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Non-perturbative determination of couplings in Polyakov loop effective theories

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Polyakov loop effective theories have been shown to successfully describe the thermodynamics of QCD. Furthermore, due to the sign problem, they represent an alternative avenue to investigate the physics at non-zero chemical potential. However, when working with these effective theories, a new set of couplings appear whose expressions in terms of the gauge coupling and N_τ are only known from strong-coupling expansions. Using the finite-cluster method, we can show how one can efficiently compute high-order expressions for correlators of Polyakov loops in the effective theory which are directly mapped to those in full lattice QCD. These can then be in turn be used to determine the effective couplings as a function of temperature. Furthermore, the inclusion of heavy quarks allows one to investigate the cold and dense regime.

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1. Introduction

The study of the phase diagram of quantum chromodynamics (QCD) represents a long-standing goal of non-perturbative approaches to gauge theories. The presence of the infamous "sign problem" represents a serious hurdle for lattice approaches at finite chemical potential. Although there has been considerable progress made in recent years using alternative approaches, a detailed quantitative understanding of the finite-density regimes of starting from the original QCD Lagrangian is still lacking.

One particular successful approach to finite-density QCD, both in the continuum and on the lattice, has been the use of three-dimensional Polyakov loop effective theories (PEFT) [1–3]. These much simpler, bosonic theories describing the many-body interactions of the order parameter for the deconfinement transition accurately describe the thermodynamics of the phase transition. Furthermore, due to the fact that in going from the full QCD action to the effective theory many degrees of freedom have been integrated out, these models are known to have much milder sign problems. It is with this in mind that one hopes that these theories can play a further role in elucidating the physics of QCD at finite-density.

The effective theory can be obtained directly by integrating out the spatial gauge links in a strong-coupling expansion [3]. In doing so, one sees how n-body terms between Polyakov loops in various representations are systematically generated. The insight gained from this approach is that one immediately sees how interactions over both increasingly large distances and increasingly larger number of Polyakov loops are parametrically suppressed. This in agreement with the conjecture made by Svetitsky and Jaffe [4]. However, the strong-coupling estimates for the couplings are known to receive large corrections in the vicinity of the deconfinement phase transition. Furthermore, a detailed accounting of long-range couplings is necessary in order to describe correlation functions within the PEFT.

For Yang-Mills theory, previous studies successfully showed how one could non-perturbatively map out the gauge coupling dependence of the various terms in the effective theory [5]. This approach exploits the fact that there are multiple representations of the effective theory. In one representation, each coupling can be expessed exactly as a correlation function of Polyakov loops in full QCD which can be computed non-perturbatively. In the second representation, inspired by resummation procedures in strong-coupling expansions, the above-mentioned Polyakov loop correlators can be expressed as a rapidly convering series in the effective couplings. These perturbative series can be computed to arbitrarily high order using techniques successful in studying high-temperature expansions in lattice spin models. Using this formalism, the addition of fermions can be incorporated in a quite straightforward manner. It is in this direction that the work described here is focused on.

2. Setup

2.1 Basics

In the construction of the effective theory starts one starts from the full lattice QCD partition function and separates the integration over temporal and spatial gauge links

$$Z = \int [dU][d\psi d\bar{\psi}] e^{-(S_G+S_F)} = \int [dU_0][dU_i] \, (\det Q)^{N_f} \;\; e^{-S_G}, \tag{1}$$

where S_W is the Wilson plaquette gauge action and we have N_f mass-degenerate quarks described by the unimproved Wilson Dirac operator Q. The general strategy is to perform a dual expansion in both the hopping parameter κ as well as in the inverse gauge-coupling β and then perform the gauge link integration. Early strong-coupling studies of Yang-Mills theory applied linked-cluster expansions in order to directly compute the free energy as a series in β [6]. In this approach, one identifies the appropriate graphs which contribute to the free energy and then performs the integration over both spatial and temporal gauge links [7].

Alternatively, one could perform the integration just over the spatial links, leaving an effective action written in terms of the Polyakov loop [3]

$$-S_{\text{eff}} = \log \int [dU_i] e^{-S[U]} \, (\det Q)^{N_f} \, . \tag{2}$$

The terms in the effective action can be grouped into terms that are Z_3-symmetric and those that are generated by the fermion determinant which break this global symmetry. The long-term goal of this program is to identify and accurately characterize a set of terms which would allow one to study light fermions at non-zero chemical potential. In the current study, the addition of static quarks allows for the study of the heavy-quark region of the Columbia plot [8–10].

