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Dispersive representation of C- and CP-violation in $\eta \to \pi^+\pi^-\pi^0$ and $\eta' \to \eta\pi^+\pi^-$

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We present a summary of a model-independent, non-perturbative construction of the hadronic amplitudes $\eta \to \pi^+\pi^-\pi^0$ and $\eta' \to \eta\pi^+\pi^-$, allowing for C- and CP-violating asymmetries in the $\pi^+\pi^-$ distributions. These amplitudes are consistent with the constraints of analyticity and unitarity. We find that the currently most accurate Dalitz-plot distributions taken by the KLOE-2 and BESIII collaborations confine the patterns of these asymmetries to a relative per mille and per cent level, respectively. Our dispersive representation allows us to extract the individual coupling strengths of the C- and CP-violating contributions arising from effective isoscalar and isotensor operators in $\eta^{(\prime)} \to \pi^+\pi^-\pi^0$ and an effective isovector operator in $\eta' \to \eta\pi^+\pi^-$.[†]

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1. Introduction

According to Sakharov [2], one prerequisite to dynamically create the observed asymmetry between matter and antimatter during the baryogenesis is a simultaneous violation of the discrete space-time symmetries C and CP. Given the dearth of experimental evidence for sources of CP-violation beyond the weak interactions of the Standard Model (SM), it is worthwhile to investigate potential new sources, such as T-odd and P-even (TOPE) interactions. Suitable candidates to analyze these kinds of operators are certain decays of the $\eta^{(\prime)}$ mesons [3], which are eigenstates of C. These allow us to investigate TOPE forces in the absence of the weak interaction, such that the observation of a C-violating $\eta^{(\prime)}$ decay would automatically indicate physics beyond the Standard Model (BSM). An ideal stage for this endeavour is provided by studying the charge asymmetries in the Dalitz-plot distributions of $\eta \to \pi^+\pi^-\pi^0$ and $\eta' \to \eta\pi^+\pi^-$. These two decays serve as orthogonal probes sensitive to different classes of TOPE operators driven by isospin transitions $\Delta I = 0, 2$ and $\Delta I = 1$, respectively. For a rigorous construction of the BSM amplitudes consistent with the fundamental principles of analyticity (a mathematical description of causality) and unitarity (a consequence of probability conservation) we rely on techniques from dispersion theory, using the so-called Khuri–Treiman representations [4], allowing for a model-independent resummation of re-scattering effects to all orders.

2. Dispersive representation of $\eta \to \pi^+\pi^-\pi^0$

As G-parity demands, the SM contribution to $\eta \to \pi^+\pi^-\pi^0$ is driven by a transition of total isospin $\Delta I = 1$, whereas the C-violating mechanisms underlie isoscalar $\Delta I = 0$ or isotensor $\Delta I = 2$ operators, such that the generalized $\eta \to \pi^+\pi^-\pi^0$ amplitude has to be of the form [5]

$$\mathcal{M}(s,t,u) = \mathcal{M}_0^{\not{C}}(s,t,u) + \xi\, \mathcal{M}_1^{C}(s,t,u) + \mathcal{M}_2^{\not{C}}(s,t,u)\,, \qquad \xi = \frac{\left(M_{K^+}^2 - M_{K^0}^2\right)_{\text{QCD}}}{3\sqrt{3}F_\pi^2}\,, \tag{1}$$

which is split into a contribution for each total isospin denoted by the respective index. In accordance with Refs. [6, 7], we factorized out the isospin-breaking normalization of the SM amplitude $\xi = -0.140(9)$. As isospin is an accidental (approximate) symmetry of the strong interactions and as we do not know anything about the isospin structure of the BSM operators, there is no reason to assume isospin to be a useful symmetry for them, too, and hence imply any kind of hierarchy between isoscalar and isotensor C-violation on the underlying, fundamental level.

2.1 Reconstruction theorem

Neglecting the discontinuities of D- and higher partial waves, one may express each amplitude of *total* isospin, on the right-hand side of Eq. (1), in terms of functions depending on only one kinematical variable, the relative angular momentum, and isospin of the $\pi\pi$ intermediate state [5]:

$$\begin{aligned}
\mathcal{M}_1^{C}(s,t,u) &= \mathcal{F}_0(s) + (s-u)\,\mathcal{F}_1(t) + (s-t)\,\mathcal{F}_1(u) + \mathcal{F}_2(t) + \mathcal{F}_2(u) - \frac{2}{3}\mathcal{F}_2(s)\,, \\
\mathcal{M}_0^{\not{C}}(s,t,u) &= (t-u)\,\mathcal{G}_1(s) + (u-s)\,\mathcal{G}_1(t) + (s-t)\,\mathcal{G}_1(u)\,, \\
\mathcal{M}_2^{\not{C}}(s,t,u) &= 2(u-t)\,\mathcal{H}_1(s) + (u-s)\,\mathcal{H}_1(t) + (s-t)\,\mathcal{H}_1(u) - \mathcal{H}_2(t) + \mathcal{H}_2(u)\,.
\end{aligned} \tag{2}$$

