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HVP contribution to $g-2$

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Hadronic vacuum polarization currently yields the dominant uncertainty in the Standard-Model prediction for the anomalous magnetic moment of the muon. While the phenomenological approach is only as accurate as the hadronic cross sections used as input, there are several aspects related to chiral dynamics that can be used as cross checks, including $\pi\pi$ dynamics and the chiral anomaly. In the talk I gave an overview over such aspects, including recent work to extrapolate the isovector HVP contribution to unphysical quark masses.

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1. Introduction

Hadronic vacuum polarization (HVP) currently gives the biggest uncertainty in the Standard-Model prediction for the anomalous magnetic moment of the muon [1–28]

$$a_\mu^{\text{SM}} = 116\,591\,810(43) \times 10^{-11}, \tag{1}$$

i.e., when deriving the HVP contribution from $e^+e^- \to$ hadrons cross-section data, the resulting uncertainty in [1, 6–12]

$$a_\mu^{\text{HVP}}\big|_{e^+e^-} = 6\,931(40) \times 10^{-11} \tag{2}$$

dominates the uncertainty quoted in Eq. (1), leading to a 4.2σ difference to experiment [29–33]

$$a_\mu^{\text{exp}} = 116\,592\,061(41) \times 10^{-11}. \tag{3}$$

The consensus value (2) takes into account differences in methodology when combining data sets (especially in the presence of tensions), includes constraints from analyticity and unitarity, and reflects tensions among the data sets by an additional systematic error. Details can be found in Ref. [1]; here, we concentrate on specific aspects of the HVP contribution that are related to chiral dynamics, i.e., the role of the chiral anomaly in $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$ in Sec. 2 and the use of (unitarized) chiral perturbation theory (ChPT) to guide the chiral extrapolation of lattice-QCD results in Sec. 3.[1]

2. Chiral anomaly

The $\gamma^* \to 3\pi$ matrix element can be decomposed as

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma \mathcal{F}(s,t,u;q^2), \tag{4}$$

where the Mandelstam variables describe the momentum dependence of the final-state pions and q^2 the invariant mass of the virtual photon. At low energies, the scalar function $\mathcal{F}(s,t,u;q^2)$ is then constrained by the Wess–Zumino–Witten anomaly [43–47], with the chiral prediction

$$\mathcal{F}(0,0,0;0) = F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3} = 32.23(10)\,\text{GeV}^{-3}. \tag{5}$$

Here, chiral corrections related to the renormalization of the pion decay constant F_π have already been absorbed, while additional quark-mass renormalization increases the value at the physical point by $\simeq 7\%$ [48, 49]. Such constraints have been implemented in fits to data, see Fig. 1, using dispersive representations based on Khuri–Treiman methods [50] in which the chiral anomaly enters as a subtraction constant and thereby adds information especially in the threshold region, where data are scarce. Similar representations have been applied to the pion pole in hadronic light-by-light scattering [21, 22, 51] and $\pi^0 \to e^+e^-$ [52, 53], and it is thus natural to ask how precisely the corresponding low-energy theorems have been tested.

[1] We do not address the 2.1σ tension between Eq. (2) and Ref. [34] (nor the one for the intermediate window [35–37]), see Refs. [38–41] for the consequences of the corresponding change in HVP, but only study the role of chiral symmetry in controlling the quark-mass dependence of the HVP contribution. Apart from the $\pi\pi$ channel, two-pion dynamics also play a role for the $\bar{K}K$ channel, as a constraint on the isovector spectral function [42].

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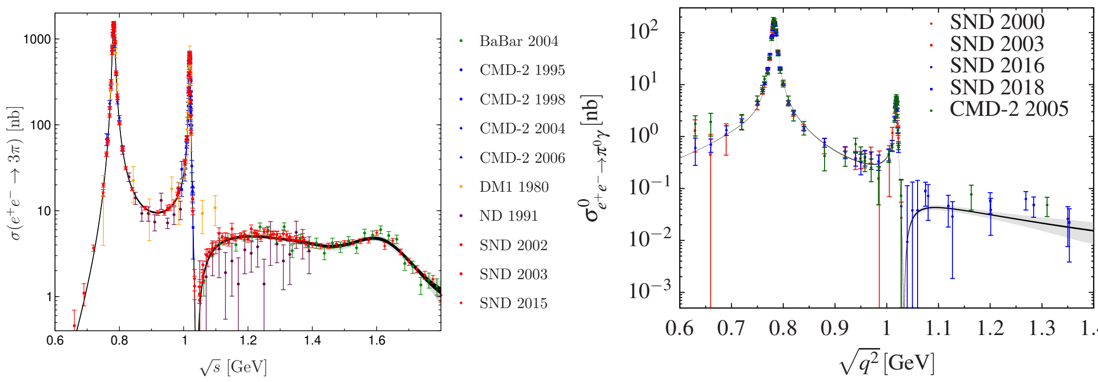

