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Elliptic, hyperbolic, complex gamma functions and QFT in various dimensions

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We present new relations for integrals of complex gamma functions. We show that starting from properties of the elliptic hypergeometric integrals, and using some limiting procedures, one can get vast number of identities for integrals of products of the hyperbolic and complex gamma functions. The latter integrals have physical interpretation of partition functions for different types of quantum field theories dualities in various dimensions, and the corresponding limiting procedures can be viewed as dimensional reduction and multiplet decoupling.

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1. Introduction

There are many problems where elliptic, hyperbolic and complex gamma functions enter the description of physical quantities. Let us mention some of them. It is shown in [7] that the contribution of $4D$ $N = 1$ chiral multiplet to superconformal index is given by the elliptic gamma function, and full superconformal index of $4D$ $N = 1$ gauge theories is given by the elliptic hypergeometric integrals [20]. There is a similar story in three dimensions. It is shown in [11] that the hyperbolic gamma function, known also as the Faddeev modular dilogarithm [8], gives partition function of the $3D$ $N = 2$ chiral multiplet, and correspondingly full partition function of $3D$ $N = 2$ gauge theories is given by hyperbolic hypergeometric integrals. Another vast topic of application of the hyperbolic gamma functions is the two-dimensional Liouville field theory. The complex gamma functions are building blocks of matrix elements of operators in $\mathrm{SL}(2,\mathbb{C})$ spin chain models [3], which are given again by the proper integrals.

Thus we would like to note that there are two parallel pictures of relations between different integrals of products of any of the mentioned gamma functions: physical and mathematical. It was established in nineties that there are many duality relations between different $4D$ $N = 1$ gauge theories called Seiberg dualities, relating strong and week coupling regimes to each other. Superconformal index [7] was designed precisely with purpose to test them, since it is the renormgroup invariant. Thus, we have a correspondence between dualities in four dimensions and integral identities for elliptic gamma functions. Similarly there are many mirror symmetries between $N = 2$ supersymmetric gauge theories in $3D$, and the right quantity to test them is the partition function, since for $3D$ $N = 2$ gauge theories it is invariant under the renormgroup flow. So, here we have a correspondence between mirror symmetries in $3D$ and integral identities for hyperbolic gamma functions. And, finally, there are alternative ways to calculate matrix elements for operators in $\mathrm{SL}(2,\mathbb{C})$ spin chain models, which in turn mathematically are expressed as integral identities for complex gamma functions. Thus, the integral identities for all mentioned types of gamma functions encode an important physical information.

Curiously all three pictures are related from the mathematical point of view. First it is established in [16] that hyperbolic gamma function can be derived from the elliptic gamma function in the limit when both basic parameters go to 1. Then it was found in [1, 17], that complex gamma function can be derived from the hyperbolic one in the limit when sum of the quasiperiods goes to zero. So, we can hope that starting from the identities for the elliptic hypergeometric integrals we can obtain the corresponding integral identities for the hyperbolic and complex gamma functions. The key identities for the elliptic hypergeometric integrals were derived in [19–21]. The limits to the hyperbolic identities from the elliptic ones were addressed in [14, 23, 24]. Here we present the last step, the derivation of the integral identities for the complex gamma functions from the hyperbolic ones [17].

The paper is organized as follows. In section 2, we review elliptic hypergeometric functions. In section 3, we derive integral identities for the hyperbolic gamma functions. We also show, how the asymptotics of the hyperbolic gamma function allows one to produce vast number of new identities by taking some parameters to infinity in a certain smart way. Finally in section 4, we present new integral identities for the complex gamma functions.

2. Elliptic gamma function

The standard elliptic gamma function $\Gamma(z;p,q)$ can be defined as an infinite product:

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}, \quad |p|,|q|<1, \quad z\in\mathbb{C}^*. \tag{1}$$

It satisfies equations

$$\Gamma(z;p,q)=\Gamma(z;q,p), \quad \Gamma(qz;p,q)=\theta(z;p)\Gamma(z;q,p,) \quad \Gamma(pz;p,q)=\theta(z;q)\Gamma(z;q,p) \tag{2}$$

with the short theta function $\theta(z;q)=(z;q)_\infty(qz^{-1};q)_\infty$, where $(a;q)_\infty=\prod_{j=0}^{\infty}(1-aq^j)$.