2.2 Representations of Effective Action

Following [5] and [11], the full effective Polyakov loop action of Yang-Mills theory should take the form

$$S_{\text{symm}} = \sum_{\mathbf{x},\mathbf{r}} \sum_{n} {\sum_{\{\mathbf{x}_i,\mathbf{r}_i\}}}' c^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \prod_{i=1}^{n} L_{\mathbf{r}_i,\mathbf{x}+\mathbf{x}_i}, \tag{3}$$

where $L_{\mathbf{r},\mathbf{x}}$ refers to the trace of the Polyakov loop at spatial lattice site $\mathbf{x}$ in the irreducible representation of $SU(3)$ labeled by $\mathbf{r} \equiv (r_0, r_1)$. The terms in (3) respect both the O_3 symmetry of the cubic lattice as well as the global Z_N center symmetry. The couplings, $c^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}}$, can in principle be calculated in a strong-coupling expansion and receive corrections in powers of κ^2 when quarks are introduced. In practice, however, this form of the effective action is impractical to deal with as correlations between Polyakov loops at arbitrary separation do not exist as soon as one truncates. This can be seen by exponentiating the action in (3) and directly performing a character expansion

$$e^{-S_{\text{symm}}} = \tilde{\mathcal{N}} \left(1 + \sum_{\mathbf{x},\mathbf{r}} \sum_{n} {\sum_{\{\mathbf{x}_i,\mathbf{r}_i\}}}' \tilde{\lambda}^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} L_{\mathbf{r},\mathbf{x}} \prod_{i}^{n} L_{\mathbf{r}_i,\mathbf{x}+\mathbf{x}_i} \right), \tag{4}$$

where $\tilde{\mathcal{N}}$ is a normalization constant and the couplings $\tilde{\lambda}$ can be expressed in terms of the previously introduced c's. The key observation is that by employing the orthogonality of the characters of the irreducible representations one can directly relate these to correlation functions in full QCD

$$\tilde{\lambda}^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \propto \langle L_{\mathbf{r},\mathbf{x}} \prod_i^n L_{\mathbf{r}_i,\mathbf{x}+\mathbf{x}_i} \rangle_{\text{QCD}}. \tag{5}$$

Thus, there exists a way to directly measure couplings non-perturbatively.

Inspired by resummations of diagrams in the linked-cluster expansion derivation of the effective action, one can introduce an alternative form which is known as the "log"-action. This takes the following form

$$e^{-S_{\text{symm}}} = \mathcal{N}_0 \prod_{\mathbf{x},\mathbf{r},n} {\prod_{\{\mathbf{r}_i,\mathbf{x}_i\}}}' \left[1 + \lambda^{\mathbf{r}}_{\{\mathbf{x}_i,\mathbf{r}_i\}} \left(L_{\mathbf{r},\mathbf{x}} \prod_i^n L_{\mathbf{r}_i,\mathbf{x}+\mathbf{x}_i} + \text{c.c.} \right) \right], \tag{6}$$

where $\mathcal{N}_0$ is a normalization constant and an additional set of couplings, λ_i, has been introduced. One notices that, using this form of the action, by simply including the two-body nearest-neighbor interaction between Polyakov loops in the fundamental representation, whose coupling is denoted by λ_1, one immediately gets long-range correlations.

The idea of the non-perturbative determination of the couplings becomes clear when comparing (5) and (6). One computes a Polyakov loop correlator non-perturbatively in the full lattice theory and expresses this as a perturbative series in the couplings λ_i, which are sufficiently small so that one can hope to obtain a good estimate at sufficiently low order. To organize this perturbative expansion and to determine an appropriate truncation condition, we first look at the estimates for the couplings obtained at strong-coupling. For the most relevant two-body interactions, the couplings have the following leading-order scaling

$$\begin{aligned}
\lambda_1 &\equiv \lambda^f_{(1,0,0),\bar{f}} \propto u^{N_\tau}, \\
\lambda_2 &\equiv \lambda^f_{(1,1,0),\bar{f}} \propto u^{2N_\tau+2}, \\
\lambda_3 &\equiv \lambda^f_{(2,0,0),\bar{f}} \propto u^{2N_\tau+6}, \\
\lambda_4 &\equiv \lambda^f_{(1,1,1),\bar{f}} \propto u^{3N_\tau+4}, \\
\lambda_5 &\equiv \lambda^f_{(3,0,0),\bar{f}} \propto u^{3N_\tau+4}, \\
\lambda_a &\equiv \lambda^a_{(1,0,0),a} \propto u^{3N_\tau+4}, \\
\lambda_s &\equiv \lambda^s_{(3,0,0),\bar{s}} \propto u^{3N_\tau+4},
\end{aligned} \tag{7}$$

