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Strong CP problem

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Quantum chromodynamics, the theory of strong interaction, admits a term which explicitly violates parity P and time reflection invariance T and due to the CPT theorem, also CP. Naturalness would predict this term with strength one, whereas empirical evidence from the electric dipole moment of the neutron yields an upper bound of almost vanishing strength of this term. This is the so-called strong CP problem. In this report, I follow the path from the initial $U(1)_A$ problem of strong interaction physics, via the $U(1)_A$ anomaly and instantons, the theta vacuum, the electric dipole moment of the neutron and the necessity of explicit symmetry breaking to the Peccei-Quinn model, extended to invisible axions and axion-like particles and finally back to electric dipole moments of subatomic particles.

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1. Introduction

This is a condensed report in which I focus on the main points of my two lectures in the PhD school "Aspects of Symmetry" and omit some parts that were either not directly related to my task of lecturing on the strong CP problem or that are already well documented elsewhere, see *e.g.* Refs. [1–3]. In addition, my report is heavily focused on aspects that I found interesting myself and certainly does not do general justice to this topic.

In a kind of dielectic approach, I tried to guide the audience through this topic by following the spirit of the fairy tale about *the hare and the hedgehog*, where the hedgehog(s) stood for a question or a problem that arose, while (the race of) the hare played the role of a proposed solution. As in the fairy tale, of course, each new proposed solution then inherently induced a new problem.

Instead of giving a detailed introduction, I would like to list these problems and the proposed solutions here in sequential order, which at the same time reflects the contents of this report.

2. The first problem: why are there only $N_f^2 - 1$ pseudo-Goldstone bosons?

According to the Goldstone theorem the mesons of the pseudoscalar octet of flavor $SU(3)$ are the pseudo-Goldstone bosons linked to the spontaneous symmetry breaking (SSB) of the chiral flavor group $SU(3)_L \times SU(3)_R$ to $SU(3)_V$ of the Standard Model (SM) and its strong interaction theory, Quantum Chromodynamics (QCD). In the chiral limit, the QCD Lagrangian is even invariant under the larger group[1]

$$\begin{aligned} &U(N_\mathrm{f})_L \times U(N_\mathrm{f})_R = SU(N_\mathrm{f})_L \times SU(N_\mathrm{f})_R \times U(1)_V \times U(1)_A \\ &\xrightarrow{SSB} \{SU(N_\mathrm{f})_L \times SU(N_\mathrm{f})_R / SU(N_\mathrm{f})_V\} \times U(1)_V \times U(1)_A \,. \end{aligned} \tag{1}$$

The associated vacuum remains unchanged only under $SU(N_\mathrm{f}) \times U(1)_V$ signaling invariance under the (vectorial) flavor group $SU(N_\mathrm{f})_V$ and the conservation of baryon number linked to the invariance under a global $U(1)_V$ phase transformation. The quotient group $SU(N_\mathrm{f})_L \times SU(N_\mathrm{f})_R/SU(N_\mathrm{f})$ characterizes the chiral dynamics of the Goldstone bosons which is "hidden" after the breakdown of chiral flavor group. But what is the fate of the additional (axial) unitary group, $U(1)_A$, of the QCD Lagrangian? Is it possible that it is spontaneously broken? The answer is no. If there were an extra pseudo-Goldstone boson, say η^0, it would have to satisfy, according to Weinberg [4, 5], the following bound

$$m_{\eta^0} \overset{!}{<} \sqrt{3} m_\pi \approx 240\,\mathrm{MeV}\,, \tag{2}$$

while empirically [3] the following inequalities hold:

$$\begin{array}{lllll} \text{for } N_\mathrm{f} = 2: & m_{\pi^0} \approx 135\,\mathrm{MeV} \lesssim m_{\pi^\pm} \approx 139\,\mathrm{MeV} & \ll & m_\eta \approx 548\,\mathrm{MeV}\,, \\ \text{and for } N_\mathrm{f} = 3: & m_{\pi_0} \lesssim m_{\pi^\pm} < m_{K^\pm} \lesssim m_{K^0} = m_{\bar K^0} < m_\eta & \ll & m_{\eta'} \approx 958\,\mathrm{MeV}\,, \end{array} \tag{3}$$

which manifestly contradict the Weinberg bound (2).