Figure 1: Fits to the $e^+e^- \to 3\pi$ (left) and $e^+e^- \to \pi^0\gamma$ (right) cross section. Figures taken from Refs. [9, 12].

For $\pi^0 \to \gamma\gamma$, the comparison between experiment, $F_{\pi\gamma\gamma} = 0.2754(21)\,\text{GeV}^{-1}$ [54], and the leading chiral prediction, $F_{\pi\gamma\gamma} = 1/(4\pi^2 F_\pi) = 0.2745(3)\,\text{GeV}^{-1}$, demonstrates agreement at subpercent precision, to the extent that higher-order chiral corrections [55] are becoming difficult to reconcile. In contrast, for the $3\pi\gamma$ anomaly only experimental tests at the level of 10% are currently available [56, 57]. This situation could be improved by using the $\rho(770)$ in $\gamma\pi^- \to \pi^-\pi^0$ as a lever [49, 58], to fully exploit the statistics from Primakoff measurements. More recently, also first results in lattice QCD have become available [59–61], but the extraction $F_{3\pi}$ at the physical point still requires an extrapolation in the pion mass [62, 63].

3. Chiral extrapolation

The chiral extrapolation of HVP results is part of the systematic error budget of any lattice calculation, even close to the physical point some interpolation will almost certainly be required. While the corresponding source of error does not appear dominant compared to, e.g., the continuum limit, see Refs. [34, 64–73], the amount of effort invested in these calculations motivates a study of the rigorous constraints by which such an extrapolation (or interpolation) in the pion mass can be guided. Moreover, the dependence on the pion mass determines an important source of isospin breaking [74], yielding a large negative effect that cancels against other positive corrections.

By far the biggest HVP contribution comes from the $I=1$ ud isospin-symmetric correlator, which phenomenologically corresponds to mainly 2π states, with the first inelastic admixture from 4π. However, it was shown in Ref. [75] that a purely perturbative approach is not possible: the ChPT expansion gives

$$a_\mu^{I=1} = \frac{\alpha^2}{24\pi^2}\bigg(-\log\frac{M_\pi^2}{m_\mu^2} - \frac{31}{6} + 3\pi^2\sqrt{\frac{M_\pi^2}{m_\mu^2}} + \mathcal{O}\bigg(\frac{M_\pi^2}{m_\mu^2}\log^2\frac{M_\pi^2}{m_\mu^2}\bigg)\bigg), \tag{6}$$

in such a way that only for $M_\pi < m_\mu$ a convergent behavior is expected (see, however, Ref. [76] for the application of ChPT methods to finite-volume corrections). Accordingly, some information on the ρ meson needs to be provided, which can be achieved within the inverse-amplitude method (IAM) combined with a dispersive representation of 2π intermediate states [77].