where $f \equiv (1,0)$, $a \equiv (1,1)$, and $s \equiv (2,0)$ refer to the fundamental, adjoint, and sextet representations while the barred versions denote the complex conjugate representations. Here, all β dependence is contained in u, which is the character expansion coefficient of the fundamental representation for the Wilson gauge action normalized by the trivial representation. The coefficient of N_τ in the exponent in denotes the order to which a single power of λ_i contributes to the perturbative expansion of the correlators. By inspection, one sees that, for two-body interactions in the fundamental representation, this coefficient corresponds to the "taxi-driver" distance between the

two Polyakov loops. Higher-body interactions are expected to be subleading with respect to the couplings listed above and are thus neglected to a leading approximation.

As mentioned previously, one would like to extend this non-perturbative analysis of the effective theory to the deconfinement transition in the heavy-quark region. This necessitates the introduction of terms which break the center symmetry. In the log-action, static quarks can be introduced by adding a local term which arises from the following decomposition of the fermion determinant

$$\det Q = \det\left(\mathbb{I} - \kappa \mathbf{T}\right) \det\left(\mathbb{I} - \left(\mathbb{I} - \kappa \mathbf{T}\right)^{-1} \kappa \mathbf{S}\right), \tag{8}$$

where $\mathbf{T}$ and $\mathbf{S}$ refer to the sum of positive and negative hops in the temporal and spatial directions which are present in the Wilson Dirac operator. The first term in (8) describes static quarks winding around the temporal extent of the lattice [12] and can be expressed as

$$\det Q_{\text{stat}} = \prod_{\mathbf{x}} \left(1 + h_1 L_{f,\mathbf{x}} + h_1^2 L^*_{f,\mathbf{x}} + h_1^3\right)^2 \left(1 + \bar{h}_1 L^*_{f,\mathbf{x}} + \bar{h}_1^2 L_{f,\mathbf{x}} + \bar{h}_1^3\right)^2, \tag{9}$$

where $h_1(\mu) \equiv (2\kappa)^{N_\tau} e^{\mu/T}$ and $\bar{h}_1(\mu) = h_1(-\mu)$ with μ denoting the chemical potential, $aT = 1/N_\tau$ the temperature, and $L_{f,\mathbf{x}}$ refers to the Polyakov loop at site $\mathbf{x}$ in the fundamental representation. The second term in (8), referred to as the kinetic quark determinant, can be systematically expanded in κ^2 and generates couplings, denoted as $h_i, \bar{h}_i$, $i \geq 2$, between Polyakov loops at arbitrary separation. In our power counting scheme for the perturbative expansion of the log-action, the couplings h_1 and $\bar{h}_1$ are assigned an importance equal to a taxi-driver distance of one. In general, the form of the effective action in (3) becomes modified as additional terms are added due to the hopping parameter expansion of the fermion determinant

$$S_{\text{eff}} = S_{\text{symm}} + \sum_i S'_i, \tag{10}$$

where S'_i are the additional terms which break the center-symmetry.

2.3 Finite-Cluster method

When including the static determinant and other terms generated by the hopping parameter expansion of the kinetic determinant, the series expansions for the Polyakov loop correlators can be represented as

$$\tilde{\lambda}_\mu = \sum_{\{n_i\}} \sum_{\{m_i, \bar{m}_i\}} c^{(\mu)}_{n_1,\ldots,n_N;m_1,\bar{m}_1,\ldots,m_M,\bar{m}_M} \prod_{i=1}^{N} \lambda_i^{n_i} \prod_{i=1}^{M} h_i^{m_i} \bar{h}_i^{\bar{m}_i}, \tag{11}$$

where $\sum_i n_i \leq \tilde{N}_{\text{taxi},max}$, $\sum_i m_i \leq \tilde{N}_{\text{taxi},max}$, and $\sum_i \bar{m}_i \leq \tilde{N}_{\text{taxi},max}$, with $\tilde{N}_{\text{taxi},max}$ denoting the maximum taxi-driver distance. Here we have introduced fermionic couplings $h_i, \bar{h}_i$ corresponding to the non-symmetric terms generated by the fermion determinant in the log-action. In general, for each fermion coupling that is introduced in the log-action, we can introduce a correlator which projects out the corresponding coupling in (10)

$$\tilde{h}_\mu = \sum_{\{n_i\}} \sum_{\{m_i, \bar{m}_i\}} d^{(\mu)}_{n_1,\ldots,n_N;m_1,\bar{m}_1,\ldots,m_M,\bar{m}_M} \prod_{i=1}^{N} \lambda_i^{n_i} \prod_{i=1}^{M} h_i^{m_i} \bar{h}_i^{\bar{m}_i}. \tag{12}$$

Here $\tilde{h}_\mu$ can be related to a correlator in the full lattice theory which is not invariant under global Z_N transformations. With the introduction of the static determinant, for example, we take $\tilde{h}_1 \propto \langle L_{f,\mathbf{x}} \rangle_{\text{QCD}}$.

