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Triple (and quadruple) soft-gluon radiation in QCD hard scattering

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We consider the radiation of three soft gluons in a generic process for multiparton hard scattering in QCD. In the soft limit the corresponding scattering amplitude has a singular behaviour that is factorized and controlled by a colorful soft current. We compute the tree-level current for triple soft-gluon emission from both massless and massive hard partons. The three-gluon current is expressed in terms of maximally non-abelian irreducible correlations. We compute the soft behaviour of squared amplitudes and the colour correlations produced by the squared current. The radiation of one and two soft gluons leads to colour dipole correlations. Triple soft-gluon radiation produces in addition colour quadrupole correlations between the hard partons. We examine the soft and collinear singularities of the squared current in various energy ordered and angular ordered regions. We discuss some features of soft radiation to all-loop orders for processes with two and three hard partons. Considering triple soft-gluon radiation from three hard partons, colour quadrupole interactions break the Casimir scaling symmetry between quarks and gluons. We also present some results on the radiation of four soft gluons from two hard partons, and we discuss the colour monster contribution and its relation with the violation (and generalization) of Casimir scaling. We also compute the first correction of $(1/N_c^2)$ to the eikonal formula for multiple soft-gluon radiation with strong energy ordering from two hard gluons.

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1. Introduction

The subject of this talk is definitely on the legs' side rather than on the loops' side. It deals with tree-level emission of three and also four soft gluons from any QCD hard scattering.

There are many motivations for studying soft gluon emissions from arbitrary amplitudes. First of all, high-precision LHC data require high precision in theoretical predictions, both to test our present understanding of the Standard Model and to discover (probably tiny) signals of new physics. Furthermore, the explicit knowledge of soft/collinear factorization of scattering amplitudes is necessary in resummed calculations at next-to-next-to-next-to-leading (N^3L) order. In addition, calculation of large logarithmic terms can be used to obtain approximated fixed-order results. Finally, soft/collinear factorization provides the theoretical basis of parton shower algorithms for Monte Carlo event generators.

Our aim is to investigate the behaviour of a scattering amplitude when some external gluons become soft. We denote with p the momenta of hard particles, and with q those of the soft gluons. When gluons become soft, the amplitude diverges, and the leading divergence can be factorized as a singular soft current operator J acting on the hard amplitude with the soft gluons removed, as depicted in fig. 1.

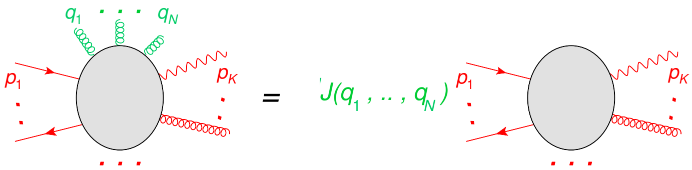

Figure 1: Schematics of the soft factorization formula.

2. Soft currents

Let me first consider the case of one photon emission in QED. The soft current is just a number which depends on the hard and soft momenta and on the charges of the hard particles (eq. (1)). A similar expression holds in QCD [1], but the abelian charge is replaced by the non-abelian colour matrices whose explicit expressions are in eq. (2)

$$\text{QED:} \qquad J(q) = J^{\mu}(q)\varepsilon_{\mu}(q) = \sum_{k=1}^{K} \frac{e_k p_k^{\mu}}{p_k \cdot q} \varepsilon_{\mu}(q) \tag{1}$$

$$\text{QCD:} \qquad J^{a}(q) = J^{a,\mu}(q)\varepsilon_{\mu}(q) = \sum_{k=1}^{K} \frac{g T_k^a p_k^{\mu}}{p_k \cdot q} \varepsilon_{\mu}(q) \qquad \begin{cases} (T_q^a)_{bc} = t_{bc}^a \\ (T_{\bar{q}}^a)_{bc} = -t_{cb}^a \\ (T_g^a)_{bc} = \mathrm{i} f^{abc} \end{cases} \tag{2}$$

$$\text{GRAV:} \qquad J(q) = J^{\mu\nu}(q)\varepsilon_{\mu\nu}(q) = \sum_{k=1}^{K} \frac{\kappa p_k^{\nu} p_k^{\mu}}{p_k \cdot q} \varepsilon_{\mu\nu}(q) \tag{3}$$