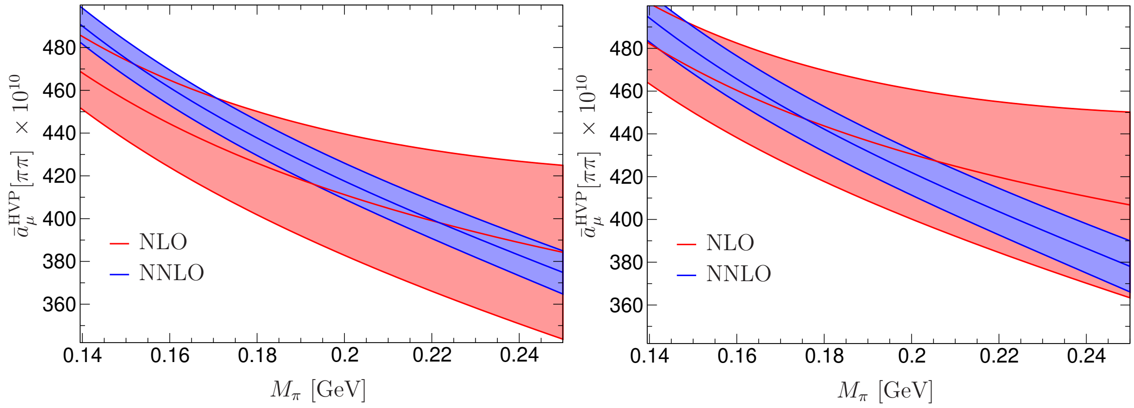

Figure 2: Pion-mass dependence of $\bar{a}_\mu^{\rm HVP}[\pi\pi]$ from the NLO (red) and NNLO (blue) IAM, for a normal (left) and conformal (right) polynomial to describe inelastic effects. Figures taken from Ref. [77].

Starting point is a dispersive decomposition of the pion vector form factor [8, 41, 78, 79]

$$F_\pi^V(s) = \underbrace{\Omega_1^1(s)}_{\text{elastic } \pi\pi \text{ scattering}} \times \underbrace{G_\omega(s)}_{\text{isospin-breaking } 3\pi \text{ cut}} \times \underbrace{G_{\rm in}(s)}_{\text{inelastic effects: } 4\pi,\,\ldots}, \tag{7}$$

in which the Omnès factor $\Omega_1^1(s)$ accounts for the elastic $\pi\pi$ scattering, the isospin-breaking 3π cut can be ignored for the isospin-symmetric correlator, and $G_{\rm in}(s)$ can be expanded in a (conformal) polynomial. For the $\pi\pi$ phase shift δ_1^1 encoded in $\Omega_1^1(s)$ [80] we employ IAM representations at one- (NLO) and two-loop (NNLO) order [81], with parameters determined from a combined fit to lattice QCD [82] and phenomenology [8]. The pion-mass dependence of $G_{\rm in}(s)$ is further constrained by the known two-loop expansion of the pion charge radius $\langle r_\pi^2\rangle$ (and shape parameter c_π) [83]. With the only new low-energy constant $r_{V1}^r = 2.0\times 10^{-5}$ estimated from resonance saturation (in agreement with lattice results for $\langle r_\pi^2\rangle$ [84, 85]), this leads to the prediction for the pion-mass dependence shown in Fig. 2. In particular, the physical point is reproduced within uncertainties.

Based on this combined dispersive + IAM representation, there are two possible strategies for application in lattice QCD:

1. Chiral low-energy constants as fit parameters:[2] in this case, the full IAM representation needs to be implemented, potentially combined with independent constraints from other lattice calculations on δ_1^1, F_π, and $\langle r_\pi^2\rangle$.

2. Simple parameterizations: we studied to which extent the full IAM result can be reproduced by simple functions in M_π. This is only possible for sufficiently smooth functions, such as the HVP integral or the space-like integrand. A purely empirical finding that emerges is that a singularity as strong as M_π^{-2} seems to be preferred in the interval $[0.14, 0.25]$ GeV, which may help inform lattice fits, but of course does not constitute an analytic approximation to the full IAM nor reflect its true chiral behavior.

[2] To account for non-2π states, an additional polynomial contribution $a_\mu^{\rm HVP}[ud, I=1, \text{non-}\pi\pi] = \zeta + M_\pi^2\xi$ will need to be added, but the infrared singularities will be totally dominated by 2π.

4. Conclusions

Control over the HVP contribution is critical to match the full sensitivity of the Fermilab experiment. In general, this requires a precise understanding of the resonance physics in the $e^+e^- \to$ hadrons spectrum, but there are instances in which perturbative insights from chiral symmetry can become advantageous. In this contribution, I discussed two such examples: the role of the chiral anomaly in $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$, and the use of unitarized ChPT, in combination with dispersive techniques, to constrain the chiral extrapolation of the $I = 1$ ud isospin-symmetric correlator.

Acknowledgments

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References

[1] T. Aoyama et al., *The anomalous magnetic moment of the muon in the Standard Model*, *Phys. Rept.* **887** (2020) 1 [2006.04822].

[2] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, *Complete Tenth-Order QED Contribution to the Muon $g-2$*, *Phys. Rev. Lett.* **109** (2012) 111808 [1205.5370].

[3] T. Aoyama, T. Kinoshita and M. Nio, *Theory of the Anomalous Magnetic Moment of the Electron*, *Atoms* **7** (2019) 28.