It is shown in [19] that when the parameters t_a satisfy constraints $|t_a|<1$ and the balancing condition $\prod_{a=1}^6 t_a = pq$, then the following integral indentity holds:

$$\frac{(p;p)_\infty(q;q)_\infty}{4\pi i}\int_{\mathbb{T}} \frac{\prod_{a=1}^6\Gamma(t_az;p,q)\Gamma(t_az^{-1};p,q)}{\Gamma(z^2;p,q)\Gamma(z^{-2};p,q)}\frac{dz}{z} = \prod_{1\le a<b\le 6}\Gamma(t_at_b;p,q). \tag{3}$$

where $\mathbb{T}$ is the unit circle of positive orientation.

Consider the V-function, an elliptic analogue of the Euler–Gauss hypergeometric function [20],

$$V(t_1,\dots,t_8;p,q) = \frac{(p;p)_\infty(q;q)_\infty}{4\pi i}\int_{\mathbb{T}} \frac{\prod_{a=1}^8\Gamma(t_az;p,q)\Gamma(t_az^{-1};p,q)}{\Gamma(z^2;p,q)\Gamma(z^{-2};p,q)}\frac{dz}{z}, \tag{4}$$

where the parameters satisfy constraints $|t_a|<1$ and the balancing condition holds $\prod_{a=1}^{8} t_a = p^2q^2$. This function has the $W(E_7)$ Weyl group symmetry transformations, whose key generating relations have been established in [21]:

$$V(t_1,\dots,t_8;p,q) = \prod_{1\le j<k\le 4}\Gamma(t_jt_k;p,q)\prod_{5\le j<k\le 8}\Gamma(t_jt_k;p,q)V(s_1,\dots,s_8;p,q), \tag{5}$$

$$s_j=\rho^{-1}t_j, \quad j=1,2,3,4 \quad \text{and} \quad s_j=\rho t_j, \quad j=5,6,7,8, \quad \rho=\sqrt{\frac{t_1t_2t_3t_4}{pq}}, \tag{6}$$

and

$$V(t_1,\dots,t_8;p,q) = \prod_{1\le j<k\le 8}\Gamma(t_jt_k;p,q)V(\sqrt{pq}/t_1,\dots,\sqrt{pq}/t_8;p,q). \tag{7}$$

As shown in [22], the V-function satisfies also the following finite difference equation called the elliptic hypergeometric equation

$$\mathcal{L}(\underline{t})(U(qt_6,q^{-1}t_7)-U(\underline{t})) + (t_6\leftrightarrow t_7) + U(\underline{t}) = 0, \tag{8}$$

where

$$U(\underline{t}) = \frac{V(t_1,\dots,t_8;p,q)}{\Gamma(t_6t_8;p,q)\Gamma(t_6t_8^{-1};p,q)\Gamma(t_7t_8;p,q)\Gamma(t_7t_8^{-1};p,q)}, \tag{9}$$

the first U-function contains parameters $qt_6, q^{-1}t_7$ instead of t_6, t_7, and

$$\mathcal{L}(\underline{t}) = \frac{\theta\left(\frac{t_6}{qt_8};p\right)\theta\left(t_6t_8;p\right)\theta\left(\frac{t_8}{t_6};p\right)}{\theta\left(\frac{t_6}{t_7};p\right)\theta\left(\frac{t_7}{qt_6};p\right)\theta\left(\frac{t_7t_6}{q};p\right)}\prod_{k=1}^{5}\frac{\theta\left(\frac{t_7t_k}{q};p\right)}{\theta(t_8t_k;p)}. \tag{10}$$

There is a beautiful four-dimensional duality interpretation [7] of the univariate elliptic beta integral evaluation formula (3). The left-hand side expression in (3) describes the superconformal index of the $4D$ supersymmetric quantum chromodynamics with $SU(2)$ gauge group and $SU(6)$ flavor group. In the deep infrared region the theory is strongly coupled, all colored particles confine, and on the right-hand side of (3) one has the superconformal index of the Wess-Zumino type model for mesonic fields lying in the 15-dimensional totally antisymmetric tensor representation of $SU(6)$. The symmetry relations (5) and (7) have similar duality interpretations as well.