Due to Bose symmetry, the isospin I of the two-pion state fixes the partial wave ℓ unambiguously by means of $\mathcal{A}_{0,2} \equiv \mathcal{A}^{\ell=0}_{I=0,2}$ and $\mathcal{A}_1 \equiv \mathcal{A}^{\ell=1}_{I=1}$, with $\mathcal{A} \in \{\mathcal{F}, \mathcal{G}, \mathcal{H}\}$. Note that the single-variable functions (SVAs) $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are completely decoupled and can be evaluated independently.

2.2 Elastic unitarity

Within the scope of this work we will exclusively study the dominant elastic rescattering effects, i.e., we restrict the evaluation of the single-variable functions to $\pi\pi$ intermediate states only. In order to obtain an amplitude with manifest unitarity, each SVA has to obey the discontinuity relation

$$\mathrm{disc}\,\mathcal{A}_I(s) = 2i\,\theta(s - 4M_\pi^2)\left[\mathcal{A}_I(s) + \hat{\mathcal{A}}_I(s)\right]\,\sin\delta_I(s)\,e^{-i\delta_I(s)}\,. \tag{3}$$

Here we introduced the so called *inhomogeneities* $\hat{\mathcal{A}}_I(s)$ that do not have a discontinuity along the right-hand cut. Let us now, for the sake of simplicity, define the angular average

$$\langle z_s^n\,\mathcal{A}_I\rangle \equiv \frac{1}{2}\int_{-1}^{1} \mathrm{d}z_s\, z_s^n\,\mathcal{A}_I\big(t(s,z_s)\big)\,. \tag{4}$$

This allows to write the inhomogeneities for the Standard-Model amplitude in the shortened form

$$\begin{aligned}
\hat{\mathcal{F}}_0(s) &= \frac{2}{9}\left[3\langle\mathcal{F}_0\rangle + 9(s-r)\,\langle\mathcal{F}_1\rangle + 3\kappa\,\langle z_s\,\mathcal{F}_1\rangle + 10\langle\mathcal{F}_2\rangle\right],\\
\hat{\mathcal{F}}_1(s) &= \frac{1}{2\kappa}\left[6\langle z_s\,\mathcal{F}_0\rangle + 9(s-r)\,\langle z_s\,\mathcal{F}_1\rangle + 3\kappa\,\langle z_s^2\,\mathcal{F}_1\rangle - 10\langle z_s\,\mathcal{F}_2\rangle\right],\\
\hat{\mathcal{F}}_2(s) &= \frac{1}{6}\left[6\langle\mathcal{F}_0\rangle - 9(s-r)\,\langle\mathcal{F}_1\rangle - 3\kappa\,\langle z_s\,\mathcal{F}_1\rangle + 2\langle\mathcal{F}_2\rangle\right],
\end{aligned} \tag{5}$$

and the ones for the C-violating contributions as

$$\begin{aligned}
\hat{\mathcal{G}}_1(s) &= -\frac{3}{\kappa}\left[3(s-r)\,\langle z_s\,\mathcal{G}_1\rangle + \kappa\,\langle z_s^2\,\mathcal{G}_1\rangle\right],\\
\hat{\mathcal{H}}_1(s) &= \frac{3}{2\kappa}\left[3(s-r)\,\langle z_s\,\mathcal{H}_1\rangle + \kappa\,\langle z_s^2\,\mathcal{H}_1\rangle + 2\langle z_s\,\mathcal{H}_2\rangle\right],\\
\hat{\mathcal{H}}_2(s) &= \frac{1}{2}\left[9(s-r)\,\langle\mathcal{H}_1\rangle + 3\kappa\,\langle z_s\,\mathcal{H}_1\rangle - 2\langle\mathcal{H}_2\rangle\right].
\end{aligned} \tag{6}$$

The general solution becomes

$$\mathcal{A}_I(s) = \Omega_I(s)\left(P_{n-1}(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x^n}\,\frac{\sin\delta_I(x)\,\hat{\mathcal{A}}_I(x)}{|\Omega_I(x)|\,(x-s)}\right), \tag{7}$$

where

$$\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\delta_I(x)}{(x-s)}\right) \tag{8}$$

is the Omnès function [8] and $P_{n-1}(s)$ a polynomial in s of order $n-1$. The coefficients of the latter—known as subtraction constants—are the only free parameters of our amplitude. Throughout this paper, we will assume all subtraction constants within the same decay amplitude representation to be *relatively real*, as the potential imaginary parts of the subtraction constants scale with the available three-body phase space, and therefore are tiny for decays such as $\eta \to 3\pi$ or $\eta' \to \eta\pi\pi$.

