-Susskind (KS) formulation [4], and more recent ones like the prepotential formulation [5–11] and the the loop-string-hadron (LSH) formulation [12, 13] derived from prepotentials.

The LSH formulation, initially developed for the SU(2) lattice gauge theories in Ref. [12], is specially interesting as it has several features that are advantageous for quantum (as well as classical) simulation in the 1+1 D case [14]. Progresses made in the recent years further encourage the continued exploration into the LSH framework [15, 16]. An exciting possibility is that the LSH approach will generalize to SU(3) and continue to offer computational advantages, hopefully providing a resource efficient formulation of QCD for its quantum simulation. A forthcoming algorithmic study confirms cost saving advantages with the LSH formulation for the SU(2) case [17], and can be speculated to still be advantageous for the SU(3) case.

In this article, we summarize the extension of the LSH formulation for an SU(3) lattice gauge theory with staggered quarks in 1+1 D space that was recently presented in our work [18]. We summarize the KS and prepotential formulation of an in 1+1 D SU(3) lattice gauge theory coupled to one flavor of staggered matter in Sec. 2, and provide a brief summary of constructing LSH framework from the prepotential Hamiltonian in Sec. 3.

## 2. The Kogut-Susskind and prepotential formulation of an SU(3) lattice gauge theory

The KS Hamiltonian for an SU(3) lattice gauge theory with one flavor of staggered fermions is given by formulating the theory on a discrete spatial lattice with continuous time. It is described by the matter and gauge degrees of freedom on lattice sites and links. The gauge degrees of freedom in this formulation are described by the chromoelectric fields and the link operators which remain after fixing the temporal gauge. We consider here a 1+1 D lattice with $N$ sites where lattice sites are denoted by $r = 1, 2, \cdots, N$, and links are labeled by the lattice sites on their left. The chromoelectric field at the left ($L$) and right ($R$) ends of link $r$ are given by $E^{\rm a}(L,r)$ and $E^{\rm a}(R,r)$, respectively, where $\rm a = 1, 2, \cdots, 8$ is the adjoint index. They satisfy the canonical commutation relations:

$$[E^{\rm a}(L/R,r), E^{\rm b}(L/R,r')] \;=\; \delta_{rr'} \sum_{\rm c=1}^{8} i\, f^{\rm abc}\, E^{\rm c}(L/R,r), \tag{1}$$

where $f^{\rm abc}$ are the structure constants for SU(3), and $L/R$ indicates that the relation holds individually for both $L$ and $R$ sides of the link. The link operator, $U(r)$, is a unitary $3\times 3$ matrix that sits on each link $r$, and it relates $E^{\rm a}(L,r)$ and $E^{\rm a}(R,r)$ via the relation $E^{\rm a}(R,r)T^{\rm a} = -U^\dagger(r)E^{\rm b}(L,r)T^{\rm b}U(r)$, where $T^{\rm a} = \lambda^{\rm a}/2$ with $\lambda^{\rm a}$ being the Gell-Mann matrices. As a consequence, the gauge invariant Casimir on either sides, $E^2(r)$, must be equal:

$$E^2(r) \equiv \sum_{\rm a} E^{\rm a}(L,r)E^{\rm a}(L,r) = \sum_{\rm a} E^{\rm a}(R,r)E^{\rm a}(R,r), \tag{2}$$

which is also the energy stored in chromoelectric field at $r$, $H_E(r)$. The one flavor staggered matter field, $\psi^\alpha(r)$, is situated at site $r$, where $\alpha = 1, 2, 3$ is the color index. It satisfies the fermionic anticommutation relations given by

$$\{\psi^\dagger_\alpha(r), \psi^\dagger_\beta(r')\} = \{\psi^\alpha(r), \psi^\beta(r')\} = 0, \qquad \{\psi^\alpha(r), \psi^\dagger_\beta(r')\} = \delta^\alpha_\beta\, \delta_{rr'}, \tag{3}$$

