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Paving a path from the refined Chern-Simons to the topological strings

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This talk was based on our joint recent work with R.Mkrtchyan [1], where we present a new expression for the partition functions of the refined Chern-Simons theory on S^3 for arbitrary simple gauge groups and investigate the possibility to rewrite them in terms of Vogel's universal parameters. This investigation is aimed at the possible establishment of dualities between refined Chern-Simons and topological strings for all simple gauge groups.

We showed that for the simply laced or ADE algebras the corresponding partition functions are universal. For the non-simply laced algebras, we managed to rewrite them in a form, which makes it possible to transform them into a product of multiple sine functions, paving a path for the future study of the corresponding dualities.

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Introduction

The story begins with the fact, discovered by R.Mkrtchyan and A.Veselov in [2, 3]. Namely, they showed that the partition function of Chern-Simons (CS) theory on a three-dimensional sphere S^3 can be presented in terms of Vogel's universal parameters α, β, γ [4]. It has been shown later, that this representation of the partition function happens to be very convenient for the further transformation of the abovementioned partition function into the Gopakumar-Vafa partition function of topological strings. Particularly, in [3, 5] the dualities between the CS theories and topological strings were established for the classical gauge groups. In [6] this result has been extended to the exceptional algebras, namely the partition functions of CS on S^3 with the exceptional gauge groups have been presented in the form of a partition function of a specific refined topological string.

The extension of the results of the understanding of the CS/topological strings dualities to the case of the *refined* CS theories has not been left untouched. In particular, D.Krefl and A.Shwartz [7] managed to prove, that the partition functions of the refined CS theories based on the A_n and D_n series of gauge algebras are universal. Taking the advantage of that universal representation dualities between the corresponding refined CS theories and some refined topological strings were established in [8].

Our work [1] on which this talk was based embodies the natural development of the aforementioned investigations.

First, we ask for an exact expression for the partition functions of the refined CS theories which would work for *each of the* simple gauge algebra, after specifying the corresponding root system. We succeed in deriving that expression through the generalization of the Kac-Peterson identity for the volume of the fundamental domain of the co-root lattice of a Lie algebra [9]. See Section 1 for more details.

Then, we ask about the possibility of representation of the partition function of the refined CS in a universal form and manage to generalize the Krefl-Shwartz result to all simply laced algebras, proving that the partition function of the refined CS for E_n, $n = 6, 7, 8$ algebras is also universal, see Section 2.

Finally, we clarify the case with the non-simply laced algebras. Namely, we prove, that in that cases there are no universal expressions in terms of Vogel's universal parameters, which would coincide with the corresponding partition functions. However, we manage to find an appropriate representation of the corresponding partition functions, which allow their further transformation into products of multiple sine functions, necessary for the future investigation of their relationship with some (refined) topological strings. The corresponding expressions are presented in Section 3.

1. The partition function of refined CS theory on S^3

The partition function of CS theory on S^3 sphere was given in Witten's seminal paper [10] as the S_{00} element of the S matrix of modular transformations. For an arbitrary gauge group, it is (see, e.g. [2, 11])

$$Z(k) = Vol(Q^\vee)^{-1}(k + h^\vee)^{-\frac{r}{2}} \prod_{\alpha_+} 2 \sin \pi \frac{(\alpha, \rho)}{k + h^\vee} \tag{1}$$

Here the so-called minimal normalization of the invariant scalar product $(,)$ in the root space is used, which implies that the square of the long roots equals 2. Other notations are: $Vol(Q^\vee)$ is the volume of the fundamental domain of the coroot lattice $Q^\vee$, the integer k is the CS coupling constant, $h^\vee$ is the dual Coxeter number of the algebra, r is the rank of the algebra, the product is taken over all positive roots α_+.

$Vol(Q^\vee)$ is equal to the square root of the determinant of the matrix of scalar products of the simple coroots, accordingly for the simply laced algebras, in the minimal normalization, it is equal to the square root of the determinant of the Cartan matrix:

$$Vol(Q^\vee) = (\det(\alpha_i^\vee, \alpha_j^\vee))^{1/2} \tag{2}$$

$$\alpha_i^\vee = \alpha_i \frac{2}{(\alpha_i, \alpha_i)}, \quad i = 1, ..., r \tag{3}$$

The same formula for the partition function, rewritten in an arbitrary normalization of the scalar product [2], is

$$Z(\kappa) = Vol(Q^\vee)^{-1} (\delta)^{-\frac{r}{2}} \prod_{\alpha_+} 2 \sin \pi \frac{(\alpha, \rho)}{\delta} \tag{4}$$

where k is now replaced by κ, $h^\vee$ by t, and $\delta = \kappa + t$. In this form the r.h.s. is invariant w.r.t. the simultaneous rescaling of the scalar product, κ, and t (and hence δ). In the minimal normalization they accept their usual values in (1).