Besides these degrees of freedoms, the only input for the dispersive $\eta \to 3\pi$ amplitude are the $\pi\pi$ scattering phase shifts $\delta_I(s)$. For the elaborate choice of the $\delta_I(s)$ and the asymptotics of the SVAs used in this analysis we refer to Ref. [1]. The resulting minimal subtraction scheme for the C-conserving Standard-Model amplitude yields

$$\begin{aligned} \mathcal{F}_0(s) &= \Omega_0(s)\left(\alpha + \beta\, s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x^2}\,\frac{\sin\delta_0(x)\,\hat{\mathcal{F}}_0(x)}{|\Omega_0(x)|\,(x-s)}\right), \\ \mathcal{F}_1(s) &= \Omega_1(s)\left(\gamma + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\sin\delta_1(x)\,\hat{\mathcal{F}}_1(x)}{|\Omega_1(x)|\,(x-s)}\right), \\ \mathcal{F}_2(s) &= \Omega_2(s)\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\sin\delta_2(x)\,\hat{\mathcal{F}}_2(x)}{|\Omega_2(x)|\,(x-s)}\right). \end{aligned} \tag{9}$$

and analogously the C-violating contributions become

$$\begin{aligned} \mathcal{G}_1(s) &= \Omega_1(s)\left(\varepsilon + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\sin\delta_1(x)\,\hat{\mathcal{G}}_1(x)}{|\Omega_1(x)|\,(x-s)}\right), \\ \mathcal{H}_1(s) &= \Omega_1(s)\left(\vartheta + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\sin\delta_1(x)\,\hat{\mathcal{H}}_1(x)}{|\Omega_1(x)|\,(x-s)}\right), \\ \mathcal{H}_2(s) &= \Omega_2(s)\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\sin\delta_2(x)\hat{\mathcal{H}}_2(x)}{|\Omega_2(x)|\,(x-s)}\right). \end{aligned} \tag{10}$$

Further, we would like to remark that the normalization of each amplitude of total isospin in Eq. (1) has a phase that is fixed unambiguously by T-violation and hermiticity. Hence, the subtraction constants ε and ϑ, which absorb these normalizations, are complex quantities with a fixed phase, resulting in a total of five degrees of freedom for $\mathcal{M}$.

2.3 Taylor invariants

As pointed out in Ref. [1], the subtraction constants of our dispersive representation are no meaningful observables. Therefore we follow the idea of Refs. [6, 7], where certain linear combinations of the subtraction constants for the SM contribution were introduced, which are identified as so-called Taylor invariants. To access those, the single-variable amplitudes $\mathcal{A}_I \in \{\mathcal{F}_I, \mathcal{G}_I, \mathcal{H}_I\}$ are expanded around $s = 0$, i.e.,

$$\mathcal{A}_I(s) = A_I^{\mathcal{A}} + B_I^{\mathcal{A}}\, s + C_I^{\mathcal{A}}\, s^2 + D_I^{\mathcal{A}}\, s^3 + \ldots\,. \tag{11}$$

From elementary considerations such as crossing symmetry and the correct behavior under time reversal, the effective BSM operators for $\Delta I = 0$ and $\Delta I = 2$ transitions at lowest contributing order are expected to be of the form

$$\begin{aligned} \mathcal{M}_0^{\not{C}}(s,t,u) &= i\, g_0\,(s-t)(u-s)(t-u) + O(p^8)\,, \\ \mathcal{M}_2^{\not{C}}(s,t,u) &= i\, g_2\,(t-u) + O(p^4)\,, \end{aligned} \tag{12}$$

where the couplings have the dimensions $[g_0] = \mathrm{GeV}^{-6}$ and $[g_2] = \mathrm{GeV}^{-2}$, respectively. Reproducing this structure with the Taylor series from above we obtain

$$g_0 = i\left(C_1^{\mathcal{G}} + 3r\, D_1^{\mathcal{G}}\right), \qquad g_2 = i\left(3A_1^{\mathcal{H}} + 3r\, B_1^{\mathcal{H}} + B_2^{\mathcal{H}} + 2r\, C_2^{\mathcal{H}}\right). \tag{13}$$

Note that the requirement $g_0, g_2 \in \mathbb{R}$ fixes the phase of the subtraction constants ε and ϑ. However, due to the sufficiently small phase space, ε and ϑ can be considered as purely imaginary constants. It has to be remarked that this simple polynomial expansion is by far less accurate than the full dispersive representation, but allows one to match the couplings in a convenient way.