and contributes to the matter self energy, $H_M(r)$. Finally, with the gauge-matter interaction energy, $H_I(r, r+1)$, the KS Hamiltonian is given by:

$$\begin{aligned} H \;&=\; H_M + H_E + H_I = \sum_{r=1}^{N} H_M(r) + \sum_{r=1}^{N'} H_E(r) + \sum_{r=1}^{N'} H_I(r, r+1) \\ &=\; \mu \sum_{r=1}^{N} (-1)^r \psi^\dagger_\alpha(r)\psi^\alpha(r) + \sum_{r=1}^{N'} E^2(r) + x \sum_{r=1}^{N'} \left[\psi^\dagger_\alpha(r)\, U^\alpha{}_\beta(r)\, \psi^\beta(r+1) + \text{H.c.}\right]. \end{aligned} \tag{4}$$

Here, repeated indices are summed over, and $N' = N - 1$ ($N$) for open (periodic) boundary condition. The dimensionless couplings $\mu$ and $x$ are related to the fermion mass and the gauge coupling, respectively (see Ref. [19] for details). The physical Hilbert space of this theory is spanned by states $|\psi\rangle$ that satisfy the Gauss's law constraints given by $G^{\rm a}(r)|\Psi\rangle = 0 \quad \forall\, {\rm a}, r$, where $G^{\rm a}(r)$s are the full generators of the local gauge transformations: $G^{\rm a}(r) = E^{\rm a}(L,r) + E^{\rm a}(R, r-1) + \psi^\dagger_\alpha(r)\ (T^{\rm a})^\alpha{}_\beta\, \psi^\beta(r)$.

The gauge invariant Hamiltonian in Eq. (4) commutes with $G^{\rm a}(r)$ for all a, $r$. The original proposal of basis by Kogut and Susskind for constructing the Hilbert space of this Hamiltonian in 1+1 D is known as the angular-momentum basis that is over-complete, and the physical Hilbert space is only a small subspace of the full Hilbert space spanned by this basis.

An alternate formulation of gauge degrees of freedom in Eq. (4) is given by a set of Schwinger bosons (or harmonic oscillators) in the the fundamental representation of the gauge group, known as prepotentials, at each end of a link. The number of independent sets at each link is determined by the rank of the gauge group. Thus, for the rank two SU(3) gauge group which has a three dimensional fundamental representation, the prepotential formulation requires two independent Schwinger boson triplets at both ends of each link. Note that, unlike the prepotential formulation of an SU(2) group, a representation created by a monomial of SU(3) prepotentials is generally not irreducible [20, 21]. In order to form an irreducible representation (irrep) using them, one needs to systematically remove all the traces to make the irrep traceless and address the multiplicity problem, see Refs. [21, 22]. Nonetheless, a monomial construction of traceless SU(3) irreps can be constructed using a set of modified Schwinger bosons, called the irreducible Schwinger bosons (ISBs), as shown in Ref. [7]. We denote them by $A^\dagger_\alpha(L,r)$ and $B^{\dagger\alpha}(L,r)$ for ISBs on the $L$ end of link $r$, and $A^\dagger_\alpha(R,r)$ and $B^{\dagger\alpha}(R,r)$ for ISBs on the $R$ end of the link $r-1$. Then, monomial states that forms a traceless general irrep $(P,Q)$ under the SU(3) gauge transformations are given by

$$|P,Q\rangle^{\vec{\beta}}_{\vec{\alpha}} = \mathcal{N} A^\dagger_{\alpha_1}\dots A^\dagger_{\alpha_P} B^{\dagger\beta_1}\dots B^{\dagger\beta_Q}|0\rangle, \tag{5}$$

