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Maxwell fish eye and Luneburg profiles for polarized light

Mher Davtyan[a,b,*]

[a] *Institute of Radiophysics and Electronics, Alikhanyan Brothers 1, 0203, Ashtarak, Armenia*
[b] *Yerevan Physics Institute, Alikhanian Brothers St. 2, 0036, Yerevan, Armenia*

E-mail: mher.davtyan@gmail.com

Propagation of light in various gradient index (GRIN) media with emphasis on closed ray trajectories of light is considered. Firstly, it is examined the case without considering the light polarization. General mechanism of deducing the expressions of corresponding deformed GRIN profiles is introduced. The deformed GRIN profiles and their corresponding refraction indices preserve the closed ray trajectories of light as well as the symmetries of the Hamiltonian. The deformed versions of Maxwell fish eye and Luneburg profiles were found using the proposed procedure.

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1. Introduction

In order to describe the propagation of light in various media the Fermat principle or the minimal action principle states

$$S_{Fermat} = \frac{1}{\lambda_0}\int dl, \qquad dl := n(\mathbf{r})|d\mathbf{r}/d\tau|d\tau \tag{1}$$

where $n(\mathbf{r})$ is the refraction index, and λ_0 is the wavelength in vacuum. This is the action of the system on the three-dimensional curved space that has the "optical metrics" of Euclidean signature (see. [1])

$$dl^2 = n^2(\mathbf{r})d\mathbf{r}\cdot d\mathbf{r}\ . \tag{2}$$

Therefore, the symmetries of the system in a certain medium coincide with the symmetries of the optical metrics of that particular medium. In systems with maximal number of functionally independent integrals of motion ($2N-1$ integrals for N-dimensional system), all the trajectories of the system appear to be closed. Closeness of the trajectories makes such profiles highly relevant in the study of cloaking and perfect imaging phenomena. The most well-known profile of this sort is the so-called "Maxwell fish eye" profile which is defined by the metrics of sphere or the upper/lower sheet of the two-sheet hyperboloid

$$n_{Mfe}(\mathbf{r}) = \frac{n_0}{|1+\kappa\mathbf{r}^2|}, \qquad \kappa = \pm\frac{1}{4r_0^2} \tag{3}$$

where plus/minus sign in the expression for κ corresponds to the sphere/pseudosphere with the radius r_0, and $n_0 > 0$. Apart from applications in cloaking and perfect imaging phenomena [2–4], Maxwell fish eye is a common profile in quantum optics with single atoms and photons [6], optical resonators [7], discrete spectrum radiation [5] etc. Moreover, there are many experimental implementations of the Maxwell fish eye lenses [8–10].

Another well-known example of a spherically symmetric GRIN medium is the Luneburg lens

$$n_{Lun}(\mathbf{r}) = n_0\sqrt{2-\left(\frac{\mathbf{r}}{r_0}\right)^2}. \tag{4}$$

Note that the refractive index of the medium decreases from $n_0\sqrt{2}$ at its center to n_0 at its surface.

In this paper, following [18] the Hamiltonian formalism for the polarized light propagating in optical medium is presented. Then, a general scheme of the deformation of isotropic refraction index profiles is introduced. It allows us to restore the initial symmetries of the system after one takes the light polarization into consideration. The deformed Maxwell Fish eye profile has the following form

$$n^s_{Mfe}(\mathbf{r}) = \frac{n_{Mfe}(\mathbf{r})}{2}\left(1+\sqrt{1-\frac{4\kappa s^2\lambda_0^2}{n_0}\frac{1}{n_{Mfe}(\mathbf{r})}}\right), \tag{5}$$

where $n_{Mfe}(\mathbf{r})$ is original Maxwell fish eye profile given by (3), and s is the polarization of light. For the linearly/circularly polarized we respectively have $s = 0/1$. Proposed deformation restores all the symmetries of the optical Hamiltonian with Maxwell fish eye profile which were broken after

the inclusion of polarization.
Using similar procedures it is shown that the deformed profile corresponding to the Luneburg refractive profile has the following form

$$n^s_{Lun}(\mathbf{r}) = \sqrt{\frac{1}{2}\left(n^2_{Lun} + \sqrt{n^4_{Lun} - \frac{4s^2 n_0^2}{r_0^2}}\right)}. \tag{6}$$

In fact, when $s \to 0$, the deformed profiles (5) and (6) revert back to their corresponding spinless expressions (3) and (4) respectively. In addition, the general form for deducing the deformations of isotropic profiles is presented.