[4] A. Czarnecki, W.J. Marciano and A. Vainshtein, *Refinements in electroweak contributions to the muon anomalous magnetic moment*, *Phys. Rev. D* **67** (2003) 073006 [hep-ph/0212229].

[5] C. Gnendiger, D. Stöckinger and H. Stöckinger-Kim, *The electroweak contributions to $(g-2)_\mu$ after the Higgs boson mass measurement*, *Phys. Rev. D* **88** (2013) 053005 [1306.5546].

[6] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, *Reevaluation of the hadronic vacuum polarisation contributions to the Standard Model predictions of the muon $g-2$ and $\alpha(m_Z^2)$ using newest hadronic cross-section data*, *Eur. Phys. J. C* **77** (2017) 827 [1706.09436].

[7] A. Keshavarzi, D. Nomura and T. Teubner, *Muon $g-2$ and $\alpha(M_Z^2)$: a new data-based analysis*, *Phys. Rev. D* **97** (2018) 114025 [1802.02995].

[8] G. Colangelo, M. Hoferichter and P. Stoffer, *Two-pion contribution to hadronic vacuum polarization*, *JHEP* **02** (2019) 006 [1810.00007].

[9] M. Hoferichter, B.-L. Hoid and B. Kubis, *Three-pion contribution to hadronic vacuum polarization*, *JHEP* **08** (2019) 137 [1907.01556].

[10] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, *A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to $\alpha(m_Z^2)$*, *Eur. Phys. J. C* **80** (2020) 241 [1908.00921].

[11] A. Keshavarzi, D. Nomura and T. Teubner, *The $g-2$ of charged leptons, $\alpha(M_Z^2)$ and the hyperfine splitting of muonium*, *Phys. Rev. D* **101** (2020) 014029 [1911.00367].

[12] B.-L. Hoid, M. Hoferichter and B. Kubis, *Hadronic vacuum polarization and vector-meson resonance parameters from $e^+e^- \to \pi^0\gamma$*, *Eur. Phys. J. C* **80** (2020) 988 [2007.12696].

[13] A. Kurz, T. Liu, P. Marquard and M. Steinhauser, *Hadronic contribution to the muon anomalous magnetic moment to next-to-next-to-leading order*, *Phys. Lett. B* **734** (2014) 144 [1403.6400].

[14] K. Melnikov and A. Vainshtein, *Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment revisited*, *Phys. Rev. D* **70** (2004) 113006 [hep-ph/0312226].

[15] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Dispersive approach to hadronic light-by-light scattering*, *JHEP* **09** (2014) 091 [1402.7081].

[16] G. Colangelo, M. Hoferichter, B. Kubis, M. Procura and P. Stoffer, *Towards a data-driven analysis of hadronic light-by-light scattering*, *Phys. Lett. B* **738** (2014) 6 [1408.2517].

[17] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Dispersion relation for hadronic light-by-light scattering: theoretical foundations*, *JHEP* **09** (2015) 074 [1506.01386].

[18] P. Masjuan and P. Sánchez-Puertas, *Pseudoscalar-pole contribution to the $(g_\mu - 2)$: a rational approach*, *Phys. Rev. D* **95** (2017) 054026 [1701.05829].

[19] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Rescattering effects in the hadronic-light-by-light contribution to the anomalous magnetic moment of the muon*, *Phys. Rev. Lett.* **118** (2017) 232001 [1701.06554].

[20] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Dispersion relation for hadronic light-by-light scattering: two-pion contributions*, *JHEP* **04** (2017) 161 [1702.07347].

[21] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold and S.P. Schneider, *Pion-pole contribution to hadronic light-by-light scattering in the anomalous magnetic moment of the muon*, *Phys. Rev. Lett.* **121** (2018) 112002 [1805.01471].

[22] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold and S.P. Schneider, *Dispersion relation for hadronic light-by-light scattering: pion pole*, *JHEP* **10** (2018) 141 [1808.04823].

[23] A. Gérardin, H.B. Meyer and A. Nyffeler, *Lattice calculation of the pion transition form factor with $N_f = 2+1$ Wilson quarks*, *Phys. Rev. D* **100** (2019) 034520 [1903.09471].

[24] J. Bijnens, N. Hermansson-Truedsson and A. Rodríguez-Sánchez, *Short-distance constraints for the HLbL contribution to the muon anomalous magnetic moment*, *Phys. Lett. B* **798** (2019) 134994 [1908.03331].

