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Family of Gauge Invariant Variables for Scalar Perturbations During Inflation

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A Family of perturbation invariant variables for scalar cosmological perturbations is discussed in this paper. We follow the idea of obtaining the well-known scalar-type perturbation invariants, i.e., the two Bardeen's potentials, the Mukhanov-Sasaki variable, and perturbation of an inflaton field, and obtained a family of scalar perturbation invariants. Based on this family, we obtained five linearly independent perturbation invariants. We expressed the Bardeen's potentials, the Mukhanov-Sasaki variable, and the perturbation of an inflaton field with respect to them.

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1. Introduction

The theory of cosmological perturbations plays a fundamental role in modeling the Universe [1], since cosmological perturbations allow us to study the origins of large structures, the anisotropies of the cosmic microwave background radiation, as well as the propagation of gravitational waves on cosmological scales.

The cosmological perturbations are connected to cosmological inflation. The theory of inflation conjectures that the Universe underwent a period of (almost) exponential expansion shortly after the Big Bang, before it transits into the radiation and matter dominated eras, which are described by conventional Big Bang theory [2, 3].

Inflation provides the primordial seeds required for structure formation. Inflation is (usually) modeled using a scalar field called inflaton. During the inflationary stage, quantum fluctuations are rapidly stretched to cosmological scales, freeze in, and become classical. Those quantum fluctuations thus transform into classical perturbations. In the radiation dominated era, those perturbations that originate during inflation manifest as perturbations in the energy density. The density inhomogeneities serve as the primordial seeds from which all observed large-scale structures have been formed. The quantum fluctuations during inflation also induce an observable temperature anisotropies in the cosmic microwave background radiation [4].

The purpose of this article is to present a study on perturbation invariants obtained concerning transformation rules of components of metric tensor corresponding to spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric.

The Bardeen's potentials and the Mukhanov-Sasaki variable are obtained after the linear combining of previously mentioned transformation rules and their derivatives with respect to spacetime coordinates [1, 5]. Following this idea, we aim to obtain a family of scalar perturbation invariants.

The paper is organized as follows: after the Introduction, the basics of scalar cosmological perturbations and gauge invariance are briefly reviewed in Section 2, after which the family of scalar perturbation invariants are introduced in Section 3. Section 4 contains a discussion of linearly independent scalar perturbation invariants. Section 5 is reserved for final remarks.

2. Basics of scalar cosmological perturbations

The FLRW spacetime for unperturbed, spatially flat, homogeneous and isotropic Universe corresponds to the metric (i.e., first quadratic form)

$$ds^2 = a^2(\eta)\big\{-d\eta^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\big\}, \tag{1}$$

for spatial coordinates x^1, x^2, x^3, the scale factor $a(\eta)$, and the conformal time η defined from the coordinate time t by $dt = a(\eta)d\eta$. The time dependence of the scale factor a depends crucially on the matter model coupled to spacetime via the Einstein equations.

The perturbations are decomposed into scalar, vector, and tensor parts, depending on their transformation behavior under the symmetry group of the spatial section, i.e., three-dimensional spatial rotations [2, 6]. This is the so-called SVT3 decomposition of cosmological perturbations (for four-dimensional SVT decomposition, SVT4, see [7] and references therein). We restrict our consideration here to scalar perturbations. The perturbed FLRW metric for perturbations of scalar types A, B, D and E is given with the following first quadratic form [6]

$$\begin{aligned} ds^2 = a^2(\eta)\Big\{ &- (1+2A)d\eta^2 + 2(\partial_i B)dx^i d\eta \\ &+ \big[\big(1 - 2(D + \frac{1}{3}(\partial_k\partial_k E))\big)\delta_{ij} + 2(\partial_i\partial_j E)\big]dx^i dx^j\Big\}, \end{aligned} \tag{2}$$

where $\partial_k\partial_k E = \delta^{kl}\partial_k\partial_l E$.

Let us consider two manifolds, M and N. The manifold M corresponds to the unperturbed Universe, the homogeneous and isotropic one, while the manifold N corresponds to the perturbed Universe. Different coordinate systems may be defined on the manifold N. Gauge transformations relate these coordinate systems. The transformation $f : M \to N$ which any point P in M transforms to a point P' in N with the same coordinates, is defined in this way. Different gauges correspond to different functions f.