2.4 Extraction of observables

We first fix our SM amplitude by a regression to the currently most precise Dalitz plot from the KLOE-2 collaboration [9]. The goodness of our fit with three *real* degrees of freedom yields $\chi^2_{\text{red}} \approx 1.054$. Our minimal subtraction scheme fulfils various constraints imposed on the SM amplitude [1] by Taylor invariants [7], polynomial Dalitz-plot parameters [9], the slope parameter of $\eta \to 3\pi^0$, and the branching ratio $\text{BR}(\eta \to 3\pi^0)/\text{BR}(\eta \to \pi^0\pi^+\pi^-)$ [10]. We take this as justification for applying the minimal subtraction scheme to the BSM contributions.

The regression of our full amplitude, with seven *real* degrees of freedom in total, amounts to a marginal improvement of $\chi^2_{\text{red}} \approx 1.048$. To investigate the Dalitz-plot asymmetries we employ the decomposition

$$\left|\mathcal{M}_c\right|^2 \approx \left|\xi\, \mathcal{M}_1^C\right|^2 + 2\text{Re}\left[\xi\, \mathcal{M}_1^C\, (\mathcal{M}_0^{\not C})^*\right] + 2\text{Re}\left[\xi\, \mathcal{M}_1^C\, (\mathcal{M}_2^{\not C})^*\right], \tag{14}$$

where we neglected all terms quadratic in C-violating amplitudes. The disentangled contributions to the Dalitz-plot distribution for our central fit results are depicted in Fig. 1. Obviously, the C-conserving SM part determined by $\mathcal{M}_1^C$ is dominating, while the two terms linear in the C-violating amplitudes $\mathcal{M}_0^{\not C}$ and $\mathcal{M}_2^{\not C}$ are suppressed by three orders of magnitude. Furthermore we find the interference effects of $\mathcal{M}_1^C$ with $\mathcal{M}_0^{\not C}$ and $\mathcal{M}_2^{\not C}$, which determine the size of the mirror symmetry breaking of the Dalitz-plot distribution under $t \leftrightarrow u$, to be of similar size. Accordingly, $\mathcal{M}_0^{\not C}$ and $\mathcal{M}_2^{\not C}$ are of the same order of magnitude. We quantify the occurring asymmetries, i.e., the left-right A_{LR}, the quadrant A_Q, and sextant A_S asymmetry parameters [11–13], by integrating over the population of the Dalitz-plot distribution in the different sectors defined by the Dalitz-plot geometry, cf. Fig. 1, and obtain

$$A_{LR} = -7.9(4.5)\,, \qquad A_Q = 1.9(2.5)\,, \qquad A_S = 2.0(3.8)\,, \tag{15}$$

where all three asymmetry parameters are given in units of 10^{-4}. For correlations of the observables presented in this work we refer to Ref. [1]. We find A_{LR}, A_Q, and A_S in good agreement with the results reported by the KLOE-2 collaboration [9]. Again, there is no hint for C-violation as all three asymmetries are compatible with zero in not more than 1.8σ. Note that the error budget in Eq. (15) is completely dominated by the statistical uncertainties of the KLOE-2 data.

Our dispersive representation allows us to extract coupling strengths g_0 and g_2 of the underlying isoscalar and isotensor BSM operators as defined Eq. (13), for which we obtain

$$g_0/\text{GeV}^{-6} = -2.8(4.5)\,, \qquad g_2/10^{-3}\,\text{GeV}^{-2} = -9.3(4.6)\,, \qquad . \tag{16}$$

Note that for the central values we find a ratio of $g_0/g_2 \approx 10^3\,\text{GeV}^{-4}$, demonstrating a much reduced sensitivity for constraining g_0. Furthermore we can utilize these coupling strengths to obtain a more general representation of the Dalitz-plot asymmetries. Carrying out the phase space