3. Hyperbolic gamma function

The function $\Gamma(z;p,q)$ has the following limiting behaviour [16]:

$$\lim_{v\to 0}\Gamma(e^{-2\pi v y};e^{-2\pi v\omega_1},e^{-2\pi v\omega_2}) = e^{\frac{-\pi(2y-\omega_1-\omega_2)}{12v\omega_1\omega_2}}\gamma^{(2)}(y;\omega_1,\omega_2), \tag{11}$$

where $\gamma^{(2)}(y;\omega_1,\omega_2)$ is the hyperbolic gamma function.

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the integral representation

$$\gamma^{(2)}(y;\omega_1,\omega_2) = \exp\left(-\int_0^{\infty}\left(\frac{\sinh(2y-\omega_1-\omega_2)x}{2\sinh(\omega_1 x)\sinh(\omega_2 x)} - \frac{2y-\omega_1-\omega_2}{2\omega_1\omega_2 x}\right)\right)\frac{dx}{x}, \tag{12}$$

and obeys the equations:

$$\frac{\gamma^{(2)}(y+\omega_1;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)} = 2\sin\frac{\pi y}{\omega_2}, \quad \frac{\gamma^{(2)}(y+\omega_2;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)} = 2\sin\frac{\pi y}{\omega_1}. \tag{13}$$

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the following asymptotics :

$$\begin{aligned}\lim_{y\to\infty}e^{\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2) = 1,\ \arg\omega_1 < \arg y < \arg\omega_2+\pi,\\ \lim_{y\to\infty}e^{-\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2) = 1,\ \arg\omega_1-\pi < \arg y < \arg\omega_2,\end{aligned} \tag{14}$$

where $B_{2,2}(y;\omega_1,\omega_2)$ is the second order Bernoulli polynomial:

$$B_{2,2}(y;\omega_1,\omega_2) = \frac{y^2}{\omega_1\omega_2} - \frac{y}{\omega_1} - \frac{y}{\omega_2} + \frac{1}{6}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right) + \frac{1}{2}. \tag{15}$$

Using (11), one can show [14] that in the limit $p,q\to 1$ relation (3) reduces to the following hyperbolic beta integral evaluation formula [14, 23]:

$$\frac{1}{2}\int_{-i\infty}^{i\infty}\frac{\prod_{k=1}^{6}\gamma^{(2)}(\mu_k+z)\gamma^{(2)}(\mu_k-z)}{\gamma^{(2)}(2z)\gamma^{(2)}(-2z)}\frac{dz}{i\sqrt{\omega_1\omega_2}} = \prod_{1\le a<b\le 6}\gamma^{(2)}(\mu_a+\mu_b), \tag{16}$$

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with the balancing condition

$$\sum_{k=1}^{6} \mu_k = \omega_1 + \omega_2 = Q\,. \tag{17}$$

Consider the function $I_h(\mu_i)$ defined by the integral

$$I_h(\mu_i) = \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^{8} \gamma^{(2)}(\mu_i \pm z)}{\gamma^{(2)}(\pm 2z)} \frac{dz}{i\sqrt{\omega_1\omega_2}} \tag{18}$$

with $\sum_{i=1}^{8} \mu_i = 2Q$. Again using (11), one can show that in the limit $p, q \to 1$ the relations (5) and (7) reduce to the following transformation properties of the function $I_h(\mu_i)$ [24]:

$$I_h(\mu_i) = \prod_{1\le i<j\le 4} \gamma^{(2)}(\mu_i + \mu_j) \prod_{5\le i<j\le 8} \gamma^{(2)}(\mu_i + \mu_j) I_h(\nu_i)\,, \tag{19}$$

where $\nu_i = \mu_i + \xi$, $i = 1, 2, 3, 4$, $\nu_i = \mu_i - \xi$, $i = 5, 6, 7, 8$, and $\xi = \frac{1}{2}(Q - \sum_{i=1}^{4} \mu_i)$, and

$$I_h(\mu_i) = \prod_{1\le i<j\le 8} \gamma^{(2)}(\mu_i + \mu_j) I_h\left(\frac{Q}{2} - \mu_i\right)\,. \tag{20}$$

