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Hadron spectroscopy and interactions from lattice QCD

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The methods to precisely calculate the hadron spectroscopy and scattering amplitudes in lattice QCD is becoming mature in recent years. There has been a lot of progress in the implementation of these methods to study resonance properties. I review the new results in the last couple of years, with an emphasis on the exotic hadrons beyond the conventional quark model. The current challenges and future opportunities are also discussed.

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1. Introduction

Hadron spectroscopy is an important part in the understanding of the strong interactions and its underlying theory – quantum chromodynamics(QCD). Simply speaking, studying spectroscopy is to obtain the masses of the hadrons. More importantly, it helps us understand how hadrons are built from the fundamental degrees of freedom of QCD — quarks and gluons. Although the QCD theory allows for the existence of any type of colorless hadrons, for a long time the hadrons observed in experiments all fitted into the simple picture of the conventional quark model, i.e., hadrons are composed of either a quark-antiquark pair or three quarks. Only since 2003, new types of hadrons were started to be observed in experiments [1–10], triggered tremendous interest and effort in the study of hadron spectroscopy. See the references [11–13] for reviews on experimental results and theory interpretations. Many experiments around the world, such as BES III, Belle-II, LHCb and the JLab 12GeV upgrade, are keeping producing rich results on hadron spectra. On the theory side, hadron spectroscopy is a very active area in phenomenology and effective field theory studies, which provide important insights [14–17]. However, studies from first principles are crucial to boost our understanding of hadron spectroscopy.

Lattice QCD is the only non-perturbative method to study hadron spectroscopy from first principles with controlled statistical and systematic uncertainties. For the ground state spectrum, lattice calculations has aimed at high precision. A milestone in these kind of calculations is the neutron-proton mass splitting obtained by the BMW collaboration [18]. In this work, the systematics in lattice calculations, including finite lattice spacing, heavier than physical quark masses, finite volume effects, isospin-breaking effects and the quantum electrodynamics(QED) corrections, are all treated carefully. The exited hadronic states are generally resonances: they decay into multiple particles. They are manifested as poles in the scattering amplitudes. A well established method to study scattering processes in lattice QCD is the Lüscher's finite volume method [19], which relates the finite-volume spectrum of a two-particle system to the scattering parameters of the two particles in the infinite volume. The finite-volume spectrum can be calculated rather straightforwardly in lattice QCD, then the scattering amplitudes can be obtained through Lüscher's method. Another formalism is the HALQCD method [20, 21], which utilizes the so-called Nambu-Bethe-Salpeter wavefunction that can be directly measured on the lattice and then solver the relevant Schr"odinger equation to get the two-particle interactions.

Lüscher's formula was originally obtained for two identical scalar particles [19]. It has been generalized to any two-particle systems — in moving frames, with arbitrary spin, different masses and multiple coupled channels [22–29]. The formulas for three-body scattering are also developing [30–32]. For the three-body scattering formalism and applications, I refer to Romero-López's topical review [33]. In the last decade, there have been significant progresses in the application of Lüscher's formula. As a benchmark, ρ resonance has been calculated with rather good precision [34–41]. One of the challenges at present is to control the systematics arising from the finite lattice spacing, unphysical light quark mass and finite volume effects. Another challenge is to handle multiple coupled channels. It requires large number of finite-volume energy levels to reliably determine the many scattering parameters in coupled channels. Most of the coupled channel scattering calculations done by now are for light mesons mainly contributed by the Hadron Spectrum Collaboration [36, 42–46], a few involves charmed mesons [47, 48].

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This review partly cover the new results on hadron spectroscopy and interactions in the last couple of years, with an emphasis on the exotic hadrons beyond the conventional quark model. It should be noted that some topics will not be discussed here. The three-body interactions is reviewed by Romero-López [**?**]. For the glueball spectrum I refer to Vadacchino's talk [49]. This review is organized as follows. In Sec. 2, I recapitulate the theory methodology utilized in the spectrum calculations. In Sec. 3, I will go over the recent results in the last couple of year, including meson-meson, meson-baryon and baryon-baryon scatterings.