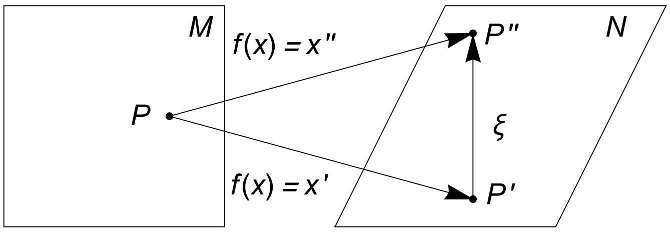

Figure 1: Transformation scheme

When changing coordinate system from $O'x'$ to $O''x''$, $x' \to x'' = x' - \xi$, the transformation rules for scalar objects A, B, D, E, (from the first square form (2)) and the perturbation of inflaton field $\delta\varphi$ that follow from the transformation of coordinates are [3]

$$A' = A'' - \frac{\partial\xi^0}{\partial\eta} - \xi^0\mathcal{H}, \tag{3}$$

$$B' = B'' - (\partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}) + \xi^0, \tag{4}$$

$$D' = D'' + \frac{1}{3}(\partial_i\xi^i) + \xi^0\mathcal{H}, \tag{5}$$

$$E' = E'' - (\partial_i^{-1}\xi^i), \tag{6}$$

$$\delta\varphi' = \delta\varphi'' - \xi^0\frac{d\bar{\varphi}}{d\eta}, \tag{7}$$

where the Hubble parameter is defined as $\mathcal{H} = \dfrac{1}{a}\dfrac{da}{d\eta}$. Let us consider the next linear combinations of equations (3-7) and their derivatives with respect to (conformal) time η and spatial coordinates x^i, $i = 1, 2, 3$

$$(3) + \mathcal{H}\big[(4) - \frac{\partial}{\partial\eta}(6)\big] + \frac{\partial}{\partial\eta}\big[(4) - \frac{\partial}{\partial\eta}(6)\big], \tag{8}$$

$$(5) + \frac{1}{3}\partial_i\partial_i(6) - \mathcal{H}\big[(4) - \frac{\partial}{\partial\eta}(6)\big], \tag{9}$$

$$(7) + \frac{d\bar{\varphi}}{d\eta}\big[(4) - \frac{\partial}{\partial\eta}(6)\big]. \tag{10}$$

The linear combinations (8, 9, 10) are equivalent to equalities

$$\Phi' = \Phi'', \quad \Psi' = \Psi'', \quad \delta\varphi'_{INV} = \delta\varphi''_{INV}, \tag{11}$$

where Φ and Ψ are the Bardeen's potentials, and $\delta\varphi_{INV}$ is the invariant perturbation of the inflaton field φ, defined as

$$\Phi \equiv A + \mathcal{H}\big(B - \frac{\partial E}{\partial\eta}\big) + \frac{\partial}{\partial\eta}\big(B - \frac{\partial E}{\partial\eta}\big), \tag{12}$$

$$\Psi \equiv D + \frac{1}{3}\partial_i\partial_i E - \mathcal{H}\big(B - \frac{\partial E}{\partial\eta}\big), \tag{13}$$

$$\delta\varphi_{INV} \equiv \delta\varphi + \frac{d\bar{\varphi}}{d\eta}\big(B - \frac{\partial E}{\partial\eta}\big). \tag{14}$$

Note that all of them are expressed in the reference system Ox. Another necessary quantity is the Mukhanov-Sasaki variable, expressed as

$$v \equiv a\Big(\delta\varphi_{INV} + \mathcal{H}^{-1}\big(\frac{d\bar{\varphi}}{d\eta}\big)\Psi\Big) = a\mathcal{H}^{-1}\big(\frac{d\bar{\varphi}}{d\eta}\big)\Big(D + \frac{1}{3}\big(\partial_i\partial_i E\big)\Big) + a\delta\varphi. \tag{15}$$

As we may see, all terms containing ξ^0, ξ^i and their derivatives, and inverse derivatives in the linear combinations (8), (9) and (10) vanish. Because the Hubble parameter $\mathcal{H}$ and $\frac{d\bar{\varphi}}{d\eta}$ depend only on time η means that terms containing

$$\xi^0, \quad \frac{\partial\xi^0}{\partial\eta}, \quad \partial_i\xi^i, \quad \partial_i^{-1}\xi^i, \quad \partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}, \tag{16}$$

as well as (possibly) higher derivative terms vanish.