where we have suppressed the direction and position arguments for brevity. Here $\mathcal{N}$ corresponds to the normalization factor, $\alpha_i$ with $i=1,\cdots,P$ and $\beta_j$ with $j=1,\cdots,Q$ indices can take integer values between 1 to 3, which determines the isospin and hypercharge of the irrep state [20, 21]. Such states satisfy $A^\dagger\cdot B^\dagger\,|P,Q\rangle^{\vec{\beta}}_{\vec{\alpha}}=0$ and $A\cdot B\,|P,Q\rangle^{\vec{\beta}}_{\vec{\alpha}}=0$, which solves the multiplicity problem (see Ref. [8] for details). Furthermore, the ISBs obey the modified commutation relations

$$[A^\alpha, A^\dagger_\beta] \simeq \left(\delta^\alpha_\beta - \frac{1}{\hat{N}_A+\hat{N}_B+2}B^{\dagger\alpha}B_\beta\right), \tag{6}$$

$$[B_\alpha, B^{\dagger\beta}] \simeq \left(\delta^\beta_\alpha - \frac{1}{\hat{N}_A+\hat{N}_B+2}A^\dagger_\alpha A^\beta\right), \tag{7}$$

$$[A^\alpha, B^{\dagger\beta}] \simeq -\frac{1}{\hat{N}_A+\hat{N}_B+2}B^{\dagger\alpha}A^\beta, \tag{8}$$

along with

$$[A^\dagger_\alpha, A^\dagger_\beta] = [A^\alpha, A^\beta] = [B_\alpha, B_\beta] = [B^{\dagger\alpha}, B^{\dagger\beta}] = [A^\alpha, B_\beta] = [A^\dagger_\alpha, B^{\dagger\beta}] = 0, \tag{9}$$

where $\simeq$ indicates that the equality holds within the vector subspace spanned by SU(3) irreps defined in Eq. (5). Note that, the ISBs with different arguments commute, and they obey the commutation relations in Eqs. (6)- (9) only if they have the same direction and position arguments. Above,

$$\hat{N}_A \equiv A^\dagger\cdot A = \sum_{\alpha=1}^{3} A^\dagger_\alpha A^\alpha \quad \text{and} \quad \hat{N}_B \equiv B^\dagger\cdot B = \sum_{\beta=1}^{3} B^{\dagger\beta} B_\beta, \tag{10}$$

are the number operators for $A-$type and $B-$type ISBs.

The ISBs can be used to expressed electric fields as [8]

$$E^{\mathrm{a}}(L,r) = A^\dagger_\alpha(L,r)\left(T^{\mathrm{a}}\right)^\alpha{}_\beta A^\beta(L,r) - B^{\dagger\alpha}(L,r)\left(T^{*\mathrm{a}}\right)_\alpha{}^\beta B_\beta(L,r), \tag{11a}$$

$$E^{\mathrm{a}}(R,r-1) = A^\dagger_\alpha(R,r)\left(T^{\mathrm{a}}\right)^\alpha{}_\beta A^\beta(R,r) - B^{\dagger\alpha}(R,r)\left(T^{*\mathrm{a}}\right)_\alpha{}^\beta B_\beta(R,r), \tag{11b}$$

such that they obey Eq. (1). This redefinition needs to satisfy Eq. (2) leading to the following constraints on the prepotential number operators defined in Eq. (10) at each end of the link:

$$\left(\hat{N}_A(L,r) - \hat{N}_B(R,r+1)\right)|\Psi\rangle = \left(\hat{N}_B(L,r) - \hat{N}_A(R,r+1)\right)|\Psi\rangle = 0 \tag{12}$$

for any state $|\Psi\rangle$ belonging to the Hilbert created by ISBs. The constraints in Eq. (12) are known as the Abelian Gauss's law constraints of the theory. With this, $H_E$ that contains the quadratic Casimir operator defined at each end of a link can be re-written in terms of the prepotential degrees of freedom as shown in Eq. (18)

