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Commutator of higher spin gauge transformation

Melik Karapetyan[a,*]

[a] *Yerevan Physics Institute,*
Alikhanian Br. Str. 2, 0036 Yerevan, Armenia

E-mail: meliq.karapetyan@gmail.com

In this paper we consider special case of local quartic interaction of higher spin gauge field with a scalar field. Using Neother's procedure as a perturbative method and Wolfram Mathematica as a computational tool, we successfully modelled all main objects in Mathematica and were able to construct interacting Lagrangian for the constrained higher spin field (physical gauge). The commutator of the linear on-field gauge transformation was obtained and analyzed. Moreover, we have demonstrated a generic approach of working with such complex objects in Wolfram Mathematica. As a result the right hand side of the commutator was classified in respect to gauge transformations coming from cubic interactions of a scalar with a different higher spin symmetric tensor fields and mixed symmetry tensor field transformations.

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*Speaker

1. Introduction

The construction of complete Higher Spin (HS) interaction Lagrangian is a problem with a permanent background interest [1–7]. The role of HS theories in the development of other theories such as AdS/CFT increased the interest in this field even more. Construction of HS interaction Lagrangian in itself is a very complex task and there is a need to develop non-trivial computing techniques for even small achievements. During the previous 10-12 years, there was significant progress in this area, especially in the understanding of the construction and structure of cubic interaction in different approaches, dimensions, and backgrounds yet, our knowledge is far from being complete and seems to be bounded to the idea that quartic interaction should be non-local [8]-[16].

In parallel with these activities, the questions of possible non-locality beyond cubic level have been discussed in Vasilev's nonlinear theory of interacting HS fields equations in AdS background (see [17, 18] and ref. there). In some exceptional cases, it seems like it is possible to construct local interactions between fields with different spins, at least as a part of a more complicated covering theory (maybe non-local), including other gauge fields and symmetries.

In this paper we consider a local quartic interaction of higher-spin gauge field with a scalar field. In this special case, the nontrivial task of construction of interacting Lagrangian for the higher spin field in physical gauge was solved using the full power of Noether's procedure. There are two interesting points worth highlighting first of which is that it is required to add additional cubic interaction of scalar with other spin gauge fields and corresponding HS gauge symmetries to successfully close the Neother's procedure. The second one is that during the construction of quartic vertex we were able to investigate the closure of commutators of two linear in gauge field gauge transformation of our HS field $\delta_1^{(\epsilon)}$ and were able to understand whether it leads to non-locality or not.

As a result, the linear on-field gauge transformation is obtained and the corresponding commutator of transformation is analyzed.

$$[\delta_1^{(\eta)}\delta_1^{(\epsilon)}] \sim \delta_1^{([\eta,\epsilon])} + \text{additional terms}$$

The right-hand side of this commutator is classified in respect to gauge transformations coming from cubic interactions with different higher spin symmetric tensor fields and with mixed symmetry tensor field transformations aimed to understand the closure of this algebra.

During research process, we have intensively used Wolfram Mathematica along with "xAct", "xTensor", "xTras" packages to model complex problems and solve them programmatically. We have developed methods and tools for working with higher spin fields in Mathematica, and these methods are generic enough to be used on other use-cases. We introduced some of the modeling approaches in Wolfram Mathematica language and included the developed methods and functions as codding snippets in the paper. These Notes are based on the long calculations which we performed using the technique and notation developed in the past in [19]-[26].