[25] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub and P. Stoffer, *Short-distance constraints on hadronic light-by-light scattering in the anomalous magnetic moment of the muon*, *Phys. Rev. D* **101** (2020) 051501 [1910.11881].

[26] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub and P. Stoffer, *Longitudinal short-distance constraints for the hadronic light-by-light contribution to $(g-2)_\mu$ with large-N_c Regge models*, *JHEP* **03** (2020) 101 [1910.13432].

[27] T. Blum et al., *The hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from lattice QCD*, *Phys. Rev. Lett.* **124** (2020) 132002 [1911.08123].

[28] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera and P. Stoffer, *Remarks on higher-order hadronic corrections to the muon $g-2$*, *Phys. Lett. B* **735** (2014) 90 [1403.7512].

[29] Muon $g-2$ collaboration, *Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL*, *Phys. Rev. D* **73** (2006) 072003 [hep-ex/0602035].

[30] Muon $g-2$ collaboration, *Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm*, *Phys. Rev. Lett.* **126** (2021) 141801 [2104.03281].

[31] Muon $g-2$ collaboration, *Measurement of the anomalous precession frequency of the muon in the Fermilab Muon $g-2$ experiment*, *Phys. Rev. D* **103** (2021) 072002 [2104.03247].

[32] Muon $g-2$ collaboration, *Magnetic Field Measurement and Analysis for the Muon $g-2$ Experiment at Fermilab*, *Phys. Rev. A* **103** (2021) 042208 [2104.03201].

[33] Muon $g-2$ collaboration, *Beam dynamics corrections to the Run-1 measurement of the muon anomalous magnetic moment at Fermilab*, *Phys. Rev. Accel. Beams* **24** (2021) 044002 [2104.03240].

[34] S. Borsanyi et al., *Leading hadronic contribution to the muon magnetic moment from lattice QCD*, *Nature* **593** (2021) 51 [2002.12347].

[35] M. Cè et al., *Window observable for the hadronic vacuum polarization contribution to the muon $g-2$ from lattice QCD*, 2206.06582.

[36] C. Alexandrou et al., *Lattice calculation of the short and intermediate time-distance hadronic vacuum polarization contributions to the muon magnetic moment using twisted-mass fermions*, 2206.15084.

[37] G. Colangelo et al., *Data-driven evaluations of Euclidean windows to scrutinize hadronic vacuum polarization*, *Phys. Lett. B* **833** (2022) 137313 [2205.12963].

[38] A. Crivellin, M. Hoferichter, C.A. Manzari and M. Montull, *Hadronic vacuum polarization: $(g-2)_\mu$ versus global electroweak fits*, *Phys. Rev. Lett.* **125** (2020) 091801 [2003.04886].

[39] A. Keshavarzi, W.J. Marciano, M. Passera and A. Sirlin, *Muon $g-2$ and $\Delta\alpha$ connection*, *Phys. Rev. D* **102** (2020) 033002 [2006.12666].

[40] B. Malaescu and M. Schott, *Impact of correlations between a_μ and α_{QED} on the EW fit*, *Eur. Phys. J. C* **81** (2021) 46 [2008.08107].

[41] G. Colangelo, M. Hoferichter and P. Stoffer, *Constraints on the two-pion contribution to hadronic vacuum polarization*, *Phys. Lett. B* **814** (2021) 136073 [2010.07943].

[42] D. Stamen, D. Hariharan, M. Hoferichter, B. Kubis and P. Stoffer, *Kaon electromagnetic form factors in dispersion theory*, *Eur. Phys. J. C* **82** (2022) 432 [2202.11106].

[43] J. Wess and B. Zumino, *Consequences of anomalous Ward identities*, *Phys. Lett. B* **37** (1971) 95.

[44] E. Witten, *Global Aspects of Current Algebra*, *Nucl. Phys. B* **223** (1983) 422.

[45] S.L. Adler, B.W. Lee, S.B. Treiman and A. Zee, *Low Energy Theorem for $\gamma+\gamma\to\pi+\pi+\pi$*, *Phys. Rev. D* **4** (1971) 3497.

[46] M.V. Terent'ev, *Process $\pi^\pm\to\pi^0\pi^\pm$ in Coulomb field and anomalous divergence of neutral axial vector current*, *Phys. Lett. B* **38** (1972) 419.