The idea contained in the linear combinations (8), (9) and (10) was the motivation for our research to obtain the family of cosmological scalar perturbation invariants.

3. Family of scalar perturbation invariants

To annul the linear combinations of summands given in (16), we need to expand the system (3-7) with expressions obtained as partial derivatives of (3-7) with respect to η and x^i, such that all terms in (16) will be present in at least two expressions of an expanded system.

The expanded system that we will study here (and use to obtain a family of scalar perturbation invariants) is

$$A' = A'' - \frac{\partial\xi^0}{\partial\eta} - \xi^0\mathcal{H}, \tag{3}$$

$$B' = B'' - \big(\partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}\big) + \xi^0, \tag{4}$$

$$D' = D'' + \frac{1}{3}\big(\partial_i\xi^i\big) + \xi^0\mathcal{H}, \tag{5}$$

$$E' = E'' - \left(\partial_i^{-1}\xi^i\right), \tag{6}$$

$$\delta\varphi' = \delta\varphi'' - \xi^0\frac{d\bar{\varphi}}{d\eta}, \tag{7}$$

$$\frac{\partial A'}{\partial\eta} = \frac{\partial A''}{\partial\eta} - \frac{\partial^2\xi^0}{\partial\eta^2} - \frac{\partial\xi^0}{\partial\eta}\mathcal{H} - \xi^0\frac{d\mathcal{H}}{d\eta}, \tag{17}$$

$$\frac{\partial B'}{\partial\eta} = \frac{\partial B''}{\partial\eta} - \left(\partial_i^{-1}\frac{\partial^2\xi^i}{\partial\eta^2}\right) + \frac{\partial\xi^0}{\partial\eta}, \tag{18}$$

$$\frac{\partial D'}{\partial\eta} = \frac{\partial D''}{\partial\eta} + \frac{1}{3}\frac{\partial}{\partial\eta}\left(\partial_i\xi^i\right) + \frac{\partial\xi^0}{\partial\eta}\mathcal{H} + \xi^0\frac{d\mathcal{H}}{d\eta}, \tag{19}$$

$$\frac{\partial E'}{\partial\eta} = \frac{\partial E''}{\partial\eta} - \left(\partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}\right), \tag{20}$$

$$\frac{\partial\delta\varphi'}{\partial\eta} = \frac{\partial\delta\varphi''}{\partial\eta} - \frac{\partial\xi^0}{\partial\eta}\frac{d\bar{\varphi}}{d\eta} - \xi^0\frac{d^2\bar{\varphi}}{d\eta^2}, \tag{21}$$

$$\frac{\partial^2 E'}{\partial\eta^2} = \frac{\partial^2 E''}{\partial\eta^2} - \left(\partial_i^{-1}\frac{\partial^2\xi^i}{\partial\eta^2}\right), \tag{22}$$

$$\left(\partial_i\partial_i E'\right) = \left(\partial_i\partial_i E''\right) - \left(\partial_i\xi^i\right), \tag{23}$$

$$\frac{\partial}{\partial\eta}\left(\partial_i\partial_i E'\right) = \frac{\partial}{\partial\eta}\left(\partial_i\partial_i E''\right) - \frac{\partial}{\partial\eta}\left(\partial_i\xi^i\right). \tag{24}$$

The terms present in this system, which should vanish, contain the following quantities

$$\xi^0, \quad \frac{\partial\xi^0}{\partial\eta}, \quad \partial_i\xi^i, \quad \partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}, \quad \partial_i^{-1}\frac{\partial^2\xi^i}{\partial\eta^2}, \quad \partial_i^{-1}\xi^i. \tag{25}$$