2. Commutator of δ_1 transformations for spin four

When applying Neother's procedure we came up with the following linear in gauge field transformations. The details can be found in [26]:

$$\delta_1^{(\varepsilon)} h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h_{\mu\nu\lambda\rho} + \partial_{(\mu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}\partial_\beta h_{\gamma|\nu\lambda\rho)} + \partial_{(\mu}\partial_\nu\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha} h_{\beta\gamma|\lambda\rho)} + \partial_{(\mu}\partial_\nu\partial_\lambda\varepsilon^{\alpha\beta\gamma} h_{\rho)\alpha\beta\gamma}. \tag{1}$$

$$\delta_1\Phi = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma\Phi, \tag{2}$$

$$\delta_0 h^{\mu\nu\lambda\rho} = \partial^{(\mu}\varepsilon^{\nu\lambda\rho)}, \tag{3}$$

The structure of this expression is similar to linear transformation obtained in [21] where nonlinear curvature for general higher spin and in particular for spin three case is considered. First of all for understanding of corresponding gauge algebra we can derive commutator of this linear δ_1 transformation (1) with zero order gauge transformation δ_0 (3). Straightforward calculations leads to the following expression:

$$\left[\delta_0^{(\omega)}\delta_1^{(\varepsilon)} - \delta_0^{(\varepsilon)}\delta_1^{(\omega)}\right] h_{\mu\nu\lambda\rho} = \partial_{(\mu}\left[\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}\partial_\beta\partial_{\gamma|}\omega_{\nu\lambda\rho)} + t_{\nu\lambda\rho)}(\varepsilon,\omega) - (\varepsilon \leftrightarrow \omega)\right], \tag{4}$$

$$t_{\nu\lambda\rho}(\varepsilon,\omega) = \partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}\partial_{\beta|}\omega_{\lambda\rho)\gamma} + \partial_{(\nu}\partial_\lambda\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha|}\omega_{\rho)\beta\gamma} + \frac{1}{3}\partial_{(\nu}\partial_\lambda\varepsilon^{\alpha\beta\gamma}\partial_{\rho)}\omega_{\alpha\beta\gamma}. \tag{5}$$

Here we should make two important comments:

First, we see that in (1) the form of the last three terms is ambiguously defined due to the freedom in the definition of the δ_1. This transformation can be modified by adding zero-order (full gradient) transformation with field-dependent parameter. Ruffly speaking we can add δ_0 transformation with linear on gauge field parameter to (1) modifying the last three terms and getting corresponding modification for tensor $t_{\nu\lambda\rho}(\varepsilon,\omega)$ in definition of the commutator (4).

Second following the ideas of [21] and extracting the same type δ_0 terms described above we can rewrite (1) in the following form:

$$\delta_1^{(\varepsilon)} h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + \partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon,h), \tag{6}$$

$$\Lambda_{\nu\lambda\rho}(\varepsilon,h) = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta h_{\gamma\nu\lambda\rho} + \frac{1}{2}\left[\partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha} h_{\beta\gamma|\lambda\rho)} - \varepsilon^{\alpha\beta\gamma}\partial_{(\nu}\partial_{|\alpha} h_{\beta\gamma|\lambda\rho)}\right] + \frac{1}{3}\left[\partial_{(\nu}\partial_\lambda\varepsilon^{\alpha\beta\gamma} h_{\rho)\alpha\beta\gamma} + \varepsilon^{\alpha\beta\gamma}\partial_{(\nu}\partial_\lambda h_{\rho)\alpha\beta\gamma} - \frac{1}{2}\partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_\lambda h_{\rho)\alpha\beta\gamma}\right], \tag{7}$$

where

$$\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) = \partial_\alpha\partial_\beta\partial_\gamma h_{\mu\nu\lambda\rho} - \frac{1}{3}\partial_{<\alpha}\partial_\beta\partial_{(\mu} h_{\nu\lambda\rho)\gamma>} + \frac{1}{3}\partial_{<\alpha}\partial_{(\mu}\partial_\nu h_{\lambda\rho)\beta\gamma>} - \partial_{(\mu}\partial_\nu\partial_\lambda h_{\rho)\alpha\beta\gamma}, \tag{8}$$

is the third for spin four gauge field (last before Curvature) Christoffel Symbol in deWit-Freedman hierarchy of connections defined in [24] The key point of the splitting (6) is the simple form of zero order on field gauge transformation of connection (8):