For physical amplitudes, with overall zero charge (colour singlets), the currents are conserved:[1]

$$\sum_{k=1}^{K} e_k = 0\,, \qquad \sum_{k=1}^{K} T_k^a \overset{\text{cs}}{=} 0 \quad\Longrightarrow\quad q_\mu J^{a,\mu} \overset{\text{cs}}{=} 0 \tag{4}$$

It is interesting to show that a soft current with the same structure exists in gravity as well, see eq. (3), with the colour charge replaced by the 4-momenta of the hard particles.

Also the current for two soft gluon emission is known since long [3]. It can be conveniently expressed as the symmetric product of two 1-gluon currents, plus a remainder $\Gamma(1,2)$ which represents the maximally non-abelian correlation between the two soft gluons:

$$\begin{aligned}
J^{a_1 a_2}_{\mu_1\mu_2}(q_1,q_2) &= J^{a_1}_{\mu_1}(q_1) * J^{a_2}_{\mu_2}(q_2) + \Gamma^{a_1 a_2}_{\mu_1\mu_2}(q_1,q_2) \qquad \left(A * B \equiv \tfrac{1}{2}(AB+BA)\right)\\
\Gamma^{a_1 a_2}_{\mu_1\mu_2}(q_1,q_2) &= \mathrm{i} f^{a_1 a_2 b} \sum_{k\in\text{hard}} T_k^b\; \gamma_k^{\mu_1\mu_2}(q_1,q_2)\\
\gamma_k^{\mu_1\mu_2}(q_1,q_2) &= \frac{1}{p_k\cdot(q_1+q_2)}\left\{\frac{p_k^{\mu_1} p_k^{\mu_2}}{2\,p_k\cdot q_1} + \frac{1}{q_1\cdot q_2}\left(p_k^{\mu_1} q_1^{\mu_2} + \frac{1}{2} g^{\mu_1\mu_2} p_k\cdot q_2\right)\right\} - (1\leftrightarrow 2)\,.
\end{aligned}$$

The colour structure of Γ is just a colour matrix contracted with a structure constant. This current is conserved, if applied on colour-singlet states, even without contracting the remaining Lorentz index with the polarization vector of the second gluon. In the abelian case only the independent product of single gluon currents survives. In QCD however we have colour correlation also from the symmetric product. By the way, the 1-gluon currents do not commute, since they contain colour matrices.

We have evaluated [4] the current for the emission of three gluons computing the relevant Feynman diagrams with gluon insertions on the external lines, where we use eikonal vertices on hard lines and exact vertices and propagators elsewhere. We find that the current, when acting on colour singlets, does not depend on the gauge and is conserved. It can conveniently be expressed as a symmetric product of three 1-gluon currents, the symmetric product of one 1-gluon current and the 2-gluon non-abelian term, plus an irreducible, maximally non-abelian remainder $\Gamma(1,2,3)$:

$$J(1,2,3) = J(1) * J(2) * J(3) + \left[\sum_{\text{cyc.123}} J(1) * \Gamma(2,3)\right] + \Gamma(1,2,3) \tag{5}$$

$$\Gamma(1,2,3) = \sum_{k\in\text{hard}} T_k^b \sum_{\text{cyc.123}} \sum_{s} f^{a_1 a_2 s} f^{s\, a_3 b}\; \gamma_k^{\mu_1\mu_2\mu_3}(q_1,q_2;q_3) \tag{6}$$

We can present the soft currents also in terms of the decomposition in colour-ordered subamplitudes

$$M(1,\cdots,n) = \sum_{\text{perm}(1,\cdots,n-1)} \mathrm{tr}(t^{a_1}\cdots t^{a_n})\; C(1,\cdots,n)$$

