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A Wilson line approach to classical and quantum effects in gravitational scattering

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The all-order structure of scattering amplitudes is greatly simplified by the use of (generalized) Wilson line operators, describing (subleading) soft emissions from straight lines extending to infinity. We discuss how these techniques, originally developed for QCD phenomenology, can be naturally applied to gravitational scattering. At the quantum level, we find a convenient way to derive the exponentiation of the (subleading) graviton Reggeization. At the classical level, the formalism provides a powerful tool for the computation of observables relevant in the gravitational wave program.

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1. The QFT road to General Relativity

The demand for highly precise theoretical predictions in gravitational wave astronomy have spurred the search for new analytic methods in general relativity. The goal is to enrich the toolbox for the computation of gravitational observables, which are indispensable for the extraction of signal from extremely noisy data. A strategy that has attracted considerable interest over the last years is given by the use of quantum field theory methods.

A common feature of these techniques is the computation of quantum scattering amplitudes, where astrophysical objects such as black holes and neutron stars are modeled as point-like particles interacting via gravitons. These are defined via the standard perturbative weak field expansion in the Newton constant G (also known as Post-Minkowskian expansion) via

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \,, \tag{1}$$

where $\kappa^2 = 32\pi G$. The efficiency of this approach, which relies on the large amount of tools developed for the computation of scattering amplitudes in particle physics, depends on the possibility to isolate the terms surviving the classical limit $\hbar \to 0$ as early as possible in the calculation.

This research area has seen a rapid growth over the last years [1]. Among the various techniques that have been exploited, a method which is particularly tied to particle physics (and QCD in particular) is given by the use of (generalized) Wilson lines. The definition of a gravitational Wilson Line (WL) on a semi-infinite straight trajectory is well-known:

$$W_p(0,\infty) = \exp\left\{\frac{-i\kappa}{2}\int_0^\infty dt\, p_\mu p_\nu\, h^{\mu\nu}(pt)\right\} = \exp\left\{\frac{-\kappa}{2}\int \frac{d^d k}{(2\pi)^d}\frac{p_\mu p_\nu}{p\cdot k}\,\tilde{h}^{\mu\nu}(k)\right\}. \tag{2}$$

When combined in a vacuum expectation value (VEV), WL operators like the one in eq. (2) generate an infinite number of soft emissions along direction p^μ. In particular, as originally proposed for QCD in [2, 3], the high energy limit of scattering amplitudes (and the related classical limit) can be elegantly formulated in terms of Wilson lines.

A generalization at subleading orders in the soft expansion, hence defined as a generalized Wilson line (GWL), has been provided in the literature both in gauge theories and gravity [4–6]. However, it remained not clear how these GWLs could be rigorously derived and how they were related to the perturbative Post-Minkowskian expansion. Here we review how GWLs can be derived from first principles in the worldline formalism and their role in the high energy limit of scattering amplitudes. In doing so, we clarify their usefulness in the computation of classical observables and the connection with similar methods discussed in the literature [7, 8].

2. From the relativistic point particle to generalized Wilson lines

The origin of the worldline representation of relativistic particles can be traced back to the Schwinger's representation of a propagator. In the free case one has

$$\frac{i}{p^2 - m^2 + i\epsilon} = \int_0^\infty dt\, e^{\,i\,(p^2-m^2+i\epsilon)t} \,,$$

[1]For an overview of the available methods see e.g. [1] and references therein.

so that the inverse propagator can be interpreted as a Hamiltonian governing the evolution of a scalar free particle in proper time t. The insertion of a (gauge or gravitational) background field follows analogously, and introduces a dependence on both x and p in the dressed propagator. Equipped with this Hamiltonian, one can set up a constrained quantization procedure (e.g. à la Dirac), thus providing a Hilbert space where the canonical fields $x^\mu(t)$ and $p^\mu(t)$ live in a one-dimensional space, i.e. the worldline.