$$\delta_0^{(\varepsilon)}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) = -4\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon_{\alpha\beta\gamma}, \tag{9}$$

and possibility in the future calculations *to identify in r.h.s of commutator different symmetries by existence of the terms in the form of Christoffel symbols or Generalized Curvatures (in some case with symmetrized derivatives) contracted with composite parameters like in (6) but with different rank and symmetry structure of the indices for composite parameters. So from now on we call such a type of terms as a "regular"* . In this way we see that expressions (1)-(9) is really looks like higher spin generalization of the gauge transformation (Lie derivative) and usual Christoffel symbol for linearized gravity[1]

$$\delta_1^{(\varepsilon)} h_{\mu\nu} = \mathfrak{L}_{\varepsilon^\lambda} h_{\mu\nu} = \varepsilon^\alpha \Gamma^{(1)}_{\alpha;\mu\nu} + \partial_{(\mu} \left(\varepsilon^\alpha h_{\nu)\alpha}\right), \tag{10}$$

$$\Gamma^{(1)}_{\alpha;\mu\nu} = \partial_\alpha h_{\mu\nu} - \partial_{(\mu} h_{\nu)\alpha}, \tag{11}$$

$$\delta_0^{(\varepsilon)} \Gamma^{(1)}_{\alpha;\mu\nu}(h) = -2\partial_\mu \partial_\nu \varepsilon_\alpha. \tag{12}$$

Using representation (6) and transformation rule (9) we can derive the following expression for commutator:

$$\begin{aligned}[][\delta_1^{(\omega)}, \delta_1^{(\varepsilon)}] h_{\mu\nu\lambda\rho} &= \varepsilon^{\alpha\beta\gamma} \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta_1^{(\omega)} h) - 4\varepsilon^{\alpha\beta\gamma} \partial_\mu \partial_\nu \partial_\lambda \partial_\rho \Lambda_{\alpha\beta\gamma}(\omega, h) \\ &+ \partial_{(\mu} \Lambda_{\nu\lambda\rho)}(\varepsilon, \delta_1^{(\omega)} h) - (\varepsilon \leftrightarrow \omega). \end{aligned} \tag{13}$$

Then taking int account that all symmetrized full gradients in r.h.s we can drop as a trivial δ_0 contribution from composite symmetric third rank gauge parameter linear in gauge field, we can first of all drop second line in (13). Then we can put four μ, ν, λ, ρ, derivatives in second term of first line from $\Lambda_{\alpha\beta\gamma}$ to parameter $\varepsilon^{\alpha\beta\gamma}$ and integrate using formula (9) and came to the following expression

$$[\delta_1^{(\omega)}, \delta_1^{(\varepsilon)}] h_{\mu\nu\lambda\rho} \sim \varepsilon^{\alpha\beta\gamma} \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta_1^{(\omega)} h) + \Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) \delta_0^{(\omega)} \Lambda^{\alpha\beta\gamma}(\varepsilon, h) - (\varepsilon \leftrightarrow \omega), \tag{14}$$

where $\sim$ means an equality up to any δ_0 variations with composed field dependent parameter described above or delta zero variation with usual parameter ε or ω from any second order on gauge field tensor. At this point it is worth to note that considering perturbative on linearized gauge field deformation of the initial gauge transformation regulated by Noether's procedure

$$\delta^{(\epsilon)} h_{\mu\nu\lambda\rho} = (\delta_0^{(\epsilon)} + \delta_1^{(\epsilon)} + \delta_2^{(\epsilon)} + \ldots) h_{\mu\nu\lambda\rho}, \tag{15}$$

for commutator on the linear level on gauge field we obtain:

$$\left\{[\delta^{(\omega)}, \delta^{(\epsilon)}] h_{\mu\nu\lambda\rho}\right\}_1 = ([\delta_1^{(\omega)}, \delta_1^{(\epsilon)}] + \delta_0^{(\omega)} \delta_2^{(\varepsilon)} - \delta_0^{(\varepsilon)} \delta_2^{(\omega)}) h_{\mu\nu\lambda\rho}. \tag{16}$$

So we see that we can factorize in right hand side of our commutator of the first order gauge transformation two type of trivial terms:

- Symmetrized full derivatives from composed gauge parameter linear in gauge fields $\partial_{(\mu} \tilde{\Lambda}_{\nu\lambda\rho)}(\varepsilon, \omega, h) - (\varepsilon \leftrightarrow \omega)$.