The asymptotics (14) implies that much more identities, containing smaller number of the hyperbolic gamma functions $\gamma^{(2)}(x)$, can be derived, by taking several parameters in relations (16), (19), (20) to infinity in a certain way. Applying for example the limit

$$\mu_k = f_k + i\xi\,, \quad k = 1, 2, 3 \quad \text{and} \quad \mu_k = g_k - i\xi\,, \quad k = 4, 5, 6\,, \quad z \to z - i\xi, \quad \xi \to -\infty \tag{21}$$

to the hyperbolic beta integral evaluation formula (16), we derive the following star-triangle formula

$$\int_{-i\infty}^{i\infty} \frac{dx}{i\sqrt{\omega_1\omega_2}} \prod_{i=1}^{3} \gamma^{(2)}(x + f_i)\gamma^{(2)}(-x + g_i) = \prod_{i,j=1}^{3} \gamma^{(2)}(f_i + g_j)\,, \tag{22}$$

with the balancing constraint $\sum_{i=1}^{3}(f_i + g_i) = Q$. In many applications to quantum field theory one has $\omega_1 = b$ and $\omega_1 = b^{-1}$. In this case the special notation for the hyperbolic gamma function is used:

$$\gamma^{(2)}(x, b, b^{-1}) \equiv S_b(x)\,. \tag{23}$$

In this notation the star-triangle identity (22) takes the form:

$$\int_{-i\infty}^{i\infty} \frac{dx}{i} \prod_{i=1}^{3} S_b(x + f_i)S_b(-x + g_i) = \prod_{i,j=1} S_b(f_i + g_j)\,. \tag{24}$$

Fixing $g_3 = Q - f_1 - f_2 - f_3 - g_1 - g_2$ in (24), and applying the limit $f_3 \to i\infty$ we obtain:

$$\int_{-i\infty}^{i\infty} e^{i\pi[(y(f_1+f_2+g_1+g_2)+f_1f_2-g_1g_2]} S_b(y + f_1)S_b(y + f_2)S_b(-y + g_1)S_b(-y + g_2)\frac{dy}{i}$$
$$= S_b(Q - f_1 - f_2 - g_1 - g_2)S_b(f_1 + g_1)S_b(f_1 + g_2)S_b(f_2 + g_1)S_b(f_2 + g_2).$$

Next, performing the degeneration $f_2 \to -i\infty$, $g_2 \to i\infty$, $f_2 + g_2 = \alpha$, we derive the pentagon identity [9]:

$$\int_{-i\infty}^{i\infty} e^{i\pi y(2\alpha+g-Q)} S_b(y) S_b(-y+g) \frac{dy}{i} = e^{i\pi(g(\alpha-Q/2)+g^2/2)} S_b(Q-\alpha-g) S_b(\alpha) S_b(g). \qquad (25)$$

Relation (25) encodes an example of $3D$ duality [6], namely mirror symmetry between $N = 2$ SQED with two chiral multiplets and XYZ model on a squashed sphere. Other integral identities have $3D$ mirror symmetry interpretation as well. We would like also to note that the degeneration procedure of taking some parameters to infinity in the physical language corresponds to taking masses of some multipltets to infinity, thus decoupling them.