2.1 Mass term of $U(3)$ pseudo-Goldstone bosons and the $U(1)_A$ problem

In detail, the pseudo-Goldstone boson matrix U reads for $N_\mathrm{f} = 3$ light flavors:

$$U = \exp\left(\frac{\mathrm{i}}{F_\pi} \sum_1^8 \lambda^a \phi^a + \frac{\mathrm{i}}{F_\mathrm{s}} \lambda^0 \eta^0\right) \equiv \mathrm{e}^{\mathrm{i}\tilde\phi/F_\pi} \tag{4}$$

where $F_\pi \approx 92.2\,\mathrm{MeV}$ is here the average axial decay constant of the octet pseudo-Goldstone bosons ϕ^a, while F_s is its counterpart for the would-be singlet pseudo-Goldstone boson η^0. The λ^a, $a = 1, \ldots, 8$ are the usual Gell-Mann matrices, while their singlet equivalent $\lambda^0 \equiv \sqrt{2/3}\,\mathrm{diag}(1,1,1)$ is defined in such way that $Tr(\lambda^a\lambda^b) = 2\delta^{ab}$ holds for all $a, b = 0, 1, \ldots, 8$. Inserting the chiral matrix $U = \exp(\mathrm{i}\tilde\phi/F_\pi)$ with

$$\tilde\phi = \begin{bmatrix} \pi^0 + \frac{1}{\sqrt3}\eta^8 & \sqrt2\pi^+ & \sqrt2 K^+ \\ \sqrt2\pi^- & -\pi^0 + \frac{1}{\sqrt3}\eta^8 & \sqrt2 K^0 \\ \sqrt2 K^- & \sqrt2 \bar K^0 & -\frac{2}{\sqrt3}\eta^8 \end{bmatrix} + \sqrt{\frac{2}{3}} \frac{F_\pi}{F_\mathrm{s}} \begin{bmatrix} \eta^0 & 0 & 0 \\ 0 & \eta^0 & 0 \\ 0 & 0 & \eta^0 \end{bmatrix} \tag{5}$$

[1]In the following, we will leave the number of light flavors $2 \le N_\mathrm{f} \le 3$ general.

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into the usual (leading-order) chiral mass term [6, 7], $\frac{F_\pi^2}{4}\mathrm{Tr}\left(2B_0\mathcal{M}(U+U^\dagger)\right)$, where $\mathcal{M}$ = diag(m_u, m_d, m_s) is the diagonal quark-mass matrix for the three light flavors u, d, and s, we get the following result

$$\begin{aligned}\tfrac{1}{2}B_0\mathrm{Tr}(\mathcal{M}\tilde{\phi}^2) &= \tfrac{1}{2}B_0 m_u\left(\pi^0+\frac{\eta^8}{\sqrt{3}}+\sqrt{\frac{2}{3}}\frac{F_\pi\eta^0}{F_\mathrm{s}}\right)^2+\tfrac{1}{2}B_0 m_d\left(-\pi^0+\frac{\eta^8}{\sqrt{3}}+\sqrt{\frac{2}{3}}\frac{F_\pi\eta^0}{F_\mathrm{s}}\right)^2\\ &+\tfrac{1}{2}B_0 m_s\left(\frac{-2\eta^8}{\sqrt{3}}+\frac{2}{3}\frac{F_\pi\eta^0}{F_\mathrm{s}}\right)^2\\ &+B_0(m_u+m_d)\pi^+\pi^-+B_0(m_u+m_s)K^+K^-+B_0(m_d+m_s)K^0\bar{K}^0\,. \end{aligned}\tag{6}$$

Thus the mass squares of the flavor-charged pseudo-Goldstone bosons $\pi^\pm$, $K^\pm$, and K^0, $\bar{K}^0$ are given by $m^2_{\pi^\pm} = B_0(m_u + m_d)$, $m^2_{K^\pm} = B_0(m_u + m_s)$, and $m^2_{K^0} = m^2_{\bar{K}^0} = B_0(m_d + m_s)$ to that order. The terms of the flavor-neutral pseudo-Goldstone bosons are still mixed and can be collected in the following mass matrix[2]

$$B_0\begin{pmatrix}\pi_0\\ \eta^8\\ \eta^0\end{pmatrix}^T\left[\begin{matrix} m_u+m_d & \frac{1}{\sqrt{3}}(m_u-m_d) & \sqrt{\frac{2}{3}}\frac{F_\pi}{F_\mathrm{s}}(m_u-md)\\ \frac{1}{\sqrt{3}}(m_u-m_d) & \frac{1}{3}(m_u+m_d+4m_s) & \frac{\sqrt{2}F_\pi}{3F_\mathrm{s}}(m_u+m_d-2m_s)\\ \sqrt{\frac{2}{3}}\frac{F_\pi}{F_\mathrm{s}}(m_u-md) & \frac{\sqrt{2}F_\pi}{3F_\mathrm{s}}(m_u+m_d-2m_s) & \frac{2F_\pi^2}{3F_\mathrm{s}^2}(m_u+m_d+m_s)\end{matrix}\right]\begin{pmatrix}\pi_0\\ \eta^8\\ \eta^0\end{pmatrix}.\tag{7}$$

If we assume $m_{u,d} \ll m_s$, we see that this mass matrix has *two* pseudo-zero modes,[3] namely [4, 5]:

$$u_{\pi^0} = (1,0,0)^T \quad \text{with} \quad B_0\, u^T_{\