[1]Note that most common definition of Christoffel symbol $\Gamma^\beta_{\mu\nu}(g) = \frac{1}{2} g^{\beta\alpha}(\partial_{(\mu} g_{\nu)\alpha} - \partial_\alpha g_{\mu\nu})$ for general metric $g_{\mu\nu}$ relates with our definition after linearization in the flat background in the following way $\Gamma^\beta_{\mu\nu}(\eta_{\mu\nu} + h_{\mu\nu}) = -\frac{1}{2} \eta^{\beta\alpha} \Gamma^{(1)}_{\alpha;\mu\nu}(h)$.

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- The terms which can be classified as a second part of r.h.s of (16): $\delta_0^{(\omega)}\delta_2^{(\varepsilon)}h_{\mu\nu\lambda\rho} - (\varepsilon \leftrightarrow \omega)$, and we can throw them out also to understand algebra of two δ_1 transformations.

Now following this simple methodology we can present final result for commutator:

$$
\begin{aligned}
&[\delta_1^{(\omega)}, \delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho} \sim \left[\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma} + T^{\alpha\beta\gamma}(\partial,\varepsilon,\omega)\right]\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)\\
&+3\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma}R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + \frac{9}{20}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\sigma}R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h)\\
&+[Rem]_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) - (\varepsilon \leftrightarrow \omega)\,,
\end{aligned}
\tag{17}
$$

where:

$$
\begin{aligned}
T^{\alpha\beta\gamma}(\partial,\varepsilon,\omega) \;=\;& \frac{1}{4}\partial^{(\alpha}\partial^{\beta}\varepsilon^{\delta\sigma\eta}\delta_0^{(\omega)}h^{\gamma)}_{\delta\sigma\eta} - \frac{5}{48}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial^{\beta}\delta_0^{(\omega)}h^{\gamma)}_{\delta\sigma\eta} + \frac{7}{16}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial_\delta\delta_0^{(\omega)}h^{\beta\gamma)}_{\sigma\eta}\\
-\;& \frac{1}{16}\partial^{\delta}\varepsilon^{\sigma\eta(\alpha}\partial^{\beta}\delta_0^{(\omega)}h^{\gamma)}_{\delta\sigma\eta} + \frac{1}{16}\partial^{\delta}\varepsilon^{\sigma\eta(\alpha}\partial_\delta\delta_0^{(\omega)}h^{\beta\gamma)}_{\sigma\eta}\,,
\end{aligned}
\tag{18}
$$

and

$$
[Rem]_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) =
$$

$$
\frac{9}{20}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) + \frac{3}{2}\partial_{(\mu}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]})
\tag{19}
$$

$$
\begin{aligned}
&-\frac{9}{40}\varepsilon^{\eta\sigma}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\alpha\gamma}(H^{(3)}_{[\beta\sigma]})\\
&+\frac{3}{8}\varepsilon^{\eta}_{\sigma\delta}\partial^{[\sigma}\partial^{\left[\delta\right.}\omega^{\left.\alpha\right]\beta]\gamma}\partial_{(\mu}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) + \frac{1}{2}\partial_{(\mu}\varepsilon^{\eta}_{\delta\sigma}\partial^{[\sigma}\partial^{\left[\delta\right.}\omega^{\left.\alpha\right]\beta]\gamma}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]})\\
&+\frac{3}{8}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}\partial_\nu\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) + \frac{1}{2}\partial_{(\mu}\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_\nu\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]})\\
&+\frac{3}{4}\partial_{(\mu}\partial_\nu\varepsilon^{\sigma\eta}_{\delta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]})
\end{aligned}
\tag{20}
$$

is remaining part of commutator contained transformation described by composed gauge parameter with mixed symmetry of indices in the form of one or two antisymmetrized pairs.