Similarly one can apply various sequences of the limiting procedures to the symmetry relations (19) and (20) producing many new integral identities. Now we demonstrate one of the most useful applications of this technique. Define the function [15]

$$J_h(g_a, f_a) = \int_{-i\infty}^{i\infty} \prod_{a=1}^{4} \gamma^{(2)}(z+f_a,;\omega_1,\omega_2)\gamma^{(2)}(-z+g_a,;\omega_1,\omega_2)\frac{dz}{i\sqrt{\omega_1\omega_2}}. \qquad (26)$$

This function appears as the most important part in construction of the fusion matrix in two-dimensional Liouville field theory [13], as well as in building of eigenfunctions of the two-particle Ruijsenaars-Schneider model [15]. So studying of the properties and symmetries of this function is an important task. Now we will show how the proper degeneration of the symmetry relations (19) and (20) leads to the symmetries of $J_h(g_a, f_a)$ function. Applying the limiting procedure

$$\mu_k = f_k + i\xi\,, \quad k = 1,2,3,4 \quad \text{and} \quad \mu_k = g_k - i\xi\,, \quad k = 5,6,7,8 \qquad (27)$$

to (20) and also shifting $z \to z - i\xi$ on both sides and taking the limit $\xi \to -\infty$ we obtain [17]

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{j=1}^{4} \gamma^{(2)}(f_j+z;\omega)\gamma^{(2)}(g_j-z;\omega)dz = \prod_{j,k=1}^{4} \gamma^{(2)}(g_j+f_k;\omega) \times \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{j=1}^{4} \gamma^{(2)}(\tfrac{1}{2}Q-f_j+z;\omega)\gamma^{(2)}(\tfrac{1}{2}Q-g_j-z;\omega)dz. \qquad (28)$$

Applying another limiting procedure

$$\begin{array}{llll} \mu_1 = f_1 + i\xi & \mu_5 = f_3 + i\xi & \mu_3 = g_1 - i\xi & \mu_7 = g_3 - i\xi \\ \mu_2 = f_2 + i\xi & \mu_6 = f_4 + i\xi & \mu_4 = g_2 - i\xi & \mu_8 = g_4 - i\xi \end{array} \qquad (29)$$

to (19) and also shifting $z \to z - i\xi$ on both sides and taking the limit $\xi \to -\infty$ we obtain [24]:

$$J_h(\underline{g}, \underline{f}) = \prod_{j,k=1}^{2} \gamma^{(2)}(g_j+f_k;\omega_1,\omega_2) \prod_{j,k=3}^{4} \gamma^{(2)}(g_j+f_k;\omega_1,\omega_2) \times J_h(g_1+\eta, g_2+\eta, g_3-\eta, g_4-\eta, f_1+\eta, f_2+\eta, f_3-\eta, f_4-\eta), \qquad (30)$$

where $\eta = \frac{1}{2}(\omega_1+\omega_2-g_1-g_2-f_1-f_2)$.

4. Complex hypergeometric functions as limits from the hyperbolic integrals

Define the complex gamma function [10]:

$$\mathbf{\Gamma}(x,n)=\frac{\Gamma(\frac{n+\mathrm{i}x}{2})}{\Gamma(1+\frac{n-\mathrm{i}x}{2})},\quad n\in\mathbb{Z},\quad x\in\mathbb{C}. \tag{31}$$

Now set

$$b=\sqrt{\frac{\omega_1}{\omega_2}}=\mathrm{i}+\delta,\quad \delta\to 0^+. \tag{32}$$

Then $\omega_1+\omega_2=2\delta\sqrt{\omega_1\omega_2}+O(\delta^2)\to 0$ and

$$\sqrt{\frac{\omega_2}{\omega_1}}=-\mathrm{i}+\delta+O(\delta^2),\quad \frac{\omega_1}{\omega_2}=-1+2\mathrm{i}\delta+\delta^2,\quad \frac{\omega_2}{\omega_1}=-1-2\mathrm{i}\delta+O(\delta^2). \tag{33}$$

One can show that in this limit uniformly on the compacta [17]:

$$\gamma^{(2)}(\mathrm{i}\sqrt{\omega_1\omega_2}(n+x\delta);\omega_1,\omega_2)\to(4\pi\delta)^{\mathrm{i}x-1}e^{\frac{\pi\mathrm{i}}{2}n^2}\mathbf{\Gamma}(x,n). \tag{34}$$

Consider

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Delta(z)\frac{dz}{\mathrm{i}\sqrt{\omega_1\omega_2}}=\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Delta(\sqrt{\omega_1\omega_2}x)\frac{dx}{\mathrm{i}},\quad x=\frac{z}{\sqrt{\omega_1\omega_2}}, \tag{35}$$

where $\Delta(z)\propto$ a product of $\gamma^{(2)}(u;\omega_1,\omega_2)$.