To complete the prepotenial reformulation of Hamiltonian in Eq. (4), the staggered matter is readily incorporated from the KS Hamiltonian with $H_M$ carried over unchanged. To couple the staggered mattered field with ISBs, the link operator in the term $\psi^\dagger_\alpha(r)\, U^\alpha{}_\beta(r)\, \psi^\beta(r+1)$ that appears $H_I$ can be written in terms of ISBs as [8]

$$\begin{aligned} U^\alpha{}_\beta(r) = {} & B^{\dagger\alpha}(L,r)\, \eta(r)\, A^\dagger_\beta(R,r+1) + A^\alpha(L,r)\, \theta(r)\, B_\beta(R,r+1) \\ & + \left(A^\dagger(L,r) \wedge B(L,r)\right)^\alpha \delta(r) \left(B^\dagger(R,r+1) \wedge A(R,r+1)\right)_\beta, \end{aligned} \tag{13}$$

where $(A^\dagger \wedge B)^\alpha \equiv \epsilon^{\alpha\gamma\delta} A^\dagger_\gamma B_\delta$ and $(B^\dagger \wedge A)_\beta \equiv \epsilon_{\beta\gamma\delta} B^{\dagger\gamma} A^\delta$, and the coefficients $\eta(r)$, $\theta(r)$, $\delta(r)$ can be further factored into a product of two operators, one for each end of the link, as

$$\eta(r) = \eta(L,r)\ \eta(R,r+1), \quad \theta(r) = \theta(L,r)\ \theta(R,r+1), \quad \text{and} \quad \delta(r) = \delta(L,r)\ \delta(R,r+1). \tag{14}$$

The unitarity and unit determinant conditions on $U(r)$ given by $U^\dagger(r)U(r) = \mathbb{1}_{3\times 3}$ and $\det U(r) = \mathbb{1}$, respectively, determine the form of these decomposed operators to be

$$\eta(L,r) = \frac{1}{\sqrt{B(L,r)\cdot B^\dagger(L,r)}}, \qquad \eta(R,r) = \frac{1}{\sqrt{A^\dagger(R,r)\cdot A(R,r)}}, \tag{15}$$

$$\theta(L,r) = \frac{1}{\sqrt{A^\dagger(L,r)\cdot A(L,r)}}, \qquad \theta(R,r) = \frac{1}{\sqrt{B(R,r)\cdot B^\dagger(R,r)}}, \tag{16}$$

$$\delta(L/R,r) = \frac{1}{\sqrt{\left(A^\dagger(L/R,r)\cdot A(L/R,r)+2\right)\, B^\dagger(L/R,r)\cdot B(L/R,r)}}. \tag{17}$$

Putting everything together, the prepotential reformulation of the KS Hamiltonian in Eq. (4) is given by

$$\begin{aligned} H = {} & H_M + H_E + H_I = \mu \sum_{r=1}^{N} (-1)^r \psi^\dagger(r)\cdot\psi(r) \\ & + \sum_{r=1}^{N'} \frac{1}{3}\Big(\big(\hat{N}_A(L,r)^2 + \hat{N}_B(L,r)^2 + \hat{N}_A(L,r)\hat{N}_B(L,r)\big) + \hat{N}_A(L,r) + \hat{N}_B(L,r)\Big) \\ & + x \sum_{r=1}^{N'} \Big( \left[\psi^\dagger\cdot B^\dagger(L)\ \eta(L)\right]_r \left[\eta(R)\ \psi\cdot A^\dagger(R)\right]_{r+1} + \left[\psi^\dagger\cdot A(L)\ \theta(L)\right]_r \left[\theta(R)\ \psi\cdot B(R)\right]_{r+1} \\ & \quad + \left[\psi^\dagger\cdot A^\dagger(L)\wedge B(L)\ \delta(L)\right]_r \left[\delta(R)\ \psi\cdot B^\dagger(R)\wedge A(R)\right]_{r+1} + \text{H.c.}\Big), \end{aligned} \tag{18}$$

where we used $[]_r$ for brevity to denote the position argument.