To be more precise when classifying terms on the right side of (17) let us consider each line separately:

1. The first line describes spin four gauge transformation with composite *symmetric rank 3 tensor parameter* in the form

$$
[\omega,\varepsilon]^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h),
\tag{21}
$$

   where

$$
[\omega,\varepsilon]^{\alpha\beta\gamma} = \varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma} + T^{\alpha\beta\gamma}(\partial,\varepsilon,\omega) - (\varepsilon \leftrightarrow \omega).
\tag{22}
$$

2. The second line also corresponds to the transformation of the spin four gauge field in respect to gauge transformation with symmetric tensor parameter. But in this case we have *symmetric*

tensor parameters of rank 4 and 5, which means that it is transformation coming from gauge field with spin 5 and 6 and our spin four gauge field participates in these transformations through the spin four gauge invariant (in zero order on field transformations) curvature. In other words we have here regular terms in the form

$$\Omega^{\eta\alpha\beta\gamma\delta}_{(4)}(\varepsilon,\omega)R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h), \tag{23}$$

$$\Omega^{\sigma\eta\alpha\beta\gamma\delta}_{(5)}(\varepsilon,\omega)\partial_{(\sigma}R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h), \tag{24}$$

where

$$\Omega^{\eta\alpha\beta\gamma\delta}_{(4)}(\varepsilon,\omega) = \frac{3}{4}\varepsilon^{\delta\sigma(\eta}\partial_{\delta}\partial_{\sigma}\omega^{\alpha\beta\gamma)} - (\varepsilon \leftrightarrow \omega), \tag{25}$$

$$\Omega^{\sigma\eta\alpha\beta\gamma\delta}_{(5)}(\varepsilon,\omega) = \frac{9}{200}\varepsilon^{\delta(\sigma\eta}\partial_{\delta}\omega^{\alpha\beta\gamma)} - \frac{3}{200}\varepsilon^{(\sigma\eta}_{\delta}\partial^{\alpha}\omega^{\beta\gamma)\delta} - (\varepsilon \leftrightarrow \omega). \tag{26}$$

3. Now we analyze the third line of (17) or eight terms in expression (20). First of all we see that in this remaining part of commutator our spin four field expressed through the reduced curvatures and Christoffel symbols. All such a objects possess one (first two lines of (20)) ore two (remaining two lines of (20)) pair of antisymmetrized indices contracted with composed gauge parameter. Therefore they could describe some mixed symmetry field gauge transformation acting on spin four symmetric gauge field. For example first term in (20) we can rewrite in the form:

$$\Omega^{[\alpha\sigma],\eta\beta\gamma}_{[2],(3)}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) \tag{27}$$

where

$$\Omega^{[\alpha\sigma],\eta\beta\gamma}_{[2],(3)} = \frac{3}{40}(\varepsilon^{\delta(\eta[\sigma}\partial_{\delta}\omega^{\alpha]\beta\gamma)} - \varepsilon^{(\eta[\sigma}_{\delta}\partial^{\alpha]}\omega^{\beta\gamma)\delta}) \tag{28}$$

and in the same way the sixth term with two pair of antisymmetrized indices we can express as

$$\Omega^{[\eta\alpha],[\sigma\beta],\gamma}_{[2],[2],(1)}\partial_{(\mu}\partial_{\nu}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) \tag{29}$$

where composit parameter is

$$\Omega^{[\eta\alpha],[\sigma\beta],\gamma}_{[2],[2],(1)} = \frac{3}{32}(\varepsilon^{\delta\left[\sigma[\eta}\partial_{\delta}\omega^{\alpha]\beta\right]\gamma} - \varepsilon^{\left[\sigma[\eta}_{\delta}\partial^{\alpha]}\omega^{\beta\right]\delta\gamma}) \tag{30}$$