In the soft limit, Berends and Giele [5] shew that the colour-stripped amplitudes $C(1,\cdots,n)$ reduce to the corresponding amplitude without soft gluons, times a soft factor $s_{12\ldots}$:

$$C(1,\underline{2},\cdots,\underline{m},m+1,\cdots,n) = s_{1,\underline{2},\cdots,\underline{m},m+1}\, C(1,m+1,\cdots,n)$$

[1] The notation $\overset{\text{cs}}{=}$ means equality of operators when acting on colour singlet states.

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We can explicitly express the soft factors $s_{i\underline{123}k}$ in terms of the kinematical coefficients of the one gluon current and of the irreducible correlations:

$$s_{i\underline{123}k} = \gamma_i(1,2;3) + \gamma_i(3,2;1) - \tfrac{1}{2}\big[\gamma_i(1,2)\, j_i(3) + j_i(1)\,\gamma_i(2,3)\big] \tag{7}$$
$$+ \gamma_i(1,2)\, j_k(3) + \tfrac{1}{2}\, j_i(1)\, j_i(2)\, j_k(3) - \tfrac{1}{6}\, j_i(1)\, j_i(2)\, j_i(3) - \left(\begin{smallmatrix}1\leftrightarrow 3\\ i\leftrightarrow k\end{smallmatrix}\right) \tag{8}$$

where $j_k^\mu(1) \equiv \frac{p_k^\mu}{p_k\cdot q_1}$.

3. Squared soft currents

The objects that are relevant for cross sections involving some number of soft gluons are the squared currents. By using Dirac's notation [6] in colour and helicity space, we can consider an amplitude which depends on colours and helicities of the outgoing particles as the components of an abstract vector in colour and helicity space. In this space, a soft current is a rectangular matrix. The squared amplitude is obtained by multiplying the amplitude by its complex conjugate, and summing over final state quantum numbers. while the modulus square of the current is a square matrix in the space of the hard partons' quantum numbers. Because of the conservation of the soft gluon currents, the square currents are explicitly gauge invariant operators when acting on colour singlet states.

The square current for one soft gluon is just a sum of colour dipoles $\boldsymbol{T}_i\cdot\boldsymbol{T}_k \equiv \sum_a T_i^a T_k^a$ representing the colour matrices of two hard particles summed over the same gluon colour index which connecs them:

$$|\boldsymbol{J}(q)|^2 \overset{\text{cs}}{=} -\sum_{i,k\in\text{hard}} \boldsymbol{T}_i\cdot\boldsymbol{T}_k\, \mathcal{S}_{ik}(q) =: W(q)\;, \qquad\qquad \mathcal{S}_{ik}(q) = \frac{p_i\cdot p_k}{p_i\cdot q\; p_k\cdot q}$$

The square current for two soft gluons can be conveniently expressed as the symmetric product of the square currents of a single gluon, plus an irreducible correlation term $W(q_1,q_2)$:

$$|\boldsymbol{J}(q_1,q_2)|^2 \overset{\text{cs}}{=} W(q_1) * W(q_2) + W(q_1,q_2)$$
$$W(q_1,q_2) = -C_A \sum_{i,k\in\text{hard}} \boldsymbol{T}_i\cdot\boldsymbol{T}_k\, \mathcal{S}_{ik}(q_1,q_2)$$

This irreducible correlation is again a sum of colour dipoles, times the adjoint Casimir, and a kinematical coefficients which has been derived in [3].