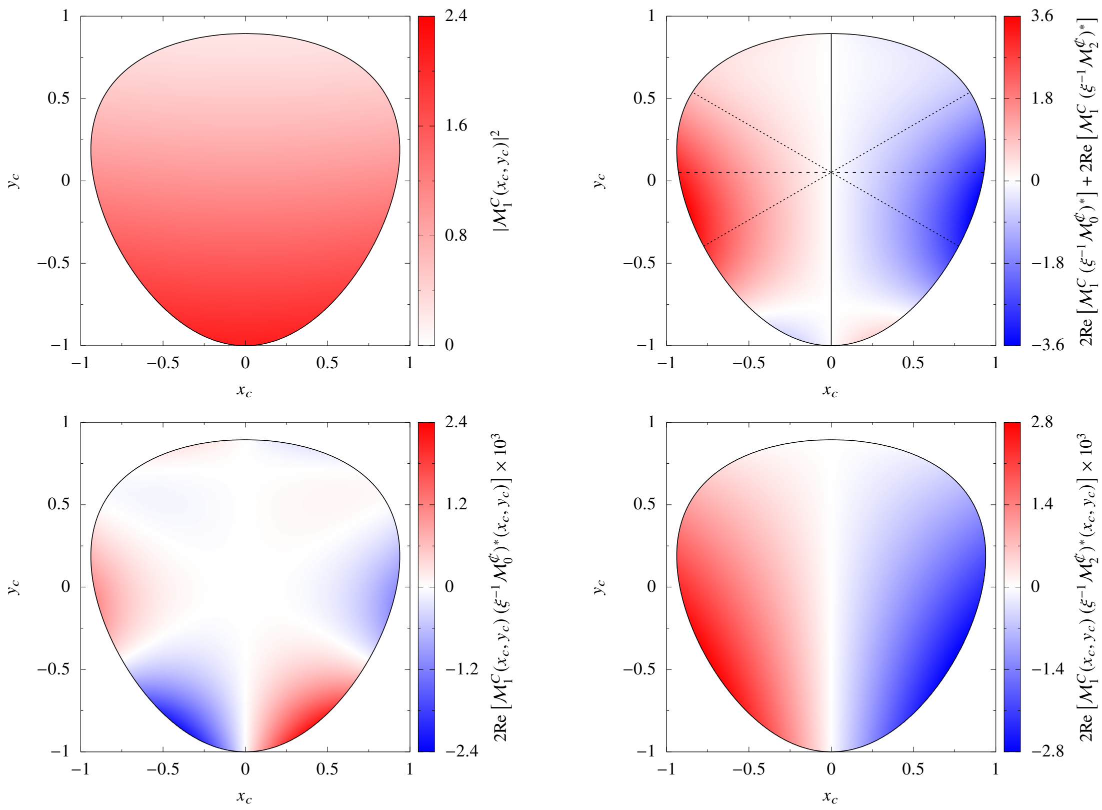

Figure 1: Decomposition of the Dalitz-plot distribution for $\eta \to \pi^+\pi^-\pi^0$ as given in Eq. (14) for the central fit result. The normalization is chosen such that $|\xi\mathcal{M}_1^C|^2$ (top left) is one in the center. Note the individual scales of each contribution. The interferences of $\mathcal{M}_1^C$ with $\mathcal{M}_0^{\not C}$ (bottom left) and $\mathcal{M}_2^{\not C}$ (bottom right) give rise to mirror symmetry breaking in the Dalitz plot. The total C-violating contributions to the full Dalitz plot is shown in the upper right, including the symmetry axes to define asymmetry parameters.

integrals individually for contributions involving interference effects of $\mathcal{M}_0^{\not C}$ or $\mathcal{M}_2^{\not C}$ in the Dalitz-plot distribution, we find that the asymmetry parameters (15) given in units of 10^{-4} are related to the BSM couplings g_0 and g_2 by

$$\begin{aligned} A_{LR} &= -0.300\, g_0 + 0.936\, g_2\,, \\ A_Q &= 0.443\, g_0 - 0.336\, g_2\,, \\ A_S &= -0.850\, g_0 + 0.043\, g_2\,. \end{aligned} \tag{17}$$

Here, g_0 and g_2 enter in units of $1\,\mathrm{GeV}^{-6}$ and $10^{-3}\,\mathrm{GeV}^{-2}$, respectively. Equation (17) shows that especially the sextant asymmetry parameter A_S is sensitive to contributions generated by $\mathcal{M}_0^{\not C}$.

Finally, we refer to Ref. [1] regarding a generalization of the dispersive analysis to $\eta' \to \pi^+\pi^-\pi^0$.

3. Dispersive representation of $\eta' \to \eta\pi\pi$

In this section we turn our attention to another class of TOPE forces by studying the decay $\eta' \to \eta\pi^+\pi^-$. As this decay preserves G-parity, transitions of even isospin $\Delta I = 0, 2$ conserve C, while odd ones violate the latter. Thus we can write the most general amplitude up to linear order in isospin breaking as

$$\mathcal{M}(s,t,u) = \mathcal{M}_0^{C}(s,t,u) + \mathcal{M}_1^{\not{C}}(s,t,u), \tag{18}$$

where for this decay, the isoscalar amplitude $\mathcal{M}_0^C$ is isospin- and C-conserving, whereas $\mathcal{M}_1^{\not{C}}$ violates both quantum numbers. Note that the decay $\eta' \to \eta\pi^+\pi^-$ is sensitive to a different class of C- and CP-violating operators from those tested in $\eta^{(\prime)} \to \pi^+\pi^-\pi^0$, namely the ones for transitions with $\Delta I = 1$.