This type of terms (first, third, fourth and sixth in (20)) with mixed symmetry composed parameters we can still call "regular". But four remaining terms of (20) (second, fifth, seventh and eighth) we cannot transform to regular form because they all have non contracted derivatives from one (non composed) gauge parameter ($\partial_\mu\varepsilon$ or $\partial_\mu\partial_\nu\varepsilon$) and we call these terms irregular because do not have at the moment interpretation of them in means of additional symmetries or equation of motion of theory under construction. But at least we can clime that all irregular terms are in the mixed symmetry parameter sector.

So we see that our *commutator of spin four linear on gauge field transformations produce regular terms coming from gauge transformation of symmetric tensors with spin* $s < 6$ and remaining irregular transformation with mixed symmetry gauge field parameters.

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3. Explicitly computing commutator using Wolfram Mathematica

In this section we will introduce whole process of modelling Higher spin objects in Wolfram Mathematica and will explicitly compute the commutator using Mathematica code. We will introduce other objects as well such as generalized Christopher's symbols and more. Here we mainly used $xTras$ package form $xAct$ bundle.

Setup

We start our project by simply defining all the required object which will be used along the way, such as manifold, tangent bundle, metric, higher spin fields and gauge parameters.

```
<< xAct`xTras`;

In[]:= DefManifold[M, dim, IndexRange[a, m]];

In[]:= AddIndices[TangentM, {n, o, p, q, r, s, t}]

DefMetric[-1, metric[-a, -b], PD, PrintAs → "η", FlatMetric → True, SymbolOfCovD → {",", "∂"}]

SetOptions[SymmetryOf,
  ConstantMetric → True];
DefTensor[H[-a, -b, -c, -d], M, Symmetric[{-a, -b, -c, -d}], PrintAs → "h"]
DefTensor[ϵ[a, b, c], M , Symmetric[{a, b, c}], PrintAs → "ϵ"];
DefTensor[ω[a, b, c], M , Symmetric[{a, b, c}], PrintAs → "ω"]
DefTensor[A[a], M , PrintAs → "a"]
DefTensor[B[a], M , PrintAs → "b"]

In[]:= DefTensor[ρ[a, b, c, d], M , Symmetric[{a, b, c, d}], PrintAs → "ρ"]

DefTensor[σ[a, b, c], M , Symmetric[{a, b, c}], PrintAs → "σ"]
DefTensor[T[-a, -b, -c, -d], M, Symmetric[{-a, -b, -c, -d}], PrintAs → "T"]
```

Figure 1

In the above code snipped we defined the manifold M without curvature, also we have defined gauge field H as a symmetric tensor of rank four. We have also defined ϵ, ω gauge parameters as symmetric tensors of the third rank and some auxiliary symmetric tensors and vectors which will be used along the way.

Next we define the generating function for the first order variation.

```
In[29]:= (*The full first order variation*)
δ1HG[H_, ϵ_] := Module[{ANS},
   ANS[d_, e_, f_, l_] := Module[{a, b, c, A, B, C, D, res},
     A = ϵ[a, b, c] * PD[-a]@PD[-b]@PD[-c]@H[d, e, f, l];
     B = PD[d]@ ϵ[a, b, c] * PD[-a]@PD[-b]@H[e, f, l, -c] // Symmetrize[#, {d, e, f, l}] &;
     C = PD[d]@PD[e]@ϵ[a, b, c] * PD[-a]@H[f, l, -b, -c] // Symmetrize[#, {d, e, f, l}] &;
     D = PD[d]@PD[e]@PD[f]@ϵ[a, b, c] * H[l, -a, -b, -c] // Symmetrize[#, {d, e, f, l}] &;
     res = A + 4 * B + 6 * C + 4 * D // CollectTensors;
     ANS[d, e, f, l] = res;
     res
    ];
   ANS
  ]
δ1H = δ1HG[H, ϵ];
δ1H[i, j, k, l]
```