We computed the square of the three soft gluon current. It is also conveniently expressed as a sum of symmetric products of object with one or two soft gluons, plus an irreducible correlation $W(1,2,3)$:

$$|\boldsymbol{J}(q_1,q_2,q_3)|^2 \overset{\text{cs}}{=} W(q_1) * W(q_2) * W(q_3) + \Big[\sum_{\text{cyc.123}} W(q_1) * W(q_2,q_3)\Big] + W(q_1,q_2,q_3)\;.$$

$W(q_1, q_2, q_3)$ involves not only colour dipoles but also colour quadrupoles, the latter being four colour matrices of hard partons connected by two structure constants, as in fig. 2-a:

$$W(q_1, q_2, q_3) = -C_A^2 \sum_{i,k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, \mathcal{S}_{ik}(q_1, q_2, q_3) + \sum_{iklm} Q_{iklm} \, \mathcal{S}_{iklm}(q_1, q_2, q_3) \tag{9}$$

$$Q_{iklm} \equiv \tfrac{1}{2} f^{ab,cd} \Big(T_l^a \{T_i^c, T_k^d\} T_m^b \; + \; \text{h.c.} \Big) \tag{10}$$

where we have identified a particular form of quadrupoles — see eq. (10) — that never reduces to dipoles when some index of the hard particles are equal.

Figure 2: (a-left): quadrupoles' colour structure; (b-right): colour monster's structure.

The kinematical coefficients of the dipole and quadrupole terms are rather cumbersome, and can be read in ref. [4]. They considerably simplify in the case of strong energy ordering (SEO) of the soft gluons. In particular, the dipole kinematical coefficient is remarkably symmetric with respect to the permutations of the three soft momenta:

$$\begin{aligned} S_{ik}^{\text{SEO}} = &\frac{2(p_i \cdot p_k)^3}{3(p_i \cdot q_1)(p_k \cdot q_1)(p_i \cdot q_2)(p_k \cdot q_2)(p_i \cdot q_3)(p_k \cdot q_3)} \\ &- \frac{2(p_i \cdot p_k)^2}{(q_1 \cdot q_2)(p_i \cdot q_1)(p_k \cdot q_2)(p_i \cdot q_3)(p_k \cdot q_3)} \\ &+ \frac{2 p_i \cdot p_k}{(q_1 \cdot q_3)(q_2 \cdot q_3)(p_i \cdot q_1)(p_k \cdot q_2)} \; + \text{perms.}\ \{1, 2, 3\} \end{aligned}$$

while the kinematical coefficient of the quadrupole is remarkably symmetric in the exchange of the two softest momenta.

We consider now the collinear singularities of the squared currents. We all know that the square amplitude is singular — more precisely, not integrable — when momenta of two or more of external massless legs become collinear.

In the case of one soft gluon collinear to parton B, it is easy to show the absence of colour correlations because of colour coherence, and the soft current is proportional to the Casimir of the B hard parton.

In the case of three soft gluons, the expansion in irreducible correlations reduces the collinear singularities of W's, so that only $W(2, 3)$ and $W(1, 2, 3)_{\text{dipole}}$ are singular, according to the following scheme:

- $W(2, 3)$

  c_1 double-collinear limit of the two soft gluons (exact $P^{\mu\nu}_{g_1 g_2}$)

  c_2 triple-collinear limit of the two soft gluons and a hard parton

- $W(1,2,3)_{\text{dipole}}$

  c_3 double-collinear limit of two soft gluons (exact)

  c_4 triple-collinear limit of the three soft gluons (exact $P^{\mu\nu}_{g_1g_2g_3}$)

  c_5 quadruple-collinear limit of the three soft gluons and a hard parton (soft $P^{ss'}_{g_1g_2g_3C}$)

the last result being new, to our knowledge. $W(1,2,3)_{\text{quadrupole}}$ has no collinear singularity at all.

4. Three hard partons

Up to now, the number of hard legs was arbitrary. Now we consider the special case of amplitudes with three hard partons $|ABC\rangle$ plus soft gluons, and possibly other colourless particles.

Because of flavour conservation, we can have only three partonic configurations: In the case of $|gq\bar{q}\rangle$, there is only one colour singlet state. Therefore the squared current, which conserves colour, acts on a one-dimensional state, so that the soft factorization becomes a multiplication by a c-number, the eigenvalue of the soft current on this state. This fact is valid to all perturbative orders in the amplitudes and in the current.