2. Methodology

Using Lüscher's method to calculate scattering amplitudes requires computing the energy eigenstates in a finite-box, preferably with large number of energy levels in a certain kinematic range. One starts from constructing a set of interpolating operators $\{O_i, i = 1, 2, 3, \cdots, N\}$ carrying the same quantum numbers of interest, then compute the correlation functions

$$C_{ij}(t) = \langle 0|O_i(t)O_j^\dagger(0)|0\rangle = \sum_n \langle 0|O_i|n\rangle\langle n|O_j^\dagger|0\rangle e^{-E_n t}, \tag{1}$$

Solving the generalized eigenvalue problem(GEVP)

$$C(t)v^n(t) = \lambda^n(t)C(t_0)v^n(t), \tag{2}$$

the energies can be extracted from the time dependence of the eigenvalues $\lambda^n(t) \sim e^{-E_n(t-t_0)}$ [50]. Operators that have good overlap onto the states of interest is essential to get the correct and complete energy levels. For example, in order to correctly identify the ρ resonance, both quark bilinear operators and the $\pi\pi$ operators should be included [36, 38]. In practice, one should build a reasonably large set of operators with diverse structures. Two-particle operators in moving frames also helps us get more energy levels. The $SO(3)$ rotation symmetry in continuum space breaks down to cubic group on lattice. Therefore one should build the operators that transform in the irreducible representations(irreps) of cubic group rather than with certain angular momentum J. The operator construction methods have been discussed in some literature [51–57]. A novel quark smearing method called distillation method [58, 59] is very advantageous in the spectrum calculation. This method enables us to greatly improve the precision with affordable cost and conveniently compute the correlation functions of many interpolating operators. The disconnect diagrams can also be computed straightforwardly with this method. It has been applied in many spectrum and scattering studies in recent years.

The next step is to obtain the scattering information through L"uscher's finite volume method. In the simplest case, considering the s-wave scattering of two identical scalar hadrons with mass m, the L"uscher's formula that related the infinite-volume scattering phase shift $\delta_0(p)$ and the finite-volume energy reads

$$p\cot\delta_0(p) = \frac{2}{L\sqrt{\pi}}\mathcal{Z}_{00}(1; q^2), \tag{3}$$

where p is related to the finite volume energy E_L via $E_L = 2\sqrt{p^2 + m^2}$, and $q = pL/(2\pi)$. By calculating the values of E_L on lattice, one can obtain the scattering phase shift $\delta_0(p)$.

The above L"uscher's formula has been generalized to any two-particle systems. The general formula, usually referred to as quantization condition, can be written as

$$\det[K(E_L) + F^{-1}(E_L, L)] = 0, \tag{4}$$

where K is the K-matrix for the infinite-volume scattering amplitude, F is a known geometric function that reflects finite-volume information. The determinant is in the space of channels and partial waves. Note that the quantization condition provides one constrain for a specific energy. In the case of coupled channels, there are more than one scattering parameters. Therefore, we can not determine all scattering parameters at this specific energy from the quantization condition. We need to parameterize the energy dependence of the scattering matrix over some energy range. For example, the scattering matrix can be parameterized by the effective range expansion in the energy range that is close to the threshold:

$$p^{2l+1} \cot \delta_l(p) = \frac{1}{a_l} + \frac{1}{2} r_l p^2, \tag{5}$$

where a_l is the scattering length of partial wave l and r_l is the effective range. Various parameterization schemes have been implemented in the coupled-channel calculations [36, 42–46]. In principle, different parameterization should induce the same resonance properties.

3. Recent results

In this section, I will review recent progresses in lattice calculation of hadron spectroscopy and interactions. I will start from the meson-meson scattering, mainly on the heavy meson scattering channels that related to exotic hadrons. Then I will discuss the scattering involves baryons, including meson-baryon and baryon-baryon scatterings.

3.1 Meson-meson scattering

3.1.1 $D\bar{D}^*$ scattering and T_{cc}

The LHCb collaboration reported a very narrow structure T_{cc} in the $D^0 D^0 \pi^+$ invariant mass spectrum [8, 9]. According the its decay channels, the quark content should $cc\bar{u}\bar{d}$. The mass of T_{cc} is only 0.36MeV below $D^0 D^{*+}$ threshold and the width is 47.8keV. There are some phenomenology models suggest that T_{cc} is a bound state formed by D and D^* [60–66]. Currently there are two lattice calculations have investigated this state [67, 68].