Before computing the correlators in (10) and (11), we first recast the partition function of the log-action on a graph G as a sum of locally-computable weights on a set of graphs whose vertices (and not necessarily edges) coincide with those of G

$$\tilde{Z}(G) = 1 + \sum_{g \in G \setminus \emptyset} \tilde{\phi}(g), \tag{13}$$

where the weight of a subgraph is given as

$$\begin{aligned} \tilde{\phi}(g) \quad = \quad & \frac{1}{z_0^{|V(g)|}} \int \prod_{v \in V(g)} dL_v \left(\det Q_{\text{stat},v}\right)^{N_f} \prod_{l \in E(g)} \prod_{\text{r}(l)} \lambda_i(l) \left(L_{\mathbf{r}(l),v_1(l)} L_{\bar{\mathbf{r}}(l),v_2(l)} + \text{c.c.}\right) \\ & \times \prod_{j(l)} \Delta^{(j(l))}(l,\kappa). \end{aligned} \tag{14}$$

Here $V(g)$ denotes the set of vertices of g, $E(g)$ denotes the set of edges of g, and $\{\lambda_i(l), \mathbf{r}(l)\}$ refer to the couplings and representation associated with the given edge l. For completeness we have included the terms $\Delta^{(j)}$, which are generated by the expansion of the kinetic determinant. The normalization factor is the integral of the single-site static determinant

$$z_0 = \int dL \left(\det Q_{\text{stat,site}}\right)^{N_f} [L, L^*], \tag{15}$$

where dL is the reduced Haar measure.

Ostensibly, the graph G refers to the thermodynamic cubic lattice on which our lattice theory resides. However, it is more efficient to consider any connected graph with $|V(g)|$ vertices and $|E(g)|$ bonds whose vertices can be mapped in a one-to-one manner to those of the cubic lattice itself. The logarithm of the partition function and its derivatives with respect to external sources coupling to the appropriate representations of the trace of the Polyakov loop are what we are ultimately interested. It turns out that one can write down a recursion relation for this quantity on a graph G

$$\xi(G) = \log \tilde{Z}(G) \; - \sum_{g \in \mathcal{G}_c(G) \setminus G} \xi(g), \tag{16}$$

where $\mathcal{G}_c(G) \setminus G$ denotes the proper, connected subgraphs of G. It should be emphasized that the above relation is valid on any abstract graph G (no reference to the embedding cubic lattice is necessary). It is only the value of the partition function itself which depends on how G is embedded in the cubic lattice. One can immediately see that the base case of the recursion corresponds to the connected graph with two vertices. The idea is to solve (16) for the various ξ's, after which one can use them to express the logarithm of the partition function. This is done through the following expression

$$\frac{\log \tilde{Z}}{V} = \sum_{l=1}^{\tilde{N}_{\text{taxi},max}} \sum_{G \in C_l} \sum_{g \in \{\mathcal{G}_c(G)\}} \sum_{p \in \mathcal{P}_g} \frac{W(g;p)}{S(g)} \xi(g^{(p)}_{\Lambda_s}), \tag{17}$$

where the first sum is over the number of bonds, the second sum is over all of the connected graphs with l bonds, the third sum is over the connected subgraphs of G, and the final sum is over partitions of the bonds of g into the various allowed types: from nearest-neighbor to fifth nearest-neighbor. Here we have introduced the symmetry factor of the graph $S(g)$ and the modified weak embedding number $W(g;p)$, which counts the number of ways the vertices of a graph can be mapped to the cubic lattice in such a way that the images of the pairs of vertices which are connected by a bond are separated by distances which satisfy the given partition p. The relation (17) is valid, of course, only up to $\tilde{N}_{\text{taxi},max}$. This approach is known as the finite-cluster method and has been used to great success in the study of lattice spin systems [13, 14], and more recently has been used to study various aspects of Polyakov loop effective theories [15].