In the case of three hard gluons, we can have two distinct colour-singlet states: the colour-antisymmetic one, where the gluons are in an $|f^{abc}\rangle$ state, and the colour-symmetric $|d^{abc}\rangle$ state. Since they have opposite charge conjugation, and the soft current conserves charge conjugation, it turns out that the square current is a 2x2 diagonal matrix in the basis of these two states. The eigenvalues on these states are equal for one and two soft gluons, but differ for three or more soft gluons, because of the quadrupole contribution, which vanishes on the $|d\rangle$ state but not on the $|f\rangle$ state.

Without entering into many details, singlet state of three hard partons are eigenstates of all dipole operators, and the eigenvalue is a linear combination of Casimir coefficients. Therefore, the one and two gluon square currents are just c-numbers containing the Casimir of particle A, which is always a gluon in our notations, and of particle B, which can be either a quark or a gluon. These currents obey the so-called Casimir scaling: moving from the $|gq\bar{q}\rangle$ state to the $|ggg\rangle$ state amounts to replace C_F with C_A.

The same holds for the dipole terms of the three gluon square current. However, the quadrupole term violates Casimir scaling, because of its peculiar action on the three different hard states. Note that, increasing the number of colours, the dipole terms grow like N_c^3 while the quadrupole ones just like N_c, Another peculiarity of the quadrupole term is that its kinematical coefficient is collinear safe with respect to angular integration over soft-gluon momenta.

We have also shown that, when three hard gluons emit N soft gluon with strong energy ordering $E_1 \ll E_2 \ll \dots E_N$, the squared current, to leading order in the number of colours, has a very simple form, in terms of a multi-eikonal function F_{eik} introduced by Bassetto-Ciafaloni-Marchesini (BCM) [1]:

$$|\mathcal{M}(p_k,q_i)|^2 \simeq |\mathcal{M}(p_k)|^2 \; |\boldsymbol{J}(q_i)|^{2,\mathsf{SEO}}_{ggg_f} \left\{1+O\left(N_c^{-2}\right)\right\}$$

$$|\boldsymbol{J}(q_i)|^{2,\mathsf{SEO}}_{ggg} = C_A^N \; p_A{\cdot}p_B \;\; p_B{\cdot}p_C \;\; p_C{\cdot}p_A \; F_{\text{eik}}(p_A,p_B,p_C,q_1,\cdots,q_N)$$

$$F_{\text{eik}}(k_1,\cdots,k_M) \equiv \left[(k_1\cdot k_2)(k_2\cdot k_3)\dots(k_{M-1}\cdot k_M)(k_M\cdot k_1)\right]^{-1} + \text{ineq. perm}\{k_1,\cdots,k_M\}$$

5. Two hard partons

Finally, we consider soft gluon emission from two hard partons (plus any colourless particles) in a colour-singlet state $|BC\rangle$. There are just two such states: $|q\bar{q}\rangle$ and $|gg\rangle$. In both cases, the colour space is one-dimensional and we have again c-number factorization. Non-abelian effects are in $SU(N_c)$ colour coefficients. The eigenvalues of the squared current on such states obey Casimir scaling up to three soft gluons:

$$\begin{aligned}|\boldsymbol{J}(q_1,q_2,q_3)|^2_{BC} &= C_B^3\, w_{BC}(q_1)\; w_{BC}(q_2)\; w_{BC}(q_3)\\ &+ C_B^2 C_A [w_{BC}(q_1)\; w_{BC}(q_2,q_3) + \text{cyc.perm.}(123)]\\ &+ C_B C_A^2\, w_{BC}(q_1,q_2,q_3)\end{aligned}$$