## 3. The loop-string-hadron formulation

The LSH approach ultimately replaces the SU(3)-covariant fields used in the ISB formulation with the degrees of freedom that are intrinsically SU(3)-invariant. They are constructed using the degrees of freedom in Eq. (18) by observing their transformation properties under the SU(3) gauge group. The ISB triplets $A^\dagger_\alpha$ and $B^{\dagger\alpha}$ transform as fundamental, $(1,0)$, or anti-fundamental, $(0,1)$, irrep, respectively, which can be seen from Eq. (5). For the fermionic matter field, we have taken, without a loss of generality, $\psi^\dagger_\alpha(r)$ and $\psi^\alpha(r)$ to transform as $(1,0)$ and $(0,1)$ irreps, respectively. We use these fundamental and anti-fundamental fields to construct local gauge singlets via symmetric or antisymmetric contractions with the Kronecker delta, $\delta^\beta_\alpha$, or Levi-Civita symbols, $\epsilon_{\alpha\beta\gamma}$ or $\epsilon^{\alpha\beta\gamma}$, respectively. These singlets act as the on-site building blocks of the prepotential Hamiltonian in Eq. (18). Before we proceed, we relabel the $L$ and $R$ ends of links with 1 and $\underline{1}$, respectively, in anticipation of notational convenience for higher-dimensional generalizations.

We start by noting that the physical Hilbert space is built up from the prepotential vacuum at each site $r$, $|\Omega\rangle_r$, which is defined as a state that is annihilated by each of $A^\alpha(1/\underline{1})$, $B_\alpha(1/\underline{1})$ and $\psi^\alpha$ fields, where we have suppressed the position argument for brevity. The gauge-singlets that can excite physical degrees of freedom must include all singlets constructed from only creation-type operators: $A^\dagger_\alpha(1/\underline{1})$, $B^{\dagger\alpha}(1/\underline{1})$ and $\psi^\dagger_\alpha$. There are in total twelve such gauge-singlets: (i) $A^\dagger(1)\cdot B^\dagger(1)$, (ii) $A^\dagger(\underline{1})\cdot B^\dagger(\underline{1})$, (iii) $A^\dagger(1)\cdot B^\dagger(\underline{1})$, (iv) $A^\dagger(\underline{1})\cdot B^\dagger(1)$, (v) $\psi^\dagger\cdot B^\dagger(\underline{1})$, (vi) $\psi^\dagger\cdot A^\dagger(\underline{1})\wedge A^\dagger(1)$, (vii) $\psi^\dagger\cdot B^\dagger(1)$, (viii) $\psi^\dagger\cdot\psi^\dagger\wedge A^\dagger(1)$, (ix) $\psi^\dagger\cdot\psi^\dagger\wedge A^\dagger(\underline{1})$, (x) $\psi^\dagger\cdot\psi^\dagger\wedge\psi^\dagger$, (xi) $\psi^\dagger\cdot A^\dagger(1)\wedge A^\dagger(1)$, and (xii) $\psi^\dagger\cdot A^\dagger(\underline{1})\wedge A^\dagger(\underline{1})$. However, (i) and (ii) are effectively the null operators in the prepotential Hilbert space as discussed in the previous section. Moreover, Eq. (9) implies that (xi) and (xii) are zero because of the contraction of symmetric indices with the Levi-Civita tensor. Finally, the singlets containing more than three $\psi^\dagger$ operators are also zero because of the anticommutation relations in Eq. (3).

Out of the remaining operators, (iii) and (iv) contain only bosonic operators while (v)-(vii) each has a fermionic creation operator. We thus introduce two bosonic $(n_P, n_Q)$ and three fermionic $(\nu_{\underline{1}}, \nu_0, \nu_1)$ quantum numbers to characterize a state at each site $r$. $n_P$ and $n_Q$ counts the number of excitations created by the singlet in (iii) and (iv), respectively. Similarly, $\nu_{\underline{1}}$ , $\nu_0$, and $\nu_1$ represent the fermion-like excitations of singlets in (v), (vi), and (vii), respectively. Furthermore, using Eqs. (6)-(9) along with Eq. (3), one can show that the singlet in (viii) (or (ix)) acts as a simultaneous excitations of quantum number $\nu_0$ and $\nu_1$ ($\nu_{\underline{1}}$ and $\nu_0$). Similarly, the singlet in (x) simultaneously excites all three fermionic quantum numbers $\nu_{\underline{1}}$, $\nu_0$ and $\nu_1$.