The paper is organized as follows. In the *Second* section, Hamiltonian formulation of the geometric optical system given by the action (1) is presented. The Hamiltonian formalism for the polarized light propagating in optical medium is formulated and a general mechanism of deducing the deformation of isotropic refraction index profiles is proposed. In addition, a general compact formula which describes the proposed mechanism is introduced. The duality between the Maxwell fish eye and Coulomb profiles is presented in the *Third* section. In the *Fourth* section, applying the proposed scheme, the duality between Lunerburg lens and the the potential of the harmonic oscillator is illustrated. Moreover, the deformation of the Luneburg profile is explicitly found. In the *final* section several concluding remarks are presented. Through the text the following notations are used: $r := |\mathbf{r}|$, $\mathbf{r} := (x_1, x_2, x_3)$, $\mathbf{p} := (p_1, p_2, p_3)$, $p := |\mathbf{p}|$.

2. Hamiltonian formalism

Due to reparametrization-invariance of the action (1), the Hamiltonian constructed by the standard Legendre transformation is identically zero. However, the constraint between momenta and coordinates appears there

$$\Phi := \frac{\mathbf{p}^2}{n^2(\mathbf{r})} - \lambda\!\!\!^{-}{}_0^{-2} = 0. \tag{7}$$

Hence, in accordance with the Dirac's constraint theory [12] the respective Hamiltonian system is defined by the canonical Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{x_i, x_j\} = 0, \tag{8}$$

and by the Hamiltonian

$$\mathcal{H}_0 = \alpha(\mathbf{p}, \mathbf{r})\Phi = \alpha(\mathbf{p}, \mathbf{r})\left(\frac{p^2}{n^2(\mathbf{r})} - \lambda\!\!\!^{-}{}_0^{-2}\right) \approx 0. \tag{9}$$

Here α is the Lagrangian multiplier which could be an arbitrary function of coordinates and momenta, and $i, j = 1, 2, 3$. The notation "weak zero", $\mathcal{H}_0 \approx 0$, means that when writing down the Hamiltonian equations of motion, we should take into account the constraint (7) only after the differentiation,

$$\frac{df(\mathbf{r}, \mathbf{p})}{d\tau} = \{f, \mathcal{H}_0\} = \{f, \alpha\}\Phi + \alpha\{f, \Phi\} \approx \alpha\{f, \Phi\}. \tag{10}$$

The arbitrariness in the choice of the function α reflects the reparametrization-invariance of (1). For the description of the equations of motion in terms of arc-length of the original Euclidian space one should choose (see, e.g. [13])

$$\alpha = \frac{n^2(\mathbf{r})}{p + \lambda_0^{-1} n(\mathbf{r})}, \qquad \Rightarrow \quad \mathcal{H}_{\text{Opt}} = p - \lambda_0^{-1} n(\mathbf{r}). \tag{11}$$

With this choice, the equations of motion take the conventional form [14]

$$\frac{d\mathbf{p}}{dl} = \lambda_0^{-1} \nabla n(\mathbf{r}), \qquad \frac{d\mathbf{r}}{dl} = \frac{\mathbf{p}}{p}, \tag{12}$$

where $dl := \alpha(\mathbf{r}, \mathbf{p}) d\tau$ is the element of arc-length. These equations describe the motion of a wave package with center coordinate $\mathbf{r}$ and momentum $\mathbf{p}$ in the medium with refraction index $n(\mathbf{r})$.

Assume we have the Hamiltonian system given by the Poisson bracket (8) and by the Hamiltonian

$$H = \frac{p^2}{g(\mathbf{r})} + V(\mathbf{r}). \tag{13}$$

In accordance with the Mopertuit principle, after fixing the energy surface $H = E$, we can relate its trajectories with the optical Hamiltonian (9) with the refraction index

$$n(\mathbf{r}) = \lambda_0 \sqrt{2g(\mathbf{r})(E - V(\mathbf{r}))}. \tag{14}$$

Clearly, the optical Hamiltonian (9) (as well as the Hamiltonian (11)) with the refraction index (14) inherits all symmetries and constants of motion of the Hamiltonian (13). Canonical transformations preserve the symmetries of the Hamiltonians and their level surfaces. Hence, we are able to construct the physically non-equivalent optical Hamiltonians (and refraction indices) with the same symmetry algebra.