It is then natural to work out a path integral representation for a propagator dressed with soft radiation in this language. More specifically, one can show that, after truncating the external free line à la LSZ, the propagator for a scalar particle in a gravitational background from an initial state of position x_i to a final state of momentum p_f reads [9]

$$(p_f^2 - m^2 + i\epsilon)\, \langle p_f |\, (2i(H - i\epsilon))^{-1}\, |x_i\rangle \tag{3}$$

$$= e^{i p_f x_i} \int\limits_{x(0)=0} \mathfrak{D}x\mathfrak{D}a\mathfrak{D}b\mathfrak{D}c \exp\left(i \int\limits_0^\infty dt\, e^{-\epsilon t}\, L[x,a,b,c] \right), \tag{4}$$

where

$$L[x,a,b,c] = -\frac{1}{2}\left((\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) g_{\mu\nu}(x) + i(\dot{x} + p_f)^\mu g_{\mu\nu}(x) V^\nu - \frac{1}{4} V^\mu g_{\mu\nu}(x) V^\nu \right). \tag{5}$$

Here a, b, c are ghost fields, whose goal is to remove ultraviolet divergences generated by the x-dependence in the metric $g_{\mu\nu}(x)$. The term V^μ instead is a regularization-dependent counterterm defined in px-ordering as $V^\mu \equiv \partial_\nu g^{\mu\nu}(x) + g^{\mu\nu}(x)(\partial_\nu \ln(\sqrt{-g(x)}))$. In fact, the Hamiltonian that has been used to derive eq. (4) is

$$H_{px} = \frac{1}{2}\left(-p_\mu p_\nu g^{\mu\nu} + m^2 + i p_\mu V^\mu \right), \tag{6}$$

and it slightly differs from similar calculations available in the literature in the more standard Weyl-ordering [10].

The path integral in eq. (4) can be solved order by order in the soft limit. At next-to-soft (or next-to-eikonal) level we get

$$\begin{aligned} \widetilde{W}_p(0,\infty) = \exp\Bigg\{ & \frac{i\kappa}{2} \int_0^\infty dt \left[-p_\mu p_\nu + i p_\nu \partial_\mu - \frac{i}{2}\eta_{\mu\nu} p^\alpha \partial_\alpha + \frac{i}{2} t p_\mu p_\nu \partial^2 \right] h^{\mu\nu}(pt) \\ & + \frac{i\kappa^2}{2} \int_0^\infty dt \int_0^\infty ds \Bigg[\frac{p^\mu p^\nu p^\rho p^\sigma}{4} \min(t,s)\, \partial_\alpha h_{\mu\nu}(pt) \partial^\alpha h_{\rho\sigma}(ps) \\ & + p^\mu p^\nu p^\rho\, \theta(t-s)\, h_{\rho\sigma}(ps) \partial_\sigma h_{\mu\nu}(pt) \;+\; p^\nu p^\sigma\, \delta(t-s)\, h^\mu{}_\sigma(ps) h_{\mu\nu}(pt) \Bigg] \Bigg\}. \end{aligned} \tag{7}$$

The first term in the first line of eq. (7) matches the WL in eq. (2) while the remaining terms provide subleading corrections that generalize eq. (2), hence the name GWL. We verified that the result in eq. (7) is consistent with a previous definition derived with less common conventions [5].

3. Amplitudes, exponentiations and classical limit

At this point we achieved the exponentiation of (subleading) soft emissions along the worldline of a single external line of a scattering process. This worldline exponentiation can be exploited to derive two different exponentiations of the full amplitude [11], after multiple GWLs are combined into a VEV in two different set-ups. The first of these exponentiations, dubbed (next-to-)soft exponentiation, is obtained by considering several GWLs meeting at the origin. For instance, for a $2 \to 2$ process, one considers a (next-to-)soft function defined as [12]

$$\widetilde{S} = \langle 0|\widetilde{W}_{p_1}(-\infty,0)\widetilde{W}_{p_2}(-\infty,0)\widetilde{W}_{p_3}(0,\infty)\widetilde{W}_{p_4}(0,\infty)|0\rangle = \exp(i\mathcal{W}) \, , \tag{8}$$

where the diagrams contributing to $\mathcal{W}$ are known in the literature as webs.