In Ref. [67], the authors calculated the isospin-0 DD^* scattering amplitude on ensembles with pion mass $M_\pi \sim 280$ MeV and lattice spacing $a \sim 0.086$ fm. The finite-volume energies are computed using the two-meson operators in the rest frame as well as moving frames with total momentum $P = 1, \sqrt{2}$, and 2(in units of $2\pi/L$). The scattering amplitudes for partial waves $l = 0, 1$ are obtained via Lüscher's method. A virtual bound state pole is found in the s-wave DD^* scattering channel. The dependence of the pole position on quark masses is discussed, suggesting that at smaller pion mass the virtual bound state pole would become a bound state. Figure 1 shows the finite-volume energies and the extracted scattering amplitude.

Ref. [68] computed the isospin $I = 0, 1$ DD^* scattering on an anisotropic ensemble with pion mass $M_\pi \sim 350$ MeV. The temporal lattice spacing $a_t^- 1 = 6.894$ GeV and the anisotropy

$\xi = a_s/a_t = 5.3$. The two-meson operators in the rest frame are used to calculate the finite-volume energies. The scattering length and effective range are obtained through Lüscher's formula and the effective range expansion parametrization of the scattering amplitude. Figure 2 present the values of $p\cot\delta_0$ as a function of q^2 and the fit of $p\cot\delta_0$ data points to the effective range expansion. The scattering length and effective range for $I = 0$ and $I = 1$ channels are

$$a_0^{(I=0)} = 0.538(33)\text{fm}, \quad r_0^{(I=0)} = 0.99(11)\text{fm}, \tag{6}$$

$$a_0^{(I=1)} = -0.433(43)\text{fm}, \quad r_0^{(I=0)} = -3.6(1.0)\text{fm}. \tag{7}$$

The results for $I = 0$ channel are in line with the results presented in Ref. [67]. The major observation from the results is that the interaction of DD^* is attractive in the $I = 0$ channel while it is repulsive in the $I = 1$ channel. By comparing the difference in these two channels, it is observed that the charged vector ρ exchange in the hadron level contributes to the $I = 0$ and $I = 1$ correlation functions with opposite signs. As a results, it raises the DD^* energy in the $I = 1$ channel and lower the energy of the $I = 0$ channel. This implies that the DD^* interaction induced by the charged ρ exchange plays a crucial role in the formation of T_{cc}, which agrees with the phenomenological studies [65, 69].

The above two works provide valuable information about the doubly charmed tetraquark T_{cc}. However, further investigations are required to clarify the properties of T_{cc}. The quark mass $m_{u/d}$ dependence and discretization effects should be studied by doing the calculations at various values(preferably smaller) of quark mass and lattice spacing. Interpolating operator with different structures, such as diquark-antidiquark operators, should be included. Performing the calculation at physical $m_{u/d}$ will be very challenging. If T_{cc} were a DD^* bound state, the binding energy would be only 0.36MeV at the physical $m_{u/d}$, which requires super high precision to resolve such a small binding energy. Also, at the physical $m_{u/d}$, D^* decays to $D\pi$ and T_{cc} decays to $DD\pi$. Therefore, three-body interactions needs to be considered.

3.1.2 BB^* scattering and T_{bb}

Although T_{bb} has not been observed in experiments, it is likely to be a candidate of heavy tetraquark state with quark flavor $bb\bar{u}\bar{d}$ and quantum number $I(J^P) = 0(1^+)$. Several lattice studies predicted T_{bb} as a bound state [70–75], while the binding energy obtained in these studies are different. A recent work calculated the scattering of the coupled channels BB^* and B^*B^* using the HALQCD method [76] and confirmed the existence of a bound state. They also investigated possible reasons that may cause the differences of the binding energy in the previous studies. A major observation is that coupled channel effects reduce the binding energy by around 50%. Using static quark action or NRQCD for bottom quarks can also make differences in the binding energy. Figure 3 presents the results of $pcot\delta(p)$ in the BB^* channel as a function of the energy measured from the BB^* threshold. A bound state pole (the intersection of $pcot\delta(p)$ with $-\sqrt{-p^2}$) is found for each of the three pion mass $m_\pi = 701, 571$ and 416 MeV. The binding energy at physical pion mass is obtained by doing the extrapolation linearly in m_π^2 and the result is $E_{\text{binding}} = -83.0 \pm 10.2$MeV. If only consider the single channel BB^*, the obtained binding energy is $E_{\text{binding}} = -154.8 \pm 17.2$MeV, which agrees with the previous single channel calculations [72–75].