In [3] it was noticed, that from this formula for the partition function one can derive an interesting closed expression for $Vol(Q^\vee)$, which agrees with that in the Kac-Peterson's paper [9], (see eq. (4.32.2)), provided

$$Z(0) = 1 \tag{5}$$

This equality is completely natural from the physical point of view. Indeed, the CS theory is based on the unitary integrable representations of affine Kac-Moody algebras. At a given k there is a finite number of such representations, and at $k = 0$ there is not any non-trivial one.

So, from (4) and (5) we have

$$Vol(Q^\vee) = t^{-\frac{r}{2}} \prod_{\alpha_+} 2 \sin \pi \frac{(\alpha, \rho)}{t} \tag{6}$$

which, as mentioned, agrees with [9]. Below we generalize this equation by inclusion of a refinement parameter.

The generalization of the usual CS to the refined CS theory is given in [12–14]. It is based on Macdonald's deformation of e.g. the Shur polynomials, and other "deformed" formulae, given in [15–17]. In a nutshell, Macdonald's deformation yields the deformed S and T matrices of the modular transformations, and since these matrices define all observables in CS theory, one can naturally consider the "deformed" or the refined versions of all observables, i.e. the link/manifold invariants.

Particularly, the partition function of the refined CS theory on S^3 is given [12] by the S_{00} element of the refined S-matrix. In [12] an orthogonal, instead of an orthonormal basis is used sometimes. We shall use the orthonormal one only (as in [7]), so there is no difference between e.g. S_{00} and S_0^0.

We suggest the following expression for S_{00} for the refined CS theory:

$$Z(\kappa, y) = Vol(Q^\vee)^{-1}\delta^{-\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} 2\sin\pi \frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{\delta} \tag{7}$$

We assume that now $\delta = \kappa + yt$, y is the refinement parameter, which we consider to be a positive integer at this stage.

Although we could not find the $Z(\kappa, y)$ in this exact form in the literature, however, the expression (7) complies with the known formulae in different limits, e.g. at $y = 1$ it yields the corresponding formula for the non-refined case (4). It also coincides with the corresponding formulae for the refined CS theory in [7, 12, 14] for A_n, D_n algebras. The coefficient $(\alpha, \alpha)/2$ in front of the summation parameter m coincides with that in the constant term formulae in [18, 19]. Actually, for non-simply laced algebras one can introduce two refinement parameters, one for each length of the roots, see e.g. [18, 19]. However, we did not try to introduce a second parameter (and also are not aware of the physical interpretation of it), so below we consider them to be coinciding, so that we always have one refinement parameter.

The latter expression of the partition function is supported by the key feature of (7): at $\kappa = 0$ the equality $Z(0, y) = 1$ holds, which is ensured by the following generalization of the formula (6) for the same object $Vol(Q^\vee)$:

$$Vol(Q^\vee) = (ty)^{-\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} 2\sin\pi \frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{ty} \tag{8}$$

For A_n algebras this equality can be easily proved with the use of the following well-known identity, valid at an arbitrary positive integer N:

$$N = \prod_{k=1}^{N-1} 2\sin\pi \frac{k}{N} \tag{9}$$

Similarly it can be checked for all the remaining root systems, too.

Next, with (8) taken into account, we obtain the following expression of the partition function:

$$Z(\kappa, y) = \left(\frac{ty}{\delta}\right)^{\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} \frac{\sin\pi \frac{y(\alpha,\rho)-m(\alpha,\alpha)/2}{\delta}}{\sin\pi \frac{y(\alpha,\rho)-m(\alpha,\alpha)/2}{ty}} \tag{10}$$

which explicitly satisfies $Z(0, y) = 1$, since $\delta = ty$ at $\kappa = 0$.