When taking into account light polarization we should add to the scalar Lagrangian $L_0 = \mathbf{p}\dot{\mathbf{r}} - p + \lambda_0^{-1} n$ the additional term $L_1 = -s\mathbf{A}(\mathbf{p})\dot{\mathbf{p}}$, where s is spin of photon, and $\mathbf{A}$ is the the vector-potential of "Berry monopole" (i.e. the potential of the magnetic (Dirac) monopole located at the origin of momentum space) [13]

$$\mathbf{F} := \frac{\partial}{\partial \mathbf{p}} \times \mathbf{A}(\mathbf{p}) = \frac{\mathbf{p}}{p^3} \tag{15}$$

From the Hamiltonian viewpoint this means to preserve the form of the Hamiltonian (9) and replace the canonical Poisson brackets (8) by the twisted ones

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = s\varepsilon_{ijk} F_k(\mathbf{p}), \qquad \{p_i, p_j\} = 0, \tag{16}$$

where $i, j, k = 1, 2, 3$, and F_k are the components of Berry monopole (15). On this phase space the rotation generators take the form

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} + s\frac{\mathbf{p}}{p} \tag{17}$$

while the equations of motion read

$$\frac{d\mathbf{p}}{dl} = \lambda_0^{-1} \nabla n(\mathbf{r}), \qquad \frac{d\mathbf{r}}{dl} = \frac{\mathbf{p}}{p} - \frac{s}{\lambda_0} \mathbf{F} \times \nabla n(\mathbf{r}), \tag{18}$$

However, the above procedure, i.e. twisting the Poisson bracket with preservation of the Hamiltonian, violates the non-kinematical (hidden) symmetry of the system. To get the profiles admitting the symmetries in the presence of polarization, we use the following observation [16] (see [17] for its quantum counterpart). Assume we have the three-dimensional rotationally-invariant system

$$\mathcal{H}_0 = \frac{p^2}{2g(r)} + V(r), \qquad \{p_i, x_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{x_i, x_j\} = 0. \tag{19}$$

For the inclusion of interaction with magnetic monopole, we should switch from the canonical Poisson brackets to the twisted ones:

$$\{p_i, x_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = s\varepsilon_{ijk}\frac{x_k}{r^3}, \qquad \{x_i, x_j\} = 0. \tag{20}$$

The rotation generators then read

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} + s\frac{\mathbf{r}}{r} \;\;:\quad \{J_i, J_j\} = \varepsilon_{ijk}J_k. \tag{21}$$

By modifying the initial Hamiltonian to

$$\mathcal{H}_s = \frac{p^2}{2g(r)} + \frac{s^2}{2g(r)r^2} + V(r), \tag{22}$$

we find that trajectories of the system preserve their form, but the plane which they belong to, fails to be orthogonal to the axis $\mathbf{J}$. Instead, it turns to the constant angle

$$\cos\theta_0 = \frac{s}{|\mathbf{J}|}. \tag{23}$$

For the systems with hidden symmetries one can find the appropriate modifications of the hidden symmetry generators respecting the inclusion of the monopole field.

For applying this observation on the systems with polarized light, we should choose the appropriate integrable system with magnetic monopole, and then perform the canonical transformation (29) which yields the Poisson brackets for polarized light (20). Afterwards we need to solve the following equation

$$r^2 + \frac{s^2}{p^2} - 2g(p)(E - V(p)) = 0, \quad \Rightarrow \quad p = \frac{n^s_{inv}(r)}{\lambda_0}. \tag{24}$$

So, to preserve the qualitative properties of scalar wave trajectories for the propagating polarized light, we should replace it with the modified index $n^s(r)$ which is the solution (with respect to p) of the following equation:

$$p = \frac{1}{\lambda_0}n\left(\sqrt{r^2 + \frac{s^2}{p^2}}\right), \quad \Rightarrow \quad p = n^s(r), \tag{25}$$

where s is polarization of light.