3.1 Reconstruction theorem

In the ongoing, we restrict our amplitude to discontinuities in the lowest contributing partial waves, i.e., to $\ell = 0$ for $\pi\pi$ states with isospin $I = 0$, or $\ell = 1$ for those with $I = 1$, and to $\ell = 0$ for the $\eta\pi$ system with $I = 1$. With these approximations the decomposition of the Standard-Model amplitude in terms of single-variable functions takes the simple form [14]

$$\mathcal{M}_0^C(s,t,u) = \mathcal{F}_{\pi\pi}(s) + \mathcal{F}_{\eta\pi}(t) + \mathcal{F}_{\eta\pi}(u)\,, \tag{19}$$

with the abbreviations $\mathcal{F}_{\pi\pi}(s) \equiv \mathcal{F}^{\ell=0}_{I=0\,\pi\pi}(s)$ and $\mathcal{F}_{\eta\pi}(t) \equiv \mathcal{F}^{\ell=0}_{I=1\,\eta\pi}(t)$. In this notation the indices $\pi\pi$ and $\eta\pi$ denote the two-particle final state of the respective scattering process. In a similar fashion we obtain the reconstruction theorem for the C-violating amplitude

$$\mathcal{M}_1^{\not{C}}(s,t,u) = (t-u)\,\mathcal{G}_{\pi\pi}(s) + \mathcal{G}_{\eta\pi}(t) - \mathcal{G}_{\eta\pi}(u)\,. \tag{20}$$

In this equation we use the short forms $\mathcal{G}_{\pi\pi}(s) \equiv \mathcal{G}^1_{1\,\pi\pi}(s)$ and $\mathcal{G}_{\eta\pi}(t) \equiv \mathcal{G}^0_{1\,\eta\pi}(t)$.

3.2 Elastic unitarity

To ensure the conservation of probability, the single-variable functions have to obey

$$\begin{aligned}
\operatorname{disc}\mathcal{A}_{\pi\pi}(s) &= 2i\,\theta(s-4M_\pi^2)\left[\mathcal{A}_{\pi\pi}(s) + \hat{\mathcal{A}}_{\pi\pi}(s)\right]\,\sin\delta_{\pi\pi}(s)\,e^{-i\delta_{\pi\pi}(s)}\,,\\
\operatorname{disc}\mathcal{A}_{\eta\pi}(t) &= 2i\,\theta\big(t-(M_\eta+M_\pi)^2\big)\left[\mathcal{A}_{\eta\pi}(t) + \hat{\mathcal{A}}_{\eta\pi}(t)\right]\,\sin\delta_{\eta\pi}(t)\,e^{-i\delta_{\eta\pi}(t)}\,,
\end{aligned} \tag{21}$$

with $\mathcal{A} \in \{\mathcal{F},\mathcal{G}\}$ and the indices of the phase shifts labeling the respective two-particle intermediate states. Note that in case of $\mathcal{M}_0$ the $\pi\pi$-state has isospin $I = 0$, such that $\delta_{\pi\pi} = \delta_{\pi\pi}^{I=0}$ for $\mathcal{A} = \mathcal{F}$. Analogously, the C-odd contribution $\mathcal{M}_1^{\not{C}}$ is driven by a $\pi\pi$-state that has isospin $I = 1$, i.e., $\delta_{\pi\pi} = \delta_{\pi\pi}^{I=1}$ for $\mathcal{A} = \mathcal{G}$. Introducing the abbreviations

$$\begin{aligned}
\langle z_s^n\,\mathcal{A}\rangle &\equiv \frac{1}{2}\int_{-1}^{1}\mathrm{d}z_s\,z_s^n\,\mathcal{A}\big(t(s,z_s)\big)\,,\\
\langle z_t^n\,\mathcal{A}\rangle^+ \equiv \frac{1}{2}\int_{-1}^{1}\mathrm{d}z_t\,z_t^n\,\mathcal{A}\big(u(t,z_t)\big)\,, &\qquad \langle z_t^n\,\mathcal{A}\rangle^- \equiv \frac{1}{2}\int_{-1}^{1}\mathrm{d}z_t\,z_t^n\,\mathcal{A}\big(s(t,-z_t)\big)\,,
\end{aligned} \tag{22}$$

the inhomogeneities for the Standard-Model amplitude become

$$\hat{\mathcal{F}}_{\pi\pi}(s) = 2\langle\mathcal{F}_{\eta\pi}\rangle\,, \qquad \hat{\mathcal{F}}_{\eta\pi}(t) = \langle\mathcal{F}_{\pi\pi}\rangle^- + \langle\mathcal{F}_{\eta\pi}\rangle^+\,, \tag{23}$$

and the ones entering the C-violating amplitude yield

$$\hat{\mathcal{G}}_{\pi\pi}(s) = \frac{6}{\kappa_{\pi\pi}} \langle z_s\, \mathcal{G}_{\eta\pi}\rangle\,, \quad \hat{\mathcal{G}}_{\eta\pi}(t) = -\langle \mathcal{G}_{\eta\pi}\rangle^+ - \frac{3}{2}\left(r - t + \frac{\Delta}{3t}\right)\langle \mathcal{G}_{\pi\pi}\rangle^- + \frac{1}{2}\kappa_{\eta\pi}\,\langle z_t\, \mathcal{G}_{\pi\pi}\rangle^-\,. \tag{24}$$