The site-local ortho-normal LSH states in the electric basis are thus characterized as,

$$|n_P, n_Q\,;\,\nu_{\underline{1}}, \nu_0, \nu_1\rangle_r\,, \qquad n_P, n_Q \in \{0,1,2,\cdots\}, \qquad \nu_{\underline{1}}, \nu_0, \nu_1 \in \{0,1\}. \tag{19}$$

The number operators associated with these quantum numbers are denoted by $\hat{n}_P(r)$, $\hat{n}_Q(r)$, $\hat{\nu}_{\underline{1}}(r)$, $\hat{\nu}_0(r)$, and $\hat{\nu}_1(r)$. They are related to $\hat{N}_A(1/\underline{1}, r)$ and $\hat{N}_B(1/\underline{1}, r)$ as

$$\hat{P}(\underline{1}, r) = \hat{n}_P(r) + \hat{\nu}_0(r)\,(1-\hat{\nu}_1(r))\,,\;\; \hat{Q}(\underline{1}, r) = \hat{n}_Q(r) + \hat{\nu}_{\underline{1}}(r)\,(1-\hat{\nu}_0(r))\,, \tag{20}$$

$$\hat{P}(1, r) = \hat{n}_P(r) + \hat{\nu}_1(r)\,(1-\hat{\nu}_0(r))\,,\;\; \hat{Q}(1, r) = \hat{n}_Q(r) + \hat{\nu}_0(r)\,\left(1-\hat{\nu}_{\underline{1}}(r)\right), \tag{21}$$

with

$$\hat{P}(\underline{1}, r) \equiv \hat{N}_A(\underline{1}, r),\;\; \hat{Q}(\underline{1}, r) \equiv \hat{N}_B(\underline{1}, r),\;\; \hat{P}(1, r) \equiv \hat{N}_B(1, r),\;\; \hat{Q}(1, r) \equiv \hat{N}_A(1, r). \tag{22}$$

This, along with Eqs. (20) and (21), can be used to provide a pictorial representation of on-site LSH basis states in Eq. (19) as shown in the supplemental Fig. 1. In the absence of any fermionic content, the gauge boson quantum numbers $n_P$ and $n_Q$ characterize the gauge boson loops. For a non-zero value of fermionic quantum numbers $\nu_{\underline{1}}$, $\nu_0$, and $\nu_1$, if at least one of the fermionic quantum number is zero then the state either sources or sinks (or both) gauge fluxes, which we call string states, otherwise it creates a hadron state. Thus, the primary on-site degrees of freedom are loops, strings and hadron, hence the name LSH.

The LSH basis for the entire lattice is a tensor product basis of local states in Eq. (19). However, such a basis state, $|\Psi\rangle$, is a physical state only if it satisfies the Gauss's law constraints in Eq. (12). Using Eqs. (10) and (20)-(22) in Eq. (12), we can re-express the Abelian Gauss's laws constraints in LSH number operators as

$$\big(\hat{P}(1,r) - \hat{P}(\underline{1},r+1)\big)|\Psi\rangle = 0 \quad \text{and} \quad \big(\hat{Q}(1,r) - \hat{Q}(\underline{1},r+1)\big)|\Psi\rangle = 0, \tag{23}$$

for each site $r$. Equation (23) gives an algebraic constraint on the eigenvalues of $P$-type and $Q$-type number operators at neighboring sites, which amounts to conservation laws on loop segments each type between them. The physical Hilbert space is then spanned by the LSH basis states for the lattice that satisfy Eq. (23).