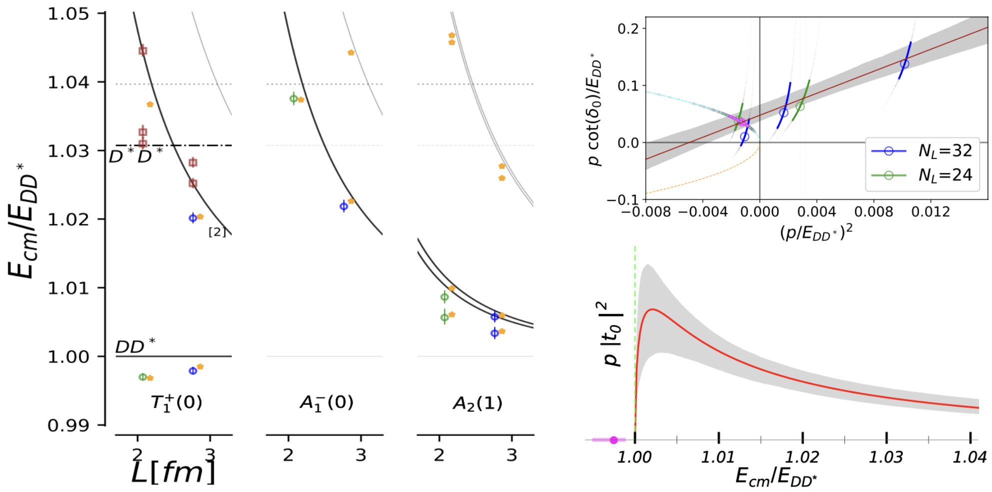

Figure 1: Left: The finite-volume energies of DD^* system in the center-of-momentum frame in various irreps, normalized by $E_{DD^*} \equiv m_D + D_{D^*}$. The lattice energy levels are shown by large circles and squares. The non-interacting energies are shown by lines. Right top: $p\cot\delta$ for the s-wave DD^* scattering(red line), also normalized by E_{DD^*}. The cyan(orange) line is $ip = |p|(-|p|)$ versus p^2. The virtual bound state occurs at the momenta where the red line and the cyan line intersect, indicated by the magenta point. Right bottom: The corresponding DD^* scattering rate $N \propto p|t_0|^2$, where t_0 is the s-wave scattering t-matrix.

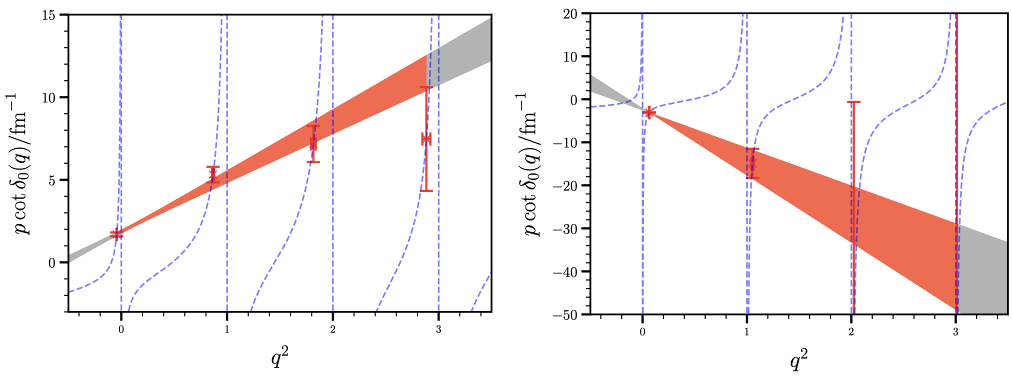

Figure 2: The values of $p\cot\delta$ as a function of p^2 for the s-wave DD^* scattering. The left panel is for isospin $I = 0$ and the right panel is for $I = 1$. The red band is the fit to the effective range expansion formula.