2. Universality of the refined CS for all simply laced algebras

The expression (10) obeys the following integral representation[1]:

$$\ln Z = -\frac{1}{4}\int_{R_+}\frac{dx}{x}\frac{\sinh(x(ty-\delta))}{\sinh(xty)\sinh(x\delta)}F_X(2x,y) \tag{11}$$

where

$$F_X(x,y) = r + \sum_{m=0}^{y-1}\sum_{\alpha_+}\left(e^{x(y(\alpha,\rho)-m(\alpha,\alpha)/2)} + e^{-x(y(\alpha,\rho)-m(\alpha,\alpha)/2)}\right) \tag{12}$$

The index X stands for the algebra. Note, that in the non-refined case, i.e. when $y = 1$ (12) coincides with the quantum dimension of the adjoint representation, which is known to be universal [2, 20]:

$$F_X(x,1) = r + \sum_{\alpha_+}\left(e^{x(\alpha,\rho)} + e^{-x(\alpha,\rho)}\right) = \chi_{ad}(x\rho) \tag{13}$$

$$\chi_{ad}(x\rho) \equiv f(x) \quad = \quad \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})} \tag{14}$$

Note that the notation α is used either for the root(s) of an algebra or for one of Vogel's parameters. Since these objects are very different, hopefully, no interpretation problem will appear.

So we can call $F_X(x,y)$ the refined quantum dimension.

The question we propose is the following: can one present the $F_X(x,y)$ in terms of Vogel's universal parameters?

In [7] Krefl and Swartz showed that for $X = A_n$ and $X = D_n$

$$F_X(x,y) = f(x,y) \tag{15}$$

where

$$f(x,y) = \frac{\sinh(x\frac{\alpha-2ty}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(xy\frac{\beta-2t}{4})}{\sinh(xy\frac{\beta}{4})}\frac{\sinh(xy\frac{\gamma-2t}{4})}{\sinh(xy\frac{\gamma}{4})} \tag{16}$$

i.e. *for the A_n and D_n series of algebras the partition function of the refined CS is universal.*

We extend this result to all simply laced algebras, claiming that

$$F_X(x,y) = f(x,y) \tag{17}$$

for any simply-laced Lie algebra X.

Take e.g. the E_6 algebra, for which the corresponding universal parameters in the minimal normalization are: $\alpha = -2, \beta = 6, \gamma = 8, t = 12$. We should calculate the sum

$$F_{E_6}(x,y) = 6 + \sum_{m=0}^{y-1}\sum_{\alpha_+} e^{x(y(\alpha,\rho)-m)} + e^{-x(y(\alpha,\rho)-m)} \tag{18}$$

First note the number of roots n_L with a given height $L = (\alpha, \rho)$ among all roots. The set of couples (L, n_L) with a non-zero n_L is

$$\begin{aligned}(-11,1),(-10,1),(-9,1),(-8,2),(-7,3),(-6,3),(-5,4),(-4,5),\\(-3,5),(-2,5),(-1,6),(0,6),(1,6),(2,5),(3,5),(4,5),(5,4),(6,3),\\(7,3),(8,2),(9,1),(10,1),(11,1)\end{aligned} \tag{19}$$

which of course is symmetric w.r.t. the $L \leftrightarrow -L$. We also include the element $(0, 6)$ in this list, which is just the first term 6 in (18). Then, using this data, we note that the sum in (18) is given by

$$F_{E_6} = \phi(11y) + \phi(8y) + \phi(7y) + \phi(5y) + \phi(4y) + \phi(y) \tag{20}$$

$$\phi(n) = \sum_{i=-n}^{n} q^i = \frac{q^{2n+1} - 1}{q^n(q-1)} \tag{21}$$

$$q = e^x \tag{22}$$

Combining the sums $\phi(11y) + \phi(8y) + \phi(5y)$ and $\phi(7y) + \phi(4y) + \phi(y)$, we get

$$\phi(11y) + \phi(8y) + \phi(5y) = \frac{(q^{9y} - 1)(q^{5y+1} - q^{-11y})}{(q-1)(q^{3y} - 1)} \tag{23}$$

$$\phi(7y) + \phi(4y) + \phi(y) = \frac{(q^{9y} - 1)(q^{y+1} - q^{-7y})}{(q-1)(q^{3y} - 1)} \tag{24}$$

$$F_{E_6} = \frac{(q^{9y} - 1)}{(q-1)(q^{3y} - 1)}(q^{4y} + 1)(q^{y+1} - q^{-11y}) = \tag{25}$$

$$\frac{(q^{9y} - 1)(q^{8y} - 1)(q^{y+1} - q^{-11y})}{(q-1)(q^{3y} - 1)(q^{4y} - 1)} \tag{26}$$

which can be easily checked to coincide with $f(x, y)$ for the universal parameters corresponding to E_6 algebra.

Literally similar calculations can be carried out for the remaining E_7, E_8 algebras, as well as for Krefl-Schwarz cases A_n, D_n, leading to the same conclusion.