The Hamiltonian in Eq. (18) can now be reformulated in terms of operators that act on the LSH basis states. Staring with $H_E$ which can be easily re-expressed in terms of LSH number operators $\hat{P}(1/\underline{1},r)$ and $\hat{Q}(1/\underline{1},r)$ using Eq. (22) as

$$H_E = \sum_{r=1}^{N'} H_E(r) = \sum_{r} \frac{1}{3}\left[\hat{P}(1,r)^2 + \hat{Q}(1,r)^2 + \hat{P}(1,r)\hat{Q}(1,r)\right] + \hat{P}(1,r) + \hat{Q}(1,r). \tag{24}$$

Next, the number operator $\psi^\dagger(r)\cdot\psi(r)$ in $H_M$ counts the total number of fermionic excitations at site $r$, which is given by $\hat{\nu}_{\underline{1}}(r) + \hat{\nu}_0(r) + \hat{\nu}_1(r)$. This leads to,

$$H_M = \sum_{r=1}^{N} H_M(r) = \sum_{r=1}^{N} \mu(-1)^r\left(\hat{\nu}_{\underline{1}}(r) + \hat{\nu}_0(r) + \hat{\nu}_1(r)\right). \tag{25}$$

Thus, $H_E$ and $H_M$ still remain diagonal in the LSH basis, and the dynamics is entirely governed by $H_I$.

To express the gauge singlet operators appearing in $H_I$ in Eq.(18) in terms of operators acting on LSH quantum numbers, we introduce the normalized lowering operators for bosonic LSH quantum number, $\Gamma_l(r)$, and for fermionic LSH quantum number, $\chi_f(r)$, along with their Hermitian conjugate raising operators. Here, $l = P, Q$, and $\Gamma_l(r)$ ($\Gamma_l^\dagger(r)$) lowers (raises) the $n_l(r)$ quantum number by one unit if $n_l(r) > 0$ ($n_l(r) \geq 0$). Similarly, with $f = \nu_{\underline{1}}, \nu_0, \nu_1$, $\chi_f(r)$ ($\chi_f^\dagger(r)$) lowers (raises) $\nu_f(r)$ by one unit if $\nu_f(r) = 1$ ($\nu_f(r) = 0$), otherwise annihilating the state. The site-local vacuum $|\Omega\rangle_r$, which is characterized by all LSH quantum numbers being zero, is annihilated by either type of lowering operators, $\Gamma_l(r)$ or $\chi_f(r)$.

With this, the gauge-matter interaction Hamiltonian, $H_I = \sum_{r=1}^{N'} H_I(r, r+1)$, is given by

$$
\begin{aligned}
H_I(r,r+1) = \sum_r x \Big[ & \hat{\chi}_1^\dagger (\hat{\Gamma}_P^\dagger)^{\nu_0} \sqrt{1-\hat{\nu}_0/(\hat{n}_P+2)} \sqrt{1-\hat{\nu}_{\underline{1}}/(\hat{n}_P+\hat{n}_Q+3)} \Big]_r \\
& \otimes \Big[ \sqrt{1+\hat{\nu}_0/(\hat{n}_P+1)} \sqrt{1+\hat{\nu}_{\underline{1}}/(\hat{n}_P+\hat{n}_Q+2)}\, \hat{\chi}_1 (\hat{\Gamma}_P^\dagger)^{1-\hat{\nu}_0} \Big]_{r+1} \\
+ x \Big[ & \hat{\chi}_{\underline{1}}^\dagger (\hat{\Gamma}_Q)^{1-\hat{\nu}_0} \sqrt{1+\hat{\nu}_0/(\hat{n}_Q+1)} \sqrt{1+\hat{\nu}_1/(\hat{n}_P+\hat{n}_Q+2)} \Big]_r \\
& \otimes \Big[ \sqrt{1-\hat{\nu}_0/(\hat{n}_Q+2)} \sqrt{1-\hat{\nu}_1/(\hat{n}_P+\hat{n}_Q+3)}\, \hat{\chi}_{\underline{1}} (\hat{\Gamma}_Q)^{\hat{\nu}_0} \Big]_{r+1} \\
+ x \Big[ & \hat{\chi}_0^\dagger (\hat{\Gamma}_P)^{1-\hat{\nu}_1} (\hat{\Gamma}_Q^\dagger)^{\hat{\nu}_{\underline{1}}} \sqrt{1+\hat{\nu}_1/(\hat{n}_P+1)} \sqrt{1-\hat{\nu}_{\underline{1}}/(\hat{n}_Q+2)} \Big]_r \\
& \otimes \Big[ \sqrt{1-\hat{\nu}_1/(\hat{n}_P+2)} \sqrt{1+\hat{\nu}_{\underline{1}}/(\hat{n}_Q+1)}\, \hat{\chi}_0 (\hat{\Gamma}_P)^{\hat{\nu}_1} (\hat{\Gamma}_Q^\dagger)^{1-\hat{\nu}_{\underline{1}}} \Big]_{r+1} + \text{H.c.} \qquad (26)
\end{aligned}
$$