3.1.3 Charmonium-like states

In the last two decades, large amount of charmonium-like states which do not fit well into the conventional quark model have been observed in experiments. The properties of these states remain unclear. Lattice calculations have investigated some of the charmonium-like states, such as the $X(3872)$ [77–79] and the charged Z_c states [79–86]. Here I introduce recent works on the scattering of $D\bar{D}$ and $D_s\bar{D}_s$ [48, 87, 88]. The finite-volume energies are obtained from the correlation matrix of a number of interpolating operators of $\bar{c}c$, $D\bar{D}$, $D_s\bar{D}_s$ and $J/\psi\omega$ with quantum numbers isospin $I = 0$ and $J^{PC} = 0^{++}$, 1^{--}, 2^{++} and 3^{--}. The infinite-volume scattering

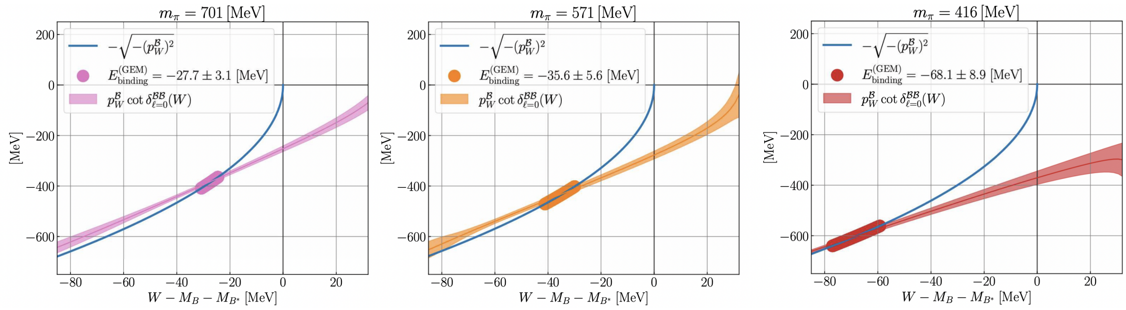

Figure 3: The results of $pcot\delta(p)$ of BB^* channel as a function of the energy measure from the the BB^* threshold for three different pion masses $m_\pi = 701$, 571 and 416 MeV. The blue curve is $-\sqrt{-p^2}$.

matrices are determined from the finite-volume energies via Lüscher's formalism. For the positive parity channels 0^{++} and 2^{++}, two coupled channels $D\bar{D}$ and $D_s\bar{D}_s$ are considered in the scattering analysis [48]. For the negative parity channels 1^{--} and 3^{--}, single channel approximation was used [88]. $J/\psi\omega$ was treated as decoupled. The masses of the charmonium(-like) resonances and bound states are related to the poles of the scattering matrix. The results are presented in Figure 4 along with the experimental values. There are several interesting observations in the results:

- There is a bound state pole in the 0^{++} $D\bar{D}$ scattering amplitude near the threshold with binding energy $-4.0^{+3.7}_{-5.0}$MeV. It is likely not a conventional charmonium state which has not been observed in experiments.
- A resonance pole is found in 2^{++} $D\bar{D}$ scattering amplitude with partial wave $l = 2$. It is most likely related to the conventional charmonium state $\chi_{c2}(3930)$.
- A broad resonance is found in the 0^{++} $D\bar{D}$ channel through coupled $D\bar{D}$-$D_s\bar{D}_s$ scattering analysis. The mass and coupling are consistent with the experimental values of $\chi_{c0}(3860)$.
- The coupled $D\bar{D}$-$D_s\bar{D}_s$ scattering with $J^{PC} = 0^{++}$ also indicates a narrow resonance slightly below the $D_s\bar{D}_s$ threshold. It mainly couples to $D_s\bar{D}_s$ and is likely related to the $X(3915)/\chi_{c)}(3930)$ observed in experiments.