In this sense, the necessary linear combination of the system should be of the form

$$\begin{aligned} &c_1\cdot(3) + c_2\cdot(4) + c_3\cdot(5) + c_4\cdot(7) + c_5\cdot(18) + c_6\cdot(19)\\ &+ c_7\cdot(20) + c_8\cdot(21) + c_9\cdot(22) + c_{10}\cdot(23) + c_{11}\cdot(24) + c_{12}\cdot(6). \end{aligned} \tag{26}$$

Explicitly, this is equivalent to

$$\begin{aligned} &c_1A' + c_2B' + c_3D' + c_{12}E' + c_4\delta\varphi' + c_5\frac{\partial B'}{\partial\eta} + c_6\frac{\partial D'}{\partial\eta} + c_7\frac{\partial E'}{\partial\eta}\\ &+ c_8\frac{\partial\delta\varphi'}{\partial\eta} + c_9\frac{\partial^2 E'}{\partial\eta^2} + c_{10}\left(\partial_i\partial_i E'\right) + c_{11}\frac{\partial}{\partial\eta}\left(\partial_i\partial_i E'\right)\\ &= c_1A'' + c_2B'' + c_3D'' + c_{12}E'' + c_4\delta\varphi'' + c_5\frac{\partial B''}{\partial\eta} + c_6\frac{\partial D''}{\partial\eta} + c_7\frac{\partial E''}{\partial\eta}\\ &+ c_8\frac{\partial\delta\varphi''}{\partial\eta} + c_9\frac{\partial^2 E''}{\partial\eta^2} + c_{10}\left(\partial_i\partial_i E''\right) + c_{11}\frac{\partial}{\partial\eta}\left(\partial_i\partial_i E''\right)\\ &+ \left(\xi^0\right)\cdot\left(-c_1\mathcal{H} + c_2 + c_3\mathcal{H} - c_4\frac{d\bar{\varphi}}{d\eta} + c_6\frac{d\mathcal{H}}{d\eta} - c_8\frac{d^2\bar{\varphi}}{d\eta^2}\right)\\ &+ \left(\frac{\partial\xi^0}{\partial\eta}\right)\cdot\left(-c_1 + c_5 + c_6\mathcal{H} - c_8\frac{d\bar{\varphi}}{d\eta}\right) + \left(\partial_i\xi^i\right)\cdot\left(\frac{1}{3}c_3 - c_{10}\right)\\ &+ \left(\partial_i^{-1}\frac{\partial\xi^i}{\partial\eta}\right)\cdot\left(-c_2 - c_7\right) + \left(\partial_i^{-1}\frac{\partial^2\xi^i}{\partial\eta^2}\right)\cdot\left(-c_5 - c_9\right) + \left(\frac{\partial}{\partial\eta}\right)\cdot\left(\frac{1}{3}c_6 - c_{11}\right) + \left(\partial_i^{-1}\xi^i\right)\cdot c_{12}. \end{aligned} \tag{27}$$

Note that the two rows on the left-hand side have the same form as the first two rows on the right-hand side. It means that expression (27) will be gauge invariant if and only if the last three rows on the right-hand side are equal to zero.

Having that in mind, it is obvious that this expression will be gauge invariant if and only if the following homogeneous system of equations for coefficients $c_1, \dots, c_{12}$ is satisfied

$$\begin{cases} -c_1\mathcal{H} + c_2 + c_3\mathcal{H} - c_4\dfrac{d\bar{\varphi}}{d\eta} + c_6\dfrac{d\mathcal{H}}{d\eta} - c_8\dfrac{d^2\bar{\varphi}}{d\eta^2} = 0 \\ -c_1 + c_5 + c_6\mathcal{H} - c_8\dfrac{d\bar{\varphi}}{d\eta} = 0 \\ \dfrac{1}{3}c_3 - c_{10} = 0 \\ -c_2 - c_7 = 0 \\ -c_5 - c_9 = 0 \\ \dfrac{1}{3}c_6 - c_{11} = 0 \\ c_{12} = 0 \end{cases} \tag{28}$$