3. Coulomb-Fisheye duality

The first illustration of the procedures described in the previous section is the relation between the Coulomb Hamiltonian which defines the so-called Coulomb refraction index profile and the free-particle Hamiltonian on the three-dimensional sphere, which defines the "Maxwell Fisheye" refraction index (see e.g. [15]). Firstly, we fix the energy surface of the Coulomb Hamiltonian and get the respective refraction index

$$H_{Coul} - E := \frac{p^2}{2} - \frac{\gamma}{r} - E = 0, \quad \Rightarrow \quad n_{Coul} = \lambda\!\!\!^{-}_0\sqrt{2(E+\gamma/r)}, \qquad \text{where} \quad \gamma > 0. \tag{26}$$

The constants of motion of the Coulomb problem (and of the respective optical Hamiltonian) are given by the rotational momentum and by the Runge-Lenz vector

$$\mathbf{L} = \mathbf{r}\times\mathbf{p}, \qquad \mathbf{A} = \mathbf{L}\times\mathbf{p} + \gamma\frac{\mathbf{r}}{r} \tag{27}$$

which form the algebra

$$\{A_i, A_j\} = -2\varepsilon_{ijk}H_{Coul}L_k, \quad \{A_i, L_j\} = \varepsilon_{ijk}A_k, \quad \{L_i, L_j\} = \varepsilon_{ijk}L_k. \tag{28}$$

Now, let us perform a simple canonical transformation,

$$(\mathbf{p}, \mathbf{r}) \to (-\mathbf{r}, \mathbf{p}). \tag{29}$$

As a result, the first equation in (26) reads

$$r^2 - \frac{2\gamma}{p} - 2E = 0 \quad \Rightarrow \quad p - \frac{2\gamma}{r^2 - 2E} = 0 \tag{30}$$

Interpreting the second equation as an optical Hamiltonian, we get the refraction index profile known as the "Maxwell Fisheye" (3) with the parameters κ, n_0 defined as follows

$$\kappa := -\frac{1}{2E}, \qquad \frac{n_0}{\lambda\!\!\!^{-}_0} := 2\epsilon\kappa\gamma, \tag{31}$$

where $\epsilon = -\mathrm{sgn}(r^2 + 1/\kappa)$.

The integrals of motion (27) result in the symmetry generators of the optical Hamiltonian with the Maxwell Fisheye refraction index

$$\mathbf{L} \to \mathbf{L}, \quad \mathbf{A} \to \frac{\mathbf{T}}{2\kappa}, \quad \mathbf{T} = \left(1 - \kappa r^2\right)\mathbf{p} + 2\kappa(\mathbf{rp})\mathbf{r} = \left(2 - \frac{n_0}{n_{Mfe}(\mathbf{r})}\right)\mathbf{p} + 2\kappa(\mathbf{rp})\mathbf{r}. \tag{32}$$

These integrals form the $so(4)$ algebra for $\kappa > 0$, and $so(1.3)$ algebra for $\kappa < 0$:

$$\{L_i, L_j\} = \varepsilon_{ijk}L_k, \quad \{T_i, L_j\} = \varepsilon_{ijk}T_k, \quad \{T_i, T_j\} = 4\kappa\varepsilon_{ijk}L_k \ . \tag{33}$$

Now, let us consider the Coulomb system with Dirac monopole which is known as "MICZ-Kepler system" [19]. It is defined by the twisted Poisson brackets (20) and by the Hamiltonian

$$H_{MICZ} = \frac{p^2}{2} + \frac{s^2}{2r^2} - \frac{\gamma}{r}. \tag{34}$$

Besides the conserved angular momentum (21), this system has the conserved Runge-Lenz vector

$$\mathbf{A}_s = \mathbf{J} \times \mathbf{p} + \gamma \frac{\mathbf{r}}{r}, \tag{35}$$

which form the following symmetry algebra (28) (with the replacement $(\mathbf{L}, \mathbf{A}) \to (\mathbf{J}, \mathbf{A}_s)$). After performing canonical transformation (29), we get

$$H_{MICZ} = E \quad \Leftrightarrow \quad r^2 + \frac{s^2}{p^2} - \frac{2\gamma}{p} - 2E = 0. \tag{36}$$

Solvinq this quadratic (on p) equation, we get the refraction index given by the expression (5), where the notation (31) is used.

The rotation generator (21) transforms to (17), and the Runge-Lenz vector (35) transforms to $\mathbf{T}_s/\kappa$, where

$$\mathbf{T}_s = \left(2 - \frac{n_0}{n^s_{Mfe}(\mathbf{r})}\right)\mathbf{p} + 2\kappa(\mathbf{rp})\mathbf{r} + \frac{2\kappa s}{n^s_{Mfe}(\mathbf{r})}\mathbf{J}. \tag{37}$$

Along with (17), these generators form the symmetry algebra of the original Maxwell Fisheye profile (33) (where $(\mathbf{L}, \mathbf{T})$ are replaced by $(\mathbf{J}, \mathbf{T}_s)$.