3.2 Meson-baryon scattering

In this subsection, I introduce a recent work on the meson-baryon scattering that is related to the hidden-charm pentaquarks.

In 2015, LHCb collaboration reported two hidden-charm pentaquark states $P_c(4450)$ and $P_c(4380)$ [6]. A later analysis based on a data sample that is an order of magnitude larger shows that the $P_c(4450)$ splits into two structures $P_c(4440)$ and $P_c(4457)$, and a third narrow peak $P_c(4312)$ emerges [7]. Numerous theoretical investigations on the nature of the P_c states followed these discoveries. Several theoretical interpretations have been proposed, including hadronic molecules [89–105], compact pentaquark states [106–108] and hadrocharmonia [109, 110]. It is readily observed that the mass of $P_c(4312)$ is close to the $\Sigma_c\bar{D}$ thresholds while the $P_c(4440)$ and $P_c(4457)$ are close to the $\Sigma_c\bar{D}^*$ threshold. The molecule interpretation naturally explains all of

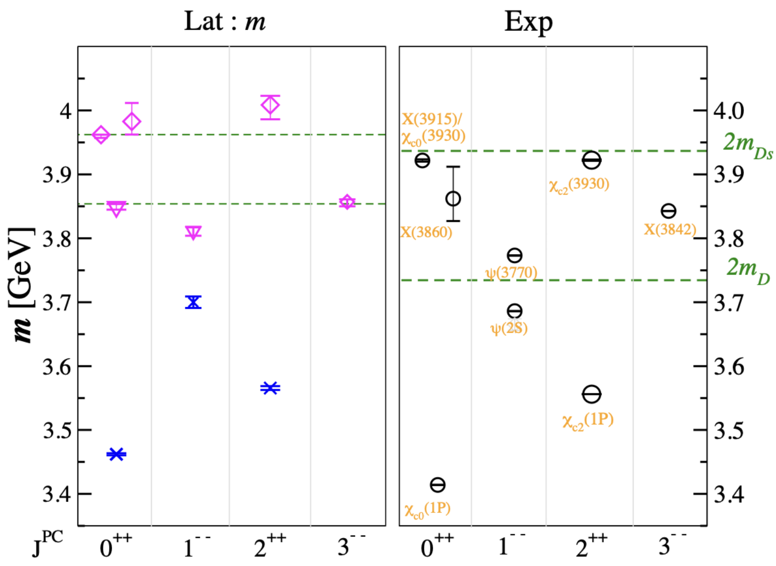

Figure 4: The masses of the charmonium(-like) states with quantum number $J^{PC} = 0^{++}, 1^{--}, 2^{++}$ and 3^{--}. The left panel shows the results from the lattice calculation and the right panel presents the experimental values. The green lines indicate the DD and D_sD_s thresholds.

the three narrow P_c states as spin multiplets of the $\Sigma_c\bar{D}^{(*)}$ bound states. In most of the literatures, the $P_c(4312)$ is explained as a $J^P = \frac{1}{2}^-$ $\Sigma_c\bar{D}$ bound state, while the $P_c(4440)$ and $P_c(4457)$ are the $\Sigma_c\bar{D}^*$ bound states with quantum numbers $\frac{1}{2}^-$ and $\frac{3}{2}^-$(or $\frac{3}{2}^-$ and $\frac{1}{2}^-$) respectively. Lattice calculations of $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ scattering will provide valuable information on the structure the P_c pentaquarks.

An exploratory study of $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ scattering on lattice has been carried out in Ref. [111]. Lüscher's finite volume method is used to extract the scattering amplitude. The finite volume energies are calculated using the $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ interpolating operators with quantum number $I(J^P) = \frac{1}{2}(\frac{1}{2}^-)$ on two ensembles with different volume and the same pion mass $\sim$ 300MeV and lattice spacing $\sim$ 0.08fm. It is observed that the mixing between $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ operators is negligible, therefore the authors performed scattering analysis separately for the $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ channels. Figure 5 shows the results of $p\cot\delta(p)$ as a function of p^2 for the two channels. Bound state poles are found in both channels. The binding energy is 6(2)(2) MeV and 7(3)(1) MeV for the $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ respectively, where the first error is statistical error and the second is the systematic error due to the lattice artifacts.