In the case of emission of N soft gluons from two hard gluons, we have checked the BCM formula in terms of the multi-eikonal function $F_{\rm eik}$, up to colour suppressed contributions. Actually, in the case of four soft gluons, we can explicitly compute the corresponding correction, because a four soft current is constrained by a three soft current. In fact, if one of the N soft gluon is much harder than the others, we can factorize the emission of the gluon q_N from the hard pair BC, and then consider the emission of the remaining $N-1$ softest gluons from the three hard particles q_N, B, C. In this way, the four soft gluon square current is constrained by the three soft gluon square current. It turns out that the irreducible correlation for four soft gluons, in the limit of $E_4 \gg E_{1,2,3}$, has a dipole contribution and a quadrupole contribution. Therefore, a term satisfies Casimir scaling, while the other does not:

$$W(q_1\cdots q_4)|_{BC} = C_B\left[C_A^3\, w_{BC}^{(L)}(q_1\cdots q_4) + \lambda_B N_c\, w_{BC}^{(S)}(q_1\cdots q_4)\right]$$

with $\lambda_F = 1/2$ on a $|gq\bar{q}\rangle$ state, $\lambda_A = 3$ on an $|f\rangle$ state and $\lambda = 0$ on an $|d\rangle$ state. Actually, this colour structure is exact, because this kinematical limit requres the computation of all Feynman diagrams. However, we can compute the kinematical coefficients only in this energy ordering approximation. The quadrupole term that violates Casimir scaling is related to the quartic Casimir. Therefore, changing the two hard particles from $q\bar{q}$ to gg can be taken into account by a generalized Casimir scaling where, in addition to change the quadratic Casimir, we change also the quartic one. In the case of strong ordering, the kinematical coefficients can be derived from the three gluon current, and we have found the first correction to the BCM formula

If you read the book on perturbative QCD by Dokshitzer, Khoze, Mueller and Troian [7], you may have noted that in chap. 6 they have examined the 4-soft-gluon radiation from two massless hard partons in strong energy ordering, and found a contribution proportional to $C_B\, N_c$ from the so-called *colour monster* diagram of fig. 2-b.

Our results are fully consistent with the colour monster, and are related to the quartic Casimir

$$C_B\, \lambda_B\, N_c = 2\,\frac{d_{AB}^{(4)}}{D_B} - \frac{1}{12}\, C_B\, C_A^3 \tag{11}$$

In particular, the collinear singularities of the subleading coefficient $w_{BC}^{(S)}$ contribute, at the inclusive level, to the soft limit $\propto \alpha_{\rm S}^4\, d_{AB}^{(4)} / \epsilon$ of the collinear evolution kernel of the parton distribution functions [8, 9]. The soft limit of the evolution kernel is proportional to the cusp anomalous dimension that violates Casimir scaling.

6. Conclusions

To conclude, we have computed tree-level current for triple gluon emission in terms of irreducible correlations for one, two and three gluons, and have obtained explicit results for colour-ordered subamplitudes. We have computed the tree-level squared current for three-soft-gluon emission, and found that the 3-gluon correlation involves colour dipoles and quadrupoles. We checked that the collinear behaviour of the squared current is consistent with collinear factorization and angular ordering features. In particular, quadrupoles are collinear safe.

We have specialized our calculation to the cases of three or two hard partons: in this cases c-number factorization holds. A remarkable results is that quadrupoles break Casimir scaling $C_F \to C_A$ when the hard pair $q\bar{q}$ is replaced by gg. We derived a generalization of the multi-eikonal BCM [1] with 3 hard gluons (the original one being for just two hard gluons).

By further specializing to the case with two hard partons, we could extended the analysis to four-soft-gluon states. We presented the full colour structure and the kinematical coefficients in energy ordering. We found the N_c^2-suppressed colour-monster contributon which is related to the quartic Casimirs, and we obtained a generalization of Casimir scaling ($C_F \to C_A\ ,\quad d_{AF}^{(4)} \to d_{AA}^{(4)}$). The Colour-monster term has collinear singularities and contributes to the cusp anomalous dimension at α_S^4. Finally, We computed the first correction $O\left(1/N_c^2\right)$ to the multi-eikonal BCM formula.

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