4. Oscillator-Luneburg duality

Another interesting illustration of the scheme, proposed at the end of the first section, is the duality between the refractive profile corresponding to the well-known Luneburg lens and the Hamiltionian of the harmonic oscillator. Assume we have the Hamiltonian of the harmonic oscillator in presence of the magnetic monopole. So, we have $V(\mathbf{r}) = \omega^2\mathbf{r}^2/2$ and hence the Hamiltonian reads

$$H = \frac{p^2}{2} + \frac{\omega^2 r^2}{2} + \frac{s^2}{2r^2} \tag{38}$$

As described in the first section, after fixing the energy surface $H = E$ of the Hamiltonian (38) and performing the canonical transformation (29) we arrive to the following equation:

$$r^2 + \omega^2 p^2 + \frac{s^2}{p^2} - 2E = 0 \tag{39}$$

By solving the equation (39) in terms of p, we interpret the solution as the deformed refractive index profile in the optical Hamiltonian. Introducing the notations

$$r_0^2 := E, \qquad n_0 := \frac{r_0}{\omega}, \tag{40}$$

we arrive to the final form of the deformed Luneburg profile

$$n^s_{Lun}(\mathbf{r}) = \sqrt{\frac{1}{2}\left(n^2_{Lun} + \sqrt{n^4_{Lun} - \frac{4s^2 n_0^2}{r_0^2}}\right)}. \tag{41}$$

Obviously, the above presented expression for the deformed Luneburg profile in the limit $s \to 0$ transforms to Eq.(4).

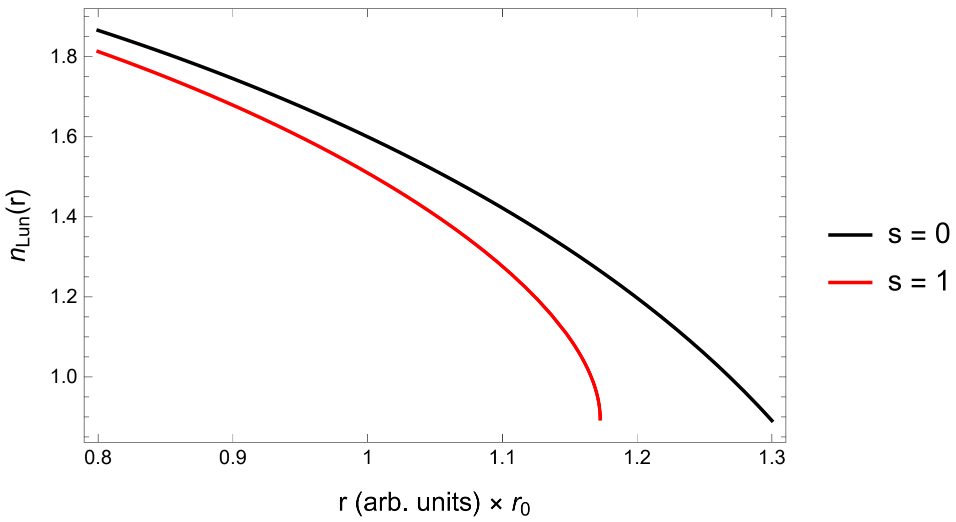

Figure 1: Luneburg refraction index profile for $s = 0$ and $s = 1$ when $n_0 = 1.5$, $r_0 = 5$.

5. Concluding Remarks

In this paper, the results regarding the Maxwell Fisheye profile obtained in [11, 18] were reviewed. In addition, a general mechanism of finding the deformed versions of different refractive profiles were proposed. Using the proposed scheme, the expressions of deformed profiles for the Maxwell Fisheye and Luneburg profiles were found. The properties of the deformed profiles were examined. For the deformed Luneburg profile the refractive index notably differs from the initial profile at the vicinity of $r \sim r_0$. The proposed modification scheme is applicable for any isotropic refraction index $n(r)$, see (25). Proposed deformation preserves the additional symmetries of the system (if any), and thus, guarantees the closeness of trajectories of polarized light.

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