[47] R. Aviv and A. Zee, *Low-energy theorem for $\gamma\to 3\pi$*, *Phys. Rev. D* **5** (1972) 2372.

[48] J. Bijnens, A. Bramon and F. Cornet, *Three Pseudoscalar Photon Interactions in Chiral Perturbation Theory*, *Phys. Lett. B* **237** (1990) 488.

[49] M. Hoferichter, B. Kubis and D. Sakkas, *Extracting the chiral anomaly from $\gamma\pi\to\pi\pi$*, *Phys. Rev. D* **86** (2012) 116009 [1210.6793].

[50] N.N. Khuri and S.B. Treiman, *Pion-Pion Scattering and $K^\pm\to 3\pi$ Decay*, *Phys. Rev.* **119** (1960) 1115.

[51] M. Hoferichter, B. Kubis, S. Leupold, F. Niecknig and S.P. Schneider, *Dispersive analysis of the pion transition form factor*, *Eur. Phys. J. C* **74** (2014) 3180 [1410.4691].

[52] M. Hoferichter, B.-L. Hoid, B. Kubis and J. Lüdtke, *Improved Standard-Model prediction for $\pi^0\to e^+e^-$*, *Phys. Rev. Lett.* **128** (2022) 172004 [2105.04563].

[53] B.-L. Hoid, *Improved Standard-Model prediction for the dilepton decay of the neutral pion*, *PoS* **CD2021** (2022) 034.

[54] PrimEx-II collaboration, *Precision measurement of the neutral pion lifetime*, *Science* **368** (2020) 506.

[55] K. Kampf and B. Moussallam, *Chiral expansions of the π^0 lifetime*, *Phys. Rev. D* **79** (2009) 076005 [0901.4688].

[56] Y.M. Antipov et al., *Investigation of $\gamma\to 3\pi$ Chiral Anomaly During Pion Pair Production by Pions in the Nuclear Coulomb Field*, *Phys. Rev. D* **36** (1987) 21.

[57] I. Giller, A. Ocherashvili, T. Ebertshäuser, M.A. Moinester and S. Scherer, *A New determination of the $\gamma\pi \to \pi\pi$ anomalous amplitude via $\pi^- e^- \to \pi^- e^- \pi^0$ data*, *Eur. Phys. J. A* **25** (2005) 229 [hep-ph/0503207].

[58] M. Hoferichter, B. Kubis and M. Zanke, *Radiative resonance couplings in $\gamma\pi \to \pi\pi$*, *Phys. Rev. D* **96** (2017) 114016 [1710.00824].

[59] R.A. Briceño et al., *The resonant $\pi^+\gamma \to \pi^+\pi^0$ amplitude from Quantum Chromodynamics*, *Phys. Rev. Lett.* **115** (2015) 242001 [1507.06622].

[60] R.A. Briceño et al., *The $\pi\pi \to \pi\gamma^\star$ amplitude and the resonant $\rho \to \pi\gamma^\star$ transition from lattice QCD*, *Phys. Rev. D* **93** (2016) 114508 [1604.03530].

[61] C. Alexandrou et al., *$\pi\gamma \to \pi\pi$ transition and the ρ radiative decay width from lattice QCD*, *Phys. Rev. D* **98** (2018) 074502 [1807.08357].

[62] M. Niehus, M. Hoferichter and B. Kubis, *Quark mass dependence of $\gamma^*\pi \to \pi\pi$*, *PoS* **CD2018** (2019) 076 [1902.10150].

[63] M. Niehus, M. Hoferichter and B. Kubis, *The $\gamma\pi \to \pi\pi$ anomaly from lattice QCD and dispersion relations*, *JHEP* **12** (2021) 038 [2110.11372].

[64] Fermilab Lattice, LATTICE-HPQCD, MILC collaboration, *Strong-Isospin-Breaking Correction to the Muon Anomalous Magnetic Moment from Lattice QCD at the Physical Point*, *Phys. Rev. Lett.* **120** (2018) 152001 [1710.11212].

[65] Budapest-Marseille-Wuppertal collaboration, *Hadronic vacuum polarization contribution to the anomalous magnetic moments of leptons from first principles*, *Phys. Rev. Lett.* **121** (2018) 022002 [1711.04980].

[66] RBC, UKQCD collaboration, *Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment*, *Phys. Rev. Lett.* **121** (2018) 022003 [1801.07224].