Below the $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$ thresholds, there are several channels that can couple to $\Sigma_c\bar{D}$ and $\Sigma_c\bar{D}^*$, i.e. $J/\Psi\pi$, $\eta_c\rho$, Λ_cD and Λ_cD^*. In principle, these coupled channels should be taken into account. One should build the interpolating operators for all these channels and obtain all the energies in the range from the lowest threshold up to the mass of the P_c states. Then use the couple-channel Lüscher's formula to determine the scattering amplitude. However, since the threshold of $J/\Psi\pi$ is far below the P_c states, it is very challenging to resolve all the relevant energies. Also, determining the scattering parameters of 6 coupled channels via Lüscher's method will be very difficult. Quark mass dependence and lattice artifacts should also be addressed in the future studies.

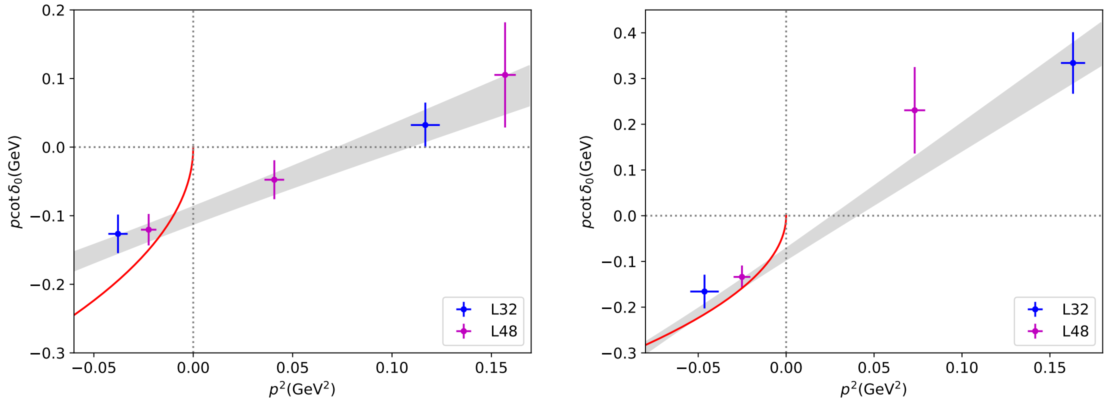

Figure 5: Fit of the $p\cot\delta_0$ to the effective range expansion for the $\Sigma_c\bar{D}$ scattering. The red curve is $ip = -|p|$ versus p^2. The grey band is the fit to the effective range expansion. The bound state pole occurs at the intersection of the the grey band and the red curve.

3.3 Baryon-baryon scattering

In this subsection, I review recent progresses of $N - N$ scattering.

Lattice QCD calculation of the two-nucleon system is a long-standing challenge because of the severe signal-to-noise problem. Up to now, all the calculations have been carried out at rather large pion mass, for which the signal-to-noise ratio decays slower than for the physical pion mass. There are discrepancies in the results of various lattice calculations. Some results suggest that there is a bound state in the $N - N$ scattering at the studied pion mass [112–116], while other results do not support the bound state scenario [56, 117, 118]. A possible reason for the discrepancy is that the differences of the energies obtained from the correlation functions of different two-nucleon interpolating operators may be non-negligible. A recent work carefully extracted the finite-volume energies from GEVP using a large set of interpolating operators including hexaquark, dibaryon and "quasi-local" operators [56]. See the reference for the detailed information about the operators. Figure 6 (taken from Ref. [56]) compared the results of the phase shift of $N-N$ scattering for isospin $I = 1$(left) and $I = 0$(right) from various lattice calculations. The black circles are the results of Ref. [56]. The brown diamonds are the results of Ref. [118], in which dibaryon operators in several moving frames are used to extract the energies. Ref. [117] also used the dibaryon operators and the results are shown in the figure as green squares. Ref. [114] and [115] used dibaryon operator at sink and hexaquark at source, the results are shown in the figure as cyan circles and yellow diamonds. It can be seen that the results of Ref. [56], which has the largest operator basis, agrees with the results using dibaryon operators at both sink and source [117, 118], while has certain discrepancies with the results using dibaryon operators at sink and hexaquark operators at source [114, 115].