3. The partition functions for the non-simply laced algebras

It appears, that the refined quantum dimension $F_X(x, y)$ is not universal in case X is a non-simply laced algebra. However, the corresponding sums can be presented in forms, appropriate for the further duality considerations [3, 6, 8]. Namely, present $F_X(x, y)$ as follows:

$$F_X = r + \sum_{m=0}^{y-1} \sum_{\alpha_+} \left(e^{x(y(\alpha,\rho) - m(\alpha,\alpha)/2)} + e^{-x(y(\alpha,\rho) - m(\alpha,\alpha)/2)} \right) = \frac{A_X}{B_X} \tag{27}$$

where X denotes an algebra of type B, C, F or G, r is its rank, B_X is a product of a number of terms of the form $q^a - 1$, and A_X is a polynomial in q.

Note, that in (27) one should explicitly mention the normalization of the scalar product. Indeed, a rescaling of the scalar product and x leaves invariant only the l.h.s. of (27), whilst the ratio A_X/B_X in the r.h.s is dependent only on x, thus changes under the corresponding rescaling. This means that when substituting the r.h.s. of (27) into the integral form of the partition function (11) one should take the parameters t and δ in the same normalization. Below we choose normalizations that allow avoiding the appearance of fractional powers of q.

Now we present F_X for all non-simply laced algebras.

Let us consider the B_n algebras. Normalization corresponds to $\alpha = -4$, i.e. the square of the long root is 4. The corresponding representation we mentioned above is

$$F_{B_n}(x,y) = \frac{A_{B_n}}{B_{B_n}} \tag{28}$$

$$A_{B_n} = q^{4ny+2} + q^{-4(n-1)y} + \tag{29}$$

$$(q+1)(q^y-1)\left(q^{2y}+1\right)\left(q^{2ny}-1\right)\left(q^{y-2ny}+q\right) - q^{4y} - q^2 \tag{30}$$

$$B_{B_n} = \left(q^2-1\right)\left(q^{4y}-1\right), \tag{31}$$

For the C_n algebras we also choose the same normalization with the square of the long root being 4. Then F_X writes as

$$F_{C_n} = \frac{A_{C_n}}{B_{C_n}} \tag{32}$$

$$B_{C_n} = (q^2-1)\left(q^{2y}-1\right) \tag{33}$$

$$A_{C_n} = (q+1)q^y\left(q^{2ny}-1\right)\left(q^{2ny+1}-1\right) + \tag{34}$$

$$\left(q^{2y}-1\right)(q^{ny}-1)\left(q^{ny+1}-1\right)\left(q^{2ny+1}-1\right) \tag{35}$$

For the F_4, with the same normalization, we have

$$F_{F_4} = \frac{A_{F_4}}{B_{F_4}} \tag{36}$$

$$B_{F_4} = (q^2-1) \tag{37}$$

$$A_{F_4} = q^{-16y}\left(q^{2y}+1\right)\left(-q^{2y}+q^{4y}+1\right)\left(q^{12y+1}-1\right)\times \tag{38}$$

$$\left(q^{5y+1}-q^{8y+1}+q^{9y+1}+q^{14y+1}+q^{5y}-q^{6y}+q^{9y}+1\right) \tag{39}$$

For the G_2 we use the normalization corresponding to the square of the long root to be equal to 6. The corresponding F_{G_2} function is

$$F_{G_2} = \frac{A_{G_2}}{B_{G_2}} \tag{40}$$

$$B_{G_2} = q^3 - 1 \tag{41}$$

$$A_{G_2} = q^{-9y}\left(q^{6y+1} - 1\right) \times \tag{42}$$

$$\left(q^{4y+1} + q^{8y+1} + q^{4y+2} - q^{6y+2} + q^{8y+2} + q^{12y+2} + q^{4y} - q^{6y} + q^{8y} + 1\right) \tag{43}$$

Conclusion

By closing the contour of integration in (11) in the upper semi plane, one obtains the output by means of a sum of contributions of poles. As it was shown in a number of cases in [3, 5, 8], the contribution of the so-called perturbative poles, i.e. those coming from $\sinh(x\delta)$, exactly coincides with the Gopakumar-Vafa partition function of the corresponding dual topological string. The corresponding contribution with the use of the newly derived expressions F_X should be examined next, aiming at the interpretation of the initial partition function in terms of some (refined) topological strings, indeed if such strings exist. Obviously, that string would be the candidate for a dual description of the corresponding CS theory.

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