The second (and independent) exponentiation is relevant in the high energy (or Regge) limit of scattering amplitudes, where two highly energetic particle separated by an impact parameter z interact with almost no recoil via the exchange of soft bosons [13–15]. The corresponding eikonal exponentiation is given by [16]

$$\begin{aligned} \mathcal{A}_{\rm E} &= \langle 0|W_{p_1}(0,-\infty,0)W_{p_2}(z,-\infty,0)W_{p_3}(0,0,\infty)W_{p_4}(z,0,\infty)|0\rangle \\ &= \exp\left[K(z)\left(i\pi s + t\log\left(\frac{s}{-t}\right)\right)\right] = e^{i\chi_{\rm E}}\left(\frac{s}{-t}\right)^{K(z)t} \, , \end{aligned} \tag{9}$$

where s and t are the Mandelstam variables. In the Regge limit $s \gg t$ the term with the eikonal phase $\chi_{\rm E}$ is leading w.r.t. the second term which contains information about the Regge trajectory of the graviton. Once again, one can generalize this procedure at subleading power with a next-to-eikonal function by replacing the WLs with GWLs.

It is then natural to ask where the classical information is stored in this language and what is the relation between the Regge and the classical limits. The GWL sheds light on this issue. In fact, one can discriminate classical and quantum bits in eq. (7) by simply restoring powers of $\hbar$ in eq. (4) and eq. (5). Then, it is immediate to see that the terms containing V^μ are suppressed in $\hbar$. Therefore, purely quantum terms are generated by quantum loops in eq. (4) and by the terms containing V^μ in eq. (5). The outcome is that the first line in eq. (7) is the sum of the (classical) WL and a purely quantum contribution, corresponding to the recoil of the hard particle. The second and the third line in eq. (7), on the other hand, represent correlations among soft emissions at different times, and are classical terms contributing at second order in the Post-Minkowskian expansion (2PM). In this way, the GWL provides a nice relation between the soft and the (classical) PM expansion.

Having discriminated classical and quantum bits in the GWL, we can now observe by direct calculation that the (next-to) eikonal phase is completely governed by the classical terms in the GWL. In fact, one has

$$e^{i\chi_{\rm NE}} = \langle 0|\widetilde{W}^{\rm cl}_{p_1}(0,-\infty,\infty)\widetilde{W}^{\rm cl}_{p_2}(z,-\infty,\infty)|0\rangle \, , \tag{10}$$

where we defined the classical GWL as

$$\widetilde{W}_p^{\rm cl}(0,\infty) = \exp\left\{ \frac{-i\kappa}{2}\int_0^\infty dt\, p_\mu p_\nu\, h^{\mu\nu}(pt) \right.$$
$$+ \frac{i\kappa^2}{2}\int_0^\infty dt \int_0^\infty ds \left[\frac{p^\mu p^\nu p^\rho p^\sigma}{4} \min(t,s)\, \partial_\alpha h_{\mu\nu}(pt)\partial^\alpha h_{\rho\sigma}(ps) \right.$$
$$\left.\left. + p^\mu p^\nu p^\rho\, \theta(t-s)\, h_{\rho\sigma}(ps)\partial_\sigma h_{\mu\nu}(pt) \,+\, p^\nu p^\sigma\, \delta(t-s)\, h^\mu{}_\sigma(ps) h_{\mu\nu}(pt) \right]\right\}. \qquad (11)$$

The scattering angle θ is then computed by differentiating $\chi_{\rm NE}$ w.r.t. the impact parameter. On the other hand, the Regge trajectory receives contributions also from the quantum parts and is subleading in the Regge limit. This argument provides a clean explanation at this order in perturbation theory of how the Regge limit corresponds to the classical limit of the scattering of two objects for large impact parameters.

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