To compute the correlators of the log-action using the finite-cluster method, we introduce sources coupled to the Polyakov loops of a given representation at the appropriate, canonical spatial locations. This is encoded in the replacement $\xi(G) \to \xi(G; J_{\mathbf{r},\mathbf{x}})$. The result for a generic n-point function follows immediately

$$\langle L_{\mathbf{r}}(\mathbf{x}) \dots L_{\mathbf{r}_{n-1}}(\mathbf{x}_{n-1}) \rangle = \sum_{l=1}^{\tilde{N}_{\text{taxi},max}} \sum_{G \in \mathcal{C}_{l,n}} \sum_{g \in \{\mathcal{G}_c(G)\}} \sum_{p \in \mathcal{P}_g} \frac{W^{(n)}(G;p)}{S(G)} \xi^{(n)}(G^{(p,n)}_{\Lambda_s}), \tag{18}$$

where the second sum is now over connected, n-rooted graphs with l bonds and where we have introduced

$$\xi^{(n)}(G^{(p,n)}_{\Lambda_s}) \equiv \frac{\partial^n \xi((G^{(p)}_{\Lambda_s}))}{\partial J_{\mathbf{r},\mathbf{x}} \dots \partial J_{\mathbf{r}_{n-1},\mathbf{x}_{n-1}}}\Big|_{J=0}. \tag{19}$$

The modified weak embedding numbers now must be computed for rooted graphs. The bulk of the computational effort in calculating (18) is spent on generating the weak embedding numbers for the rooted graphs. All of the symbolic calculations were implemented and performed using the open-source software package GiNaC, while the graph generation and transformations were performed with the help of NAUTY [16].

3. Results

We list here the formulae for the gauge couplings computed with the finite-cluster method at $\tilde{N}_{\text{taxi},max} = 4$

$$\begin{aligned}
\tilde{\lambda}_1 &= \lambda_s\lambda_a + \lambda_1 + 2\lambda_1^2\lambda_3 + 4\lambda_1^4 + 8\lambda_2\lambda_1^2 + 4\lambda_1^3 + 2\lambda_1\lambda_3 + \lambda_1\lambda_a + \lambda_s\lambda_1 + 8\lambda_2\lambda_1, && (20)\\
\tilde{\lambda}_2 &= 2\lambda_1^2 + 38\lambda_2\lambda_1^2 + 4\lambda_5\lambda_1 + 4\lambda_s\lambda_1^2 + \lambda_2 + 4\lambda_4\lambda_1 + 4\lambda_2\lambda_3 + 4\lambda_2^2 + 4\lambda_1^2\lambda_a && (21)\\
&\quad + 17\lambda_1^4 + 12\lambda_1^2\lambda_3,\\
\tilde{\lambda}_3 &= 12\lambda_1^4 + 4\lambda_2^2 + 8\lambda_5\lambda_1 + 24\lambda_2\lambda_1^2 + 2\lambda_1^2\lambda_s + \lambda_3 + 2\lambda_1^2\lambda_a + 7\lambda_1^2\lambda_3 + \lambda_1^2, && (22)\\
\tilde{\lambda}_4 &= \lambda_4 + 6\lambda_2\lambda_1 + 6\lambda_1^3, && (23)\\
\tilde{\lambda}_5 &= 2\lambda_1\lambda_3 + \lambda_5 + 3\lambda_1^3 + 2\lambda_2\lambda_1, && (24)\\
\tilde{\lambda}_a &= \lambda_a + 4\lambda_3\lambda_1^2 + 8\lambda_1^4 + 2\lambda_s\lambda_1 + 16\lambda_2\lambda_1^2, && (25)\\
\tilde{\lambda}_s &= 32\lambda_2\lambda_1 + 4\lambda_1\lambda_s + 18\lambda_1^2\lambda_3 + 4\lambda_1 + 72\lambda_2\lambda_1^2 + 36\lambda_1^4 + 16\lambda_1^3 + 9\lambda_1\lambda_a && (26)\\
&\quad + 8\lambda_1\lambda_3 + 9\lambda_a\lambda_s + \lambda_s.
\end{aligned}$$

The expressions at a similar order with $N_f = 2$ static quarks are extremely lengthy and thus we omit them.

4. Summary and Outlook

As a check, we have reproduced the results of [5] for the pure-gauge case using the modified expressions which we have obtained from the finite-cluster method. The expressions with static quarks can now be used to obtain the couplings of the log-action in the heavy-quark region. Currently, $N_f = 2$ unimproved Wilson fermion simulations are underway with $N_\tau = 6, 8$. This should enable us to determine how significant non-perturbative corrections are for both h_1 as well as the gauge couplings λ_i in the upper-right corner of the Columbia plot. Adding contributions at κ^2 and higher is a straightforward process. It remains to be seen, however, how many additional terms are needed to approach the chiral region. A further direction would be to simulate at non-zero imaginary chemical potential in the heavy-quark region. This would allow one to observe the functional dependence of the couplings on chemical potential.

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