Again we fix the input phase shifts and the high-energy behavior of each independent constituent entering the dispersive representation according to Ref. [1], leading to a representation of the corresponding SVAs involving four (real) subtraction constants in the SM amplitude,

$$\begin{aligned}
\mathcal{F}_{\pi\pi}(s) &= \Omega^0_{\pi\pi}(s)\left(\alpha + \beta\, s + \gamma\, s^2 + \frac{s^3}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x^3}\,\frac{\sin\delta^0_{\pi\pi}(x)\,\hat{\mathcal{F}}_{\pi\pi}(x)}{|\Omega^0_{\pi\pi}(s')|\,(x-s)}\right),\\
\mathcal{F}_{\eta\pi}(t) &= \Omega_{\eta\pi}(t)\left(\lambda\, t^2 + \frac{t^3}{\pi}\int_{t_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x^3}\,\frac{\sin\delta_{\eta\pi}(x)\,\hat{\mathcal{F}}_{\eta\pi}(x)}{|\Omega_{\eta\pi}(x)|\,(x-t)}\right),
\end{aligned} \tag{25}$$

and two (complex) subtraction constants in the C-violating SVAs,

$$\begin{aligned}
\mathcal{G}_{\pi\pi}(s) &= \Omega^1_{\pi\pi}(s)\left(\varrho + \frac{s}{\pi}\int_{s_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x}\,\frac{\sin\delta^1_{\pi\pi}(x)\,\hat{\mathcal{G}}_{\pi\pi}(x)}{|\Omega^1_{\pi\pi}(x)|\,(x-s)}\right),\\
\mathcal{G}_{\eta\pi}(t) &= \Omega_{\eta\pi}(t)\left(\zeta\, t + \frac{t^2}{\pi}\int_{t_{\text{th}}}^{\infty}\frac{\mathrm{d}x}{x^2}\,\frac{\sin\delta_{\eta\pi}(x)\,\hat{\mathcal{G}}_{\eta\pi}(x)}{|\Omega_{\eta\pi}(x)|\,(x-t)}\right).
\end{aligned} \tag{26}$$

The index of each Omnès function decides which scattering phase shift is used according to Eq. (8). In addition to that, one has to differentiate the case $\Omega_{\pi\pi} = \Omega^{I=0}_{\pi\pi}$ for $\mathcal{A} = \mathcal{F}$ from $\Omega_{\pi\pi} = \Omega^{I=1}_{\pi\pi}$ for $\mathcal{A} = \mathcal{G}$. Again, the phase of the subtraction constants ϱ and ζ is fixed by T-violation, so that $\mathcal{M}_1^{\mathcal{C}}$ has two *real*-valued degrees of freedom, in contrast to the C-violating isoscalar and isotensor contributions in $\eta \to 3\pi$ which are fixed by a single normalization each.

3.3 Taylor invariants

Similarly to Sect. 2.3, the subtraction constants fixing our dispersive representation have to be translated into meaningful observables. Therefore we again introduce their linear combinations as ambiguity-free Taylor invariants obtained by an expansion of the SVAs around $s, t = 0$, i.e.,

$$\begin{aligned}
\mathcal{A}_{\pi\pi}(s) &= A^{\mathcal{A}}_{\pi\pi} + B^{\mathcal{A}}_{\pi\pi}\, s + C^{\mathcal{A}}_{\pi\pi}\, s^2 + D^{\mathcal{A}}_{\pi\pi}\, s^3 + \ldots\,,\\
\mathcal{A}_{\eta\pi}(t) &= A^{\mathcal{A}}_{\eta\pi} + B^{\mathcal{A}}_{\eta\pi}\, t + C^{\mathcal{A}}_{\eta\pi}\, t^2 + D^{\mathcal{A}}_{\eta\pi}\, t^3 + \ldots\,.
\end{aligned} \tag{27}$$