Out[31]=
$$\epsilon^{abc}\, \partial_c\partial_b\partial_a h^{ijkl} + \partial_c\partial_b h^{jkl}{}_a \partial^i \epsilon^{abc} + \partial_c\partial_b h^{ikl}{}_a \partial^j \epsilon^{abc} + \partial_c h^{kl}{}_{ab} \partial^j\partial^i \epsilon^{abc} + \partial_c\partial_b h^{ijl}{}_a \partial^k \epsilon^{abc} + \partial_c h^{jl}{}_{ab} \partial^k\partial^i \epsilon^{abc} + \partial_c h^{il}{}_{ab} \partial^k\partial^j \epsilon^{abc} + h^{labc}\, \partial^k\partial^j\partial^i \epsilon_{abc} + \partial_c\partial_b h^{ijk}{}_a \partial^l \epsilon^{abc} + \partial_c h^{jk}{}_{ab} \partial^l\partial^i \epsilon^{abc} + \partial_c h^{ik}{}_{ab} \partial^l\partial^j \epsilon^{abc} + h^{kabc}\, \partial^l\partial^j\partial^i \epsilon_{abc} + \partial_c h^{ij}{}_{ab} \partial^l\partial^k \epsilon^{abc} + h^{jabc}\, \partial^l\partial^k\partial^i \epsilon_{abc} + h^{iabc}\, \partial^l\partial^k\partial^j \epsilon_{abc}$$

Figure 2

The beauty of this implementation is that it is a function of functions and due to encapsulation it can be reused with different fields and parameters. It requires arguments such as H and ϵ which are arbitrary variable tensors of rank 4 and 3 respectively (in our case of course they are gauge field and gauge parameter) and returns another function which can be treated as a tensor. On the last line of above figure is the explicit form of first order variation same as (1).

As one can see, this form is explicit but is hard to read, this is because of the symmetrization which is taken into place. To make it more readable and easy to work we will contract the indices with A vectors, as a result we will have more compact form.

```
δ1H[i, j, k, l] * A[-i] * A[-j] * A[-k] * A[-l] // CollectTensors
```

Out[40]= $4\, a^a\, a^b\, a^c\, a^i\, h_a{}^{jkl}\, \partial_i\partial_c\partial_b\epsilon_{jkl} + 6\, a^a\, a^b\, a^c\, a^i\, \partial_b\partial_a\epsilon^{jkl}\partial_l h_{cijk} + 4\, a^a\, a^b\, a^c\, a^i\, \partial_a\epsilon^{jkl}\partial_l\partial_k h_{bcij} + a^a\, a^b\, a^c\, a^i\, \epsilon^{jkl}\, \partial_l\partial_k\partial_j h_{abci}$

Figure 3

Now we have the left hand side of the (6), we move forward into modelling the first element of right hand side using the same approach.

```
In[43]:= δHG[H_, ϵ_] := Module[{ANS},
   ANS[d_, e_, f_, l_] := Module[{a, b, c, A, B, C, D, res},
     A = PD[a]@PD[b]@PD[c]@H[d, e, f, l];
     B = -PD[a]@PD[b]@PD[d]@H[e, f, l, c] // Symmetrize[#, {a, b, c}] & //
       Symmetrize[#, {d, e, f, l}] &;
     C = PD[a]@PD[d]@PD[e]@H[f, l, c, b] // Symmetrize[#, {a, b, c}] & //
       Symmetrize[#, {d, e, f, l}] &;
     D = -PD[d]@PD[e]@PD[f]@H[l, c, b, a] // Symmetrize[#, {a, b, c}] & //
       Symmetrize[#, {d, e, f, l}] &;
     res = ϵ[-a, -b, -c]
         * (A + 4 * B + 6 * C + 4 * D) // CollectTensors;
     ANS[d, e, f, l] = res;
     res
    ];
   ANS
  ]

δHϵ = δHG[H, ϵ]

In[45]:= δHϵ[i, j, k, l]
```