The solution of system (28) is

$$\begin{cases} c_1 = -\Big(c_4\big(\dfrac{d\bar{\varphi}}{d\eta}\big)^2 + c_7\dfrac{d\bar{\varphi}}{d\eta} - c_9\dfrac{d^2\bar{\varphi}}{d\eta^2} - 3c_{10}\mathcal{H}\dfrac{d\bar{\varphi}}{d\eta}\Big)\big(\mathcal{H}\dfrac{d\bar{\varphi}}{d\eta} - \dfrac{d^2\bar{\varphi}}{d\eta^2}\big)^{-1} \\ \qquad - 3c_{11}\big(\mathcal{H}\dfrac{d^2\bar{\varphi}}{d\eta^2} - \dfrac{d\mathcal{H}}{d\eta}\dfrac{d\bar{\varphi}}{d\eta}\big)\big(\mathcal{H}\dfrac{d\bar{\varphi}}{d\eta} - \dfrac{d^2\bar{\varphi}}{d\eta^2}\big)^{-1}, \\ c_2 = -c_7, \\ c_3 = 3c_{10}, \\ c_5 = -c_9, \\ c_6 = 3c_{11}, \\ c_8 = \Big(c_4\dfrac{d\bar{\varphi}}{d\eta} + c_7 - (c_9 + 3c_{10})\mathcal{H} + 3c_{11}\big(\mathcal{H}^2 - \dfrac{d\mathcal{H}}{d\eta}\big)\Big)\big(\mathcal{H}\dfrac{d\bar{\varphi}}{d\eta} - \dfrac{d^2\bar{\varphi}}{d\eta^2}\big)^{-1}, \\ c_{12} = 0, \end{cases} \tag{29}$$

for arbitrary coefficients $c_4, c_7, c_9, c_{10}, c_{11}$. The resulting family of scalar perturbation invariants is

$$\begin{aligned} \mathcal{I}(c_4, c_7, c_9, c_{10}, c_{11}) = c_1 A - c_7\big(B - \frac{\partial E}{\partial\eta}\big) + 3c_{10}\Big(D + \frac{1}{3}(\partial_i\partial_i E)\Big) + c_4\delta\varphi \\ - c_9\frac{\partial}{\partial\eta}\big(B - \frac{\partial E}{\partial\eta}\big) + 3c_{11}\frac{\partial}{\partial\eta}\Big(D + \frac{1}{3}(\partial_i\partial_i E)\Big) + c_8\frac{\partial\delta\varphi}{\partial\eta}, \end{aligned} \tag{30}$$

where $c_1 = c_1(c_4, c_7, c_9, c_{10}, c_{11})$ and $c_8 = c_8(c_4, c_7, c_9, c_{10}, c_{11})$ are given in (29). This is our main result.

The equivalent expression of the formula (30) is

$$\begin{aligned}
\mathcal{I}(c_4,c_7,c_9,c_{10},c_{11}) = c_4\cdot\Big(&-\big(\frac{d\bar\varphi}{d\eta}\big)^2\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A+\delta\varphi+\frac{d\bar\varphi}{d\eta}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}\Big)\\
&+c_7\cdot\Big(-\frac{d\bar\varphi}{d\eta}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A-\big(B-\frac{\partial E}{\partial\eta}\big)+\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}\Big)\\
&+c_9\cdot\Big(\frac{d^2\bar\varphi}{d\eta^2}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A-\frac{\partial}{\partial\eta}\big(B-\frac{\partial E}{\partial\eta}\big)-\mathcal{H}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}\Big)\\
&+c_{10}\cdot\Big(3\mathcal{H}\frac{d\bar\varphi}{d\eta}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A+3\big(D+\frac{1}{3}(\partial_i\partial_iE)\big)\\
&\qquad-3\mathcal{H}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}\Big)\\
&+c_{11}\cdot\Big(-3\big(\mathcal{H}\frac{d^2\bar\varphi}{d\eta^2}-\frac{d\mathcal{H}}{d\eta}\frac{d\bar\varphi}{d\eta}\big)\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A+3\frac{\partial}{\partial\eta}\big(D+\frac{1}{3}(\partial_i\partial_iE)\big)\\
&\qquad+3\big(\mathcal{H}^2-\frac{d\mathcal{H}}{d\eta}\big)\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}\Big).
\end{aligned}\tag{31}$$