4. Summary and outlook

Lattice QCD study of hadron spectroscopy has achieved significant progresses in the last decade, from precise determination of ground state spectrum to general studies of the excited and

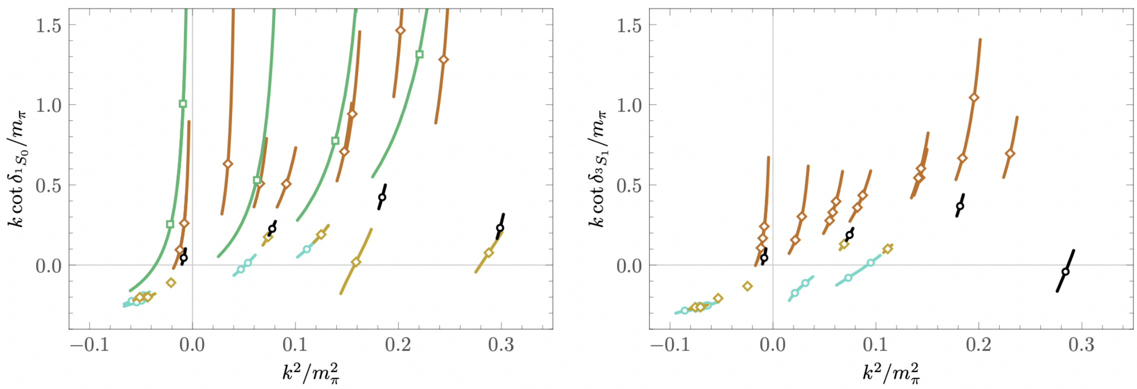

Figure 6: Comparison of the isospin $I = 1$(left) and $I = 0$(right) s-wave $N - N$ scattering phase shift from various lattice calculations [56, 114, 115, 117, 118].

exotic states that involve multi-particle interactions. There are several frontiers that need to develop in the future:

- Precision frontier.

  Besides the statistical uncertainties, lattice calculations suffers systematical uncertainties arising from the unphysical quark mass, finite volume and finite lattice spacing. Some quantities also require to consider the isospin breaking effects and QED corrections. Currently lattice calculation of the ground states have achieved high precision that is comparable to experiments and have controlled the systematics pretty well. However, studies on the resonances have not been able to control all the systematics in general. This problem largely depends on the available gauge configurations.

- Coupled channels.

  In most cases, the study of resonances or exotic states requires calculating the scattering of several coupled channels. For example, there are 6 coupled channels relevant to the hiddden-charm pentaquarks as mentioned above. The formalism to study coupled channel scattering in lattice QCD is ready. However, it is still difficult to reliably determine a large number of scattering parameters in coupled channels via Lüscher's method. One must obtain sufficient number of energy levels precisely in the relevant energy range with a large set of carefully constructed operators and variational analysis. Appropriate parametrization of the scattering amplitudes is also important in the scattering analysis.

- Multi-particle scattering.

  The study of three or more particle scattering in lattice QCD is becoming an active area in recent years. Many resonances have three particle decay modes, for example, the roper resonance $N(1440)$, the tetraquark T_{cc}, etc. The formalism for three-particle interaction in the frame work of the Lüscher's finite-volume method has made significant progresses. It has been applied in the calculations of some simple three-particle system such as $\pi\pi\pi$. In the future, the formalism needs to be extended to more general cases to include non-identical particles with spin. The roper resonance, which decay to both two particles $N\pi$ and three particles $N\pi\pi$, would be of particular interest in the application of the three-particle scattering formalism in the near future.

- Nucleon and nuclei system with near physical pion mass.

  Lattice calculation of multi-nucleon system at physical pion mass suffers severe signal-to-noise ratio problem. In addition, at physical pion mass, there will be many open channels with multi-particles. This problem probably will remain a challenge in the near future. In recent years, the quantum computing technology is developing fast. Its application in lattice QCD has also started to be explored. The new technology may help us solving this problem.

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