Out[45]= $\epsilon^{abc}\, \partial_c\partial_b\partial_a h^{ijkl} - \epsilon^{abc}\, \partial_c\partial_b\partial^i h^{jkl}{}_a - \epsilon^{abc}\, \partial_c\partial_b\partial^j h^{ikl}{}_a - \epsilon^{abc}\, \partial_c\partial_b\partial^k h^{ijl}{}_a - \epsilon^{abc}\, \partial_c\partial_b\partial^l h^{ijk}{}_a + \epsilon^{abc}\, \partial_c\partial^j\partial^i h^{kl}{}_{ab} + \epsilon^{abc}\, \partial_c\partial^k\partial^i h^{jl}{}_{ab} + \epsilon^{abc}\, \partial_c\partial^k\partial^j h^{il}{}_{ab} + \epsilon^{abc}\, \partial_c\partial^l\partial^i h^{jk}{}_{ab} + \epsilon^{abc}\, \partial_c\partial^l\partial^j h^{ik}{}_{ab} + \epsilon^{abc}\, \partial_c\partial^l\partial^k h^{ij}{}_{ab} - \epsilon^{abc}\, \partial^k\partial^j\partial^i h^{l}{}_{abc} - \epsilon^{abc}\, \partial^l\partial^j\partial^i h^{k}{}_{abc} - \epsilon^{abc}\, \partial^l\partial^k\partial^i h^{j}{}_{abc} - \epsilon^{abc}\, \partial^l\partial^k\partial^j h^{i}{}_{abc}$

Figure 4

The second term of the rhs is not important, and we will mainly focus on computing the commutator using the first term.

To do so we need to compute the variation of $\epsilon * \Gamma$ by the second gauge parameter ω. It can be done easily by plugging the $\epsilon * \Gamma$ into the generating function of variation with respect to ω parameter.

```
In[46]:= δHεω = δHG[δHε, ω];

In[47]:= δHεω[i, j, k, l]
```

Figure 5

We almost have the commutator, the last thing is to compute the variations in a reverse order, first by ω and then by ϵ and subtract from each other. To do so there is no need to compute the whole variation from scratch, it is sufficient to implement a function which can swap gauge variables and by using it we can compute the other term very quickly.

```
swap[exp_, ϵ_, ω_, σ_] := Module[{r1, r2, r3},
  r1 = MakeRule[{ω[i, j, k], σ[i, j, k]}, PatternIndices → All, MetricOn → All, UseSymmetries → True];
  r2 = MakeRule[{ϵ[i, j, k], ω[i, j, k]}, PatternIndices → All, MetricOn → All, UseSymmetries → True];
  r3 = MakeRule[{σ[i, j, k], ϵ[i, j, k]}, PatternIndices → All, MetricOn → All, UseSymmetries → True];
  exp /. r1 /. r2 /. r3
 ]

In[•]:= swapped = δHεω[i, j, k, l] // swap[#, ϵ, ω, σ] &;

(*This is the resuting commutator*)

In[•]:= comm = A[-i] * A[-j] * A[-k] * A[-l] * (swapped - δHεω[i, j, k, l]) // CollectTensors;
```

Figure 6

The *comm* term in the above figure is the commutator of gauge transformation. Now when we have the commutator successfully modelled in Mathematica the second task will be to simplify it and classify all terms. This will be done by modelling other objects such as Christofel symbols and more using the same methodology, observing the expressions and guessing the ansatz then subtracting the expression from the commutator and repeating the cycle until all terms are classified.

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