4. Linearly independent scalar perturbation invariants

To obtain linearly independent invariants from the family $\mathcal{I}(c_4,c_7,c_9,c_{10},c_{11})$, given by (30) or equivalently (31), we consider the following scalar perturbation invariants $\mathcal{I}^1=\mathcal{I}(1,0,0,0,0)$, $\mathcal{I}^2=\mathcal{I}(0,1,0,0,0)$, $\mathcal{I}^3=\mathcal{I}(0,0,1,0,0)$, $\mathcal{I}^4=\mathcal{I}(0,0,0,1,0)$, and $\mathcal{I}^5=\mathcal{I}(0,0,0,0,1)$

$$\mathcal{I}^1=-\big(\frac{d\bar\varphi}{d\eta}\big)^2\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A+\delta\varphi+\frac{d\bar\varphi}{d\eta}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta},\tag{32}$$

$$\mathcal{I}^2=-\frac{d\bar\varphi}{d\eta}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A-\big(B-\frac{\partial E}{\partial\eta}\big)+\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta},\tag{33}$$

$$\mathcal{I}^3=\frac{d^2\bar\varphi}{d\eta^2}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A-\frac{\partial}{\partial\eta}\big(B-\frac{\partial E}{\partial\eta}\big)-\mathcal{H}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta},\tag{34}$$

$$\mathcal{I}^4=3\mathcal{H}\frac{d\bar\varphi}{d\eta}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A+3\big(D+\frac{1}{3}(\partial_i\partial_iE)\big)-3\mathcal{H}\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta},\tag{35}$$

$$\begin{aligned}
\mathcal{I}^5=&-3\big(\mathcal{H}\frac{d^2\bar\varphi}{d\eta^2}-\frac{d\mathcal{H}}{d\eta}\frac{d\bar\varphi}{d\eta}\big)\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\bar\varphi}{d\eta^2}\big)^{-1}A+3\frac{\partial}{\partial\eta}\big(D+\frac{1}{3}(\partial_i\partial_iE)\big)\\
&+3\big(\mathcal{H}^2-\frac{d\mathcal{H}}{d\eta}\big)\big(\mathcal{H}\frac{d\bar\varphi}{d\eta}-\frac{d^2\varphi}{d\eta^2}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta},
\end{aligned}\tag{36}$$

such that

$$\mathcal{I}(c_4,c_7,c_9,c_{10},c_{11})=c_4\cdot\mathcal{I}^1+c_7\cdot\mathcal{I}^2+c_9\cdot\mathcal{I}^3+c_{10}\cdot\mathcal{I}^4+c_{11}\cdot\mathcal{I}^5.\tag{37}$$

Hence, expressions for the Bardeen's potentials Φ and Ψ, the scalar perturbation invariant $\delta\varphi_{INV}$, and the Mukhanov-Sasaki variable v are

$$\Phi = -\mathcal{H}\mathcal{I}^2 - \mathcal{I}^3, \tag{38}$$

$$\Psi = -\mathcal{H}\mathcal{I}^2 + \frac{1}{3}\mathcal{I}^4, \tag{39}$$

$$\delta\varphi_{INV} = \mathcal{I}^1 - \frac{d\bar{\varphi}}{d\eta}\mathcal{I}^2, \tag{40}$$

$$v = \mathcal{I}^1 + \frac{1}{3}a\frac{d\bar{\varphi}}{d\eta}\mathcal{H}^{-1}\mathcal{I}^4. \tag{41}$$

Although neither Φ, Ψ, $\delta\varphi_{INV}$ or v contain $\mathcal{I}^5$, it was necessary to be included to complete the system.

5. Final remarks

We discussed a family of perturbation invariant variables for scalar-type cosmological perturbations. In this study, we concentrated on the most commonly used gauge invariant variables, the Bardeen's potentials Φ and Ψ, the invariant perturbation of the inflaton field $\delta\varphi_{INV}$ and the Mukhanov-Sasaki variable v. This can be seen as a generalization of scalar invariant perturbations, from which the aforementioned scalar invariants can be obtained.