Rewrite

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Delta(\sqrt{\omega_1\omega_2}\,x)\frac{dx}{\mathrm{i}}=\sum_{N\in\mathbb{Z}}\int_{\mathrm{i}(N-1/2)}^{\mathrm{i}(N+1/2)}\Delta(\sqrt{\omega_1\omega_2}\,x)\frac{dx}{\mathrm{i}} \tag{36}$$

$$=\sum_{N\in\mathbb{Z}}\int_{N-1/2}^{N+1/2}\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}\,x)dx=\sum_{N\in\mathbb{Z}}\int_{-1/2}^{1/2}\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+x))dx. \tag{37}$$

Parameterise $x=y\delta$, $\delta>0$, and take the limit $\delta\to 0^+$

$$\sum_{N\in\mathbb{Z}}\int_{-1/2}^{1/2}\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+x))dx=\lim_{\delta\to 0}\sum_{N\in\mathbb{Z}}\int_{-1/2\delta}^{1/2\delta}\delta\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+y\delta))dy. \tag{38}$$

The sum over N is infinite, for $\delta\to 0^+$ the integration contour becomes $(-\infty,\infty)$. Uniformness of the limit yields for the right-hand side of (38)

$$\sum_{N\in\mathbb{Z}}\int_{-\infty}^{\infty}\left[\lim_{\delta\to 0}\delta\Delta(\mathrm{i}\sqrt{\omega_1\omega_2}(N+y\delta))\right]dy. \tag{39}$$

Apply this procedure to the general univariate hyperbolic beta integral (16). Take the integration variable z and parameters μ_k in (16) in the form:

$$z=\mathrm{i}\sqrt{\omega_1\omega_2}(N+\delta y),\quad y\in\mathbb{C},\quad N\in\mathbb{Z}+\nu,\ \nu=0,\frac{1}{2}, \tag{40}$$

$$\mu_k=\mathrm{i}\sqrt{\omega_1\omega_2}(N_k+\delta a_k),\quad \alpha_k\in\mathbb{C},\quad N_k\in\mathbb{Z}+\nu,\ \nu=0,\frac{1}{2}, \tag{41}$$

The limit $\delta \to 0^+$ of expression (17) leads to the balancing condition:

$$\sum_{k=1}^{6} a_k = -2\mathrm{i}, \quad \sum_{k=1}^{6} N_k = 0. \tag{42}$$

The parameter $\nu = 0, 1/2$ emerges because only the sums $N_k \pm N$ should be integers. Now, using (34) and (39), we obtain in the limit $\delta \to 0^+$ the complex beta integral evaluation formula [4, 17]:

$$\frac{1}{8\pi} \sum_{N \in \mathbb{Z}+\nu} \int_{-\infty}^{\infty} (y^2 + N^2) \prod_{k=1}^{6} \mathbf{\Gamma}(a_k \pm y, N_k \pm N) dy = \prod_{1 \le j < k \le 6} \mathbf{\Gamma}(a_j + a_k, N_j + N_k), \tag{43}$$

with the balancing condition (42) and $\mathbf{\Gamma}(x_1 \pm x_2, n_1 \pm n_2) := \mathbf{\Gamma}(x_1 + x_2, n_1 + n_2)\mathbf{\Gamma}(x_1 - x_2, n_1 - n_2)$. Degeneration of the $W(E_7)$-group transformation laws (19) and (20) brings us [17]:

$$\begin{aligned} &\sum_{N \in \mathbb{Z}+\nu} \int_{-\infty}^{\infty} (y^2 + N^2) \prod_{k=1}^{8} \mathbf{\Gamma}(a_k \pm y, N_k \pm N) dy \\ &= (-1)^L \prod_{1 \le j < k \le 4} \mathbf{\Gamma}(a_j + a_k, N_j + N_k) \prod_{5 \le j < k \le 8} \mathbf{\Gamma}(a_j + a_k, N_j + N_k) \sum_{N \in \mathbb{Z}+\mu} \int_{-\infty}^{\infty} (y^2 + N^2) \\ &\times \prod_{k=1}^{4} \mathbf{\Gamma}(a_k \pm y - \tfrac{1}{2}X - \mathrm{i}, N_k \pm N - \tfrac{1}{2}L) \prod_{k=5}^{8} \mathbf{\Gamma}(a_k \pm y + \tfrac{1}{2}X + \mathrm{i}, N_k \pm N + \tfrac{1}{2}L) dy, \end{aligned} \tag{44}$$

with $X := \sum_{j=1}^{4} a_j$, $L := \sum_{j=1}^{4} N_j$, and

$$\begin{aligned} &\sum_{N \in \mathbb{Z}+\nu} \int_{-\infty}^{\infty} (y^2 + N^2) \prod_{k=1}^{8} \mathbf{\Gamma}(a_k \pm y, N_k \pm N) \mathrm{d}y = \prod_{1 \le j < k \le 8} \mathbf{\Gamma}(a_j + a_k, N_j + N_k) \\ &\times \sum_{N \in \mathbb{Z}+\nu} \int_{-\infty}^{\infty} (y^2 + N^2) \prod_{k=1}^{8} \mathbf{\Gamma}(-\mathrm{i} - a_k \pm y, -N_k \pm N) \mathrm{d}y, \end{aligned} \tag{45}$$

with the balancing conditions

$$\sum_{k=1}^{8} a_k = -4\mathrm{i}, \quad a_k \in \mathbb{C}, \quad \sum_{k=1}^{8} N_k = 0, \quad N_k \in \mathbb{Z} + \nu, \ \nu = 0, \frac{1}{2}. \tag{46}$$

In the relation (44) we have two discrete parameters $\nu, \mu = 0, \frac{1}{2}$. If the integer L is even, then one has $\mu = \nu$. If L is an odd integer, then $\mu \neq \nu$. The relations of type (44) and (45) are important in $\mathrm{SL}(2, \mathbb{C})$ spin chain models [3]. And finally calculating $b \to i$ limit of (28) we obtain [17]:

$$\begin{aligned} &\sum_{N \in \mathbb{Z}} \int_{-\infty}^{\infty} \prod_{k=1}^{4} \mathbf{\Gamma}(s_k + y, N_k + N) \mathbf{\Gamma}(t_k - y, M_k - N) dy = (-1)^{\sum_{k=1}^{4} N_k} \prod_{j,k=1}^{4} \mathbf{\Gamma}(s_j + t_k, N_j + M_k) \\ &\times \sum_{N \in \mathbb{Z}} \int_{-\infty}^{\infty} \prod_{k=1}^{4} \mathbf{\Gamma}(-\mathrm{i} - t_k + y, N - M_k) \mathbf{\Gamma}(-\mathrm{i} - s_k - y, -N - N_k) dy. \end{aligned} \tag{47}$$

with the balancing conditions $\sum_{j=1}^{4}(N_j + M_j) = 0$, and $\sum_{j=1}^{4}(s_j + t_j) = -4\mathrm{i}$.

On both side of the relation (47) there appears a sum of integrals defining $6j$-symbols of $\mathrm{SL}(2,\mathbb{C})$ group [2, 5, 12], i.e. it describes a symmetry relation for these $6j$-symbols. Obviously one can similarly degenerate also the symmetry relation (30) and obtain another transformation property of $\mathrm{SL}(2,\mathbb{C})$ group $6j$-symbols [5]. One can also degenerate the finite difference equation (8), first to the hyperbolic hypergeometric functions level and afterwards to the complex hypergeometric functions thus deriving mixed difference-recurrence relations for them [5, 18].

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