[67] ETM collaboration, *Electromagnetic and strong isospin-breaking corrections to the muon $g-2$ from Lattice QCD+QED*, *Phys. Rev. D* **99** (2019) 114502 [1901.10462].

[68] E. Shintani and Y. Kuramashi, *Study of systematic uncertainties in hadronic vacuum polarization contribution to muon $g-2$ with 2+1 flavor lattice QCD*, *Phys. Rev. D* **100** (2019) 034517 [1902.00885].

[69] Fermilab Lattice, LATTICE-HPQCD, MILC collaboration, *Hadronic-vacuum-polarization contribution to the muon's anomalous magnetic moment from four-flavor lattice QCD*, *Phys. Rev. D* **101** (2020) 034512 [1902.04223].

[70] A. Gérardin et al., *The leading hadronic contribution to $(g-2)_\mu$ from lattice QCD with $N_f = 2+1$ flavours of $O(a)$ improved Wilson quarks*, *Phys. Rev. D* **100** (2019) 014510 [1904.03120].

[71] C. Aubin et al., *Light quark vacuum polarization at the physical point and contribution to the muon $g-2$*, *Phys. Rev. D* **101** (2020) 014503 [1905.09307].

[72] D. Giusti and S. Simula, *Lepton anomalous magnetic moments in Lattice QCD+QED*, *PoS* **LATTICE2019** (2019) 104 [1910.03874].

[73] C. Lehner and A.S. Meyer, *Consistency of hadronic vacuum polarization between lattice QCD and the R-ratio*, *Phys. Rev. D* **101** (2020) 074515 [2003.04177].

[74] M. Hoferichter et al., *Chiral extrapolation of hadronic vacuum polarization and isospin-breaking corrections*, *PoS* **LATTICE2022** (2022) 316 [2210.11904].

[75] M. Golterman, K. Maltman and S. Peris, *Chiral extrapolation of the leading hadronic contribution to the muon anomalous magnetic moment*, *Phys. Rev. D* **95** (2017) 074509 [1701.08685].

[76] C. Aubin, T. Blum, M. Golterman and S. Peris, *Application of effective field theory to finite-volume effects in a_μ^{HVP}*, *Phys. Rev. D* **102** (2020) 094511 [2008.03809].

[77] G. Colangelo, M. Hoferichter, B. Kubis, M. Niehus and J. Ruiz de Elvira, *Chiral extrapolation of hadronic vacuum polarization*, *Phys. Lett. B* **825** (2022) 136852 [2110.05493].

[78] G. Colangelo, M. Hoferichter, J. Monnard and J. Ruiz de Elvira, *Radiative corrections to the forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$*, *JHEP* **08** (2022) 295 [2207.03495].

[79] G. Colangelo, M. Hoferichter, B. Kubis and P. Stoffer, *Isospin-breaking effects in the two-pion contribution to hadronic vacuum polarization*, *JHEP* **10** (2022) 032 [2208.08993].

[80] F.-K. Guo, C. Hanhart, F.J. Llanes-Estrada and U.-G. Meißner, *Quark mass dependence of the pion vector form factor*, *Phys. Lett. B* **678** (2009) 90 [0812.3270].

[81] M. Niehus, M. Hoferichter, B. Kubis and J. Ruiz de Elvira, *Two-Loop Analysis of the Pion Mass Dependence of the ρ Meson*, *Phys. Rev. Lett.* **126** (2021) 102002 [2009.04479].

[82] C. Andersen, J. Bulava, B. Hörz and C. Morningstar, *The $I = 1$ pion-pion scattering amplitude and timelike pion form factor from $N_\mathrm{f} = 2 + 1$ lattice QCD*, *Nucl. Phys. B* **939** (2019) 145 [1808.05007].

[83] J. Bijnens, G. Colangelo and P. Talavera, *The Vector and scalar form-factors of the pion to two loops*, *JHEP* **05** (1998) 014 [hep-ph/9805389].

[84] X. Feng, Y. Fu and L.-C. Jin, *Lattice QCD calculation of the pion charge radius using a model-independent method*, *Phys. Rev. D* **101** (2020) 051502 [1911.04064].

[85] χQCD collaboration, *Lattice Calculation of Pion Form Factor with Overlap Fermions*, *Phys. Rev. D* **104** (2021) 074502 [2006.05431].