The matrix element for the $\Delta I = 1$ transition takes the form

$$\mathcal{M}_1^{\mathcal{C}}(s,t,u) = i\, g_1\,(t-u)\,(1 + s\,\delta g_1) + O(p^6)\,, \tag{28}$$

where in addition to the effective isovector coupling g_1, we also consider the leading s-dependent correction δg_1. In terms of the Taylor coefficients these quantities read

$$g_1 = -i\left(A^{\mathcal{G}}_{\pi\pi} + B^{\mathcal{G}}_{\eta\pi} + 3r\, C^{\mathcal{G}}_{\eta\pi}\right), \qquad \delta g_1 = -i\big(B^{\mathcal{G}}_{\pi\pi} - C^{\mathcal{G}}_{\eta\pi}\big)/g_1\,. \tag{29}$$

Note that the additional parameter δg_1 ensures that the degrees of freedom of the Taylor expansion match the ones of the dispersive representation for $\mathcal{M}_1^{\mathcal{C}}$. Both couplings are real-valued as demanded by T-violation and give rise to the phases of the subtraction constants ϱ and ζ. Anyway, the latter can be considered as purely imaginary due to the small available phase space.

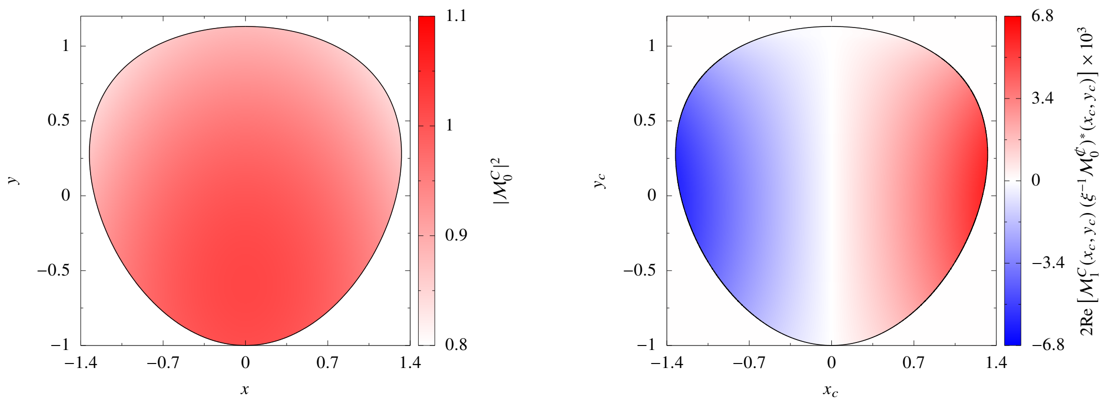

Figure 2: Dalitz-plot decomposition for $\eta' \to \eta\pi^+\pi^-$ as given in Eq. (30) for our central solution. The normalization is chosen such that the full amplitude $|\mathcal{M}|^2$ is one in its center.

3.4 Extraction of observables

Comparing our dispersive representation of the SM amplitude $\mathcal{M}_0^C$, which contains the four *real* degrees of freedom, with the $\eta' \to \eta\pi^+\pi^-$ Dalitz-plot distribution measured by the BESIII collaboration [15] we obtain $\chi^2_{\text{red}} \approx 0.994$. The additional incorporation of the C-violating isovector contribution $\mathcal{M}_1^{\mathcal{C}}$, with three *real* parameters, leaves χ^2_{red} unaffected within the given precision. The dominant contribution of the Dalitz-plot distribution arising from Eq. (18), i.e.,

$$|\mathcal{M}|^2 \approx |\mathcal{M}_0^C|^2 + 2\,\mathrm{Re}\left[\mathcal{M}_0^C\,(\mathcal{M}_1^{\mathcal{C}})^*\right]\,, \tag{30}$$

is depicted in Fig. 2. We observe a similar, however slightly flattened, hierarchy as in the case of $\eta \to 3\pi$ worked out in Sect. 2.4. The interference term giving rise to the Dalitz-plot asymmetry is constrained to be two orders of magnitude smaller than the SM contribution $|\mathcal{M}_0^C|^2$.

To finalize our analysis we quantify the asymmetry and the coupling strength of the $\Delta I = 1$ transition in $\eta' \to \eta\pi^+\pi^-$. We find left-right asymmetry in units of 10^{-3} to be

$$A_{LR} = 2.1(1.5)\,. \tag{31}$$

Thus the mirror symmetry breaking vanishes within less than 1.4σ. Furthermore, we can parameterize A_{LR} in terms of Taylor invariants

$$g_1/\text{GeV}^{-2} = 0.037(55)\,, \qquad \delta g_1/\text{GeV}^{-2} = -5.5(7.3)\,. \tag{32}$$

These couplings allow us to write the left-right asymmetry, again in units of 10^{-3}, in the compact form

$$A_{LR} = 0.13\,g_1\big(1 + 0.10\,\delta g_1\big)\,, \tag{33}$$

where g_1 and δg_1 enter in units of GeV^{-2}.

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