Here, we used 'diagonal functions' of operators that can be written as a closed-form function of the number operators. For example, if $F(m,n) \equiv \sqrt{m+n+1}$ for non-negative integers $m$ and $n$, there exists a corresponding diagonal function $F(\hat{n}_P(r), \hat{n}_Q(r))$ given by $\sqrt{\hat{n}_P(r)+\hat{n}_Q(r)+1}$. The 'conditional' bosonic ladder operators of the form $(\hat{\Gamma}_l)^{\hat{\nu}_f}$ or $(\hat{\Gamma}_l^\dagger)^{\hat{\nu}_f}$ in Eq. (26) are defined as

$$
(\hat{\Gamma}_l)^{\hat{\nu}_f} \equiv \hat{\nu}_f \hat{\Gamma}_l + (1-\hat{\nu}_f), \qquad (\hat{\Gamma}_l^\dagger)^{\hat{\nu}_f} \equiv \hat{\nu}_f \hat{\Gamma}_l^\dagger + (1-\hat{\nu}_f), \qquad l \in \{P, Q\}, \qquad f \in \{\underline{1}, 0, 1\}, \quad (27)
$$

which implies the action of $\hat{\Gamma}_l^{(\dagger)}$ on a state but only if $\nu_f = 1$. Interested reader can find a detailed discussion on deriving Eq. (26) in Sec. III D in Ref. [18].

## 4. Conclusion and outlook

The LSH framework for SU(3) gauge theory in 1+1 D presented here has all the desirable features that deemed advantageous for the SU(2) case for its classical and quantum simulation: (i) the site-local degrees of freedom are strictly gauge-invariant, (ii) the Hamiltonian is written explicitly in terms of LSH degrees of freedom and it remains local, (iii) the non-local feature of gauge theory is incorporated through two Abelian Gauss's laws that are simultaneously diagonalizable, and (iv) the gauge-matter interaction term in Eq. (26) has the same structure as its SU(2) counterpart given in Ref. [12], which is advantageous for quantum simulation implementation of Eq. (26) because it reduces the number of distinct terms from 27 (and H.c.) in the KS formulation to 3 (and H.c.).

We performed a numerical analysis and compared the eigenvalues in the LSH framework against a gauge fixed or purely fermionic formulation [23–25] for a lattice with $N = 2, \mu = 1, x = 1$ and an open boundary condition with zero background flux. The results are shown in the supplemental Fig. 2, where eigenvalues match up to machine precision, providing a numerical validation for this framework.

Further explorations regarding the SU(3) LSH formulation should have immediate relevance toward the goal of quantum simulation lattice QCD. Next steps towards that aim include generalizations to higher spatial dimensions and the extension to multiple fermion flavors. Furthermore, development of quantum simulation or tensor network algorithms using this LSH framework for SU(3) lattice gauge theory would be its interesting applications.

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