Usually, calculations begin with a particular gauge invariant scalar perturbation variable. However, in our opinion, this generalization deserves attention and should be studied further.

It is worth mentioning that besides the $\mathcal{I}(c_4, c_7, c_9, c_{10}, c_{11})$ invariant, another similar family of scalar invariants can be introduced and studied. Namely, the following linear combinations of invariants $\mathcal{I}^1, \ldots, \mathcal{I}^5$ separate terms such as A, $B - \frac{\partial E}{\partial\eta}$, $\frac{\partial}{\partial\eta}\big(B - \frac{\partial E}{\partial\eta}\big)$, $D + \frac{1}{3}(\partial_i\partial_i E)$, $\frac{\partial}{\partial\eta}\big(D + \frac{1}{3}(\partial_i\partial_i E)\big)$, $\delta\varphi$ and their derivatives

$$\begin{cases} \mathcal{J}^1 = \Big(\frac{d^2\bar{\varphi}}{d\eta^2}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-2} - \mathcal{H}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1}\Big)\mathcal{I}^1, \\ \mathcal{J}^2 = -\mathcal{I}^2 + \big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1}\mathcal{I}^1, \\ \mathcal{J}^3 = -\mathcal{I}^3 - \frac{d^2\bar{\varphi}}{d\eta^2}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-2}\mathcal{I}^1, \\ \mathcal{J}^4 = \frac{1}{3}\mathcal{I}^4 + \mathcal{H}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1}\mathcal{I}^1, \\ \mathcal{J}^5 = \frac{1}{3}\mathcal{I}^5 + \Big(\frac{d\mathcal{H}}{d\eta}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1} - \mathcal{H}\frac{d^2\bar{\varphi}}{d\eta^2}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-2}\Big)\mathcal{I}^1. \end{cases} \tag{42}$$

These invariants are expressed as

$$\mathcal{J}^1 = A - \Big(\mathcal{H}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1} - \frac{d^2\bar{\varphi}}{d\eta^2}\big(\frac{d\bar{\varphi}}{d\eta}\big)^{-2}\Big)\delta\varphi - \big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}, \tag{43}$$

$$\mathcal{J}^2 = \Big(B - \frac{\partial E}{\partial\eta}\Big) + \big(\frac{d\bar{\varphi}}{d\eta}\big)^{-1}\delta\varphi, \tag{44}$$

$$\mathcal{J}^3 = \frac{\partial}{\partial\eta}\Big(B - \frac{\partial E}{\partial\eta}\Big) - \frac{d^2\bar{\varphi}}{d\eta^2}\Big(\frac{d\bar{\varphi}}{d\eta}\Big)^{-2}\delta\varphi + \Big(\frac{d\bar{\varphi}}{d\eta}\Big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}, \tag{45}$$

$$\mathcal{J}^4 = \Big(D + \frac{1}{3}(\partial_i\partial_i E)\Big) + \mathcal{H}\Big(\frac{d\bar{\varphi}}{d\eta}\Big)^{-1}\delta\varphi, \tag{46}$$

$$\mathcal{J}^5 = \frac{\partial}{\partial\eta}\Big(D + \frac{1}{3}(\partial_i\partial_i E)\Big) + \Big(\frac{d\mathcal{H}}{d\eta}\Big(\frac{d\bar{\varphi}}{d\eta}\Big)^{-1} - \mathcal{H}\frac{d^2\bar{\varphi}}{d\eta^2}\Big(\frac{d\bar{\varphi}}{d\eta}\Big)^{-2}\Big)\delta\varphi + \mathcal{H}\Big(\frac{d\bar{\varphi}}{d\eta}\Big)^{-1}\frac{\partial\delta\varphi}{\partial\eta}, \tag{47}$$

and will be the subject of future investigations.

The next natural step is to obtain a differential equation that governs the dynamics of the family of scalar perturbations $\mathcal{I}(c_4, c_7, c_9, c_{10}, c_{11})$, and to study the behavior of its solution in the sub-Hubble and super-Hubble regimes. The following step will study the power spectrum for scalar perturbation using the slow-roll approximation. This will also be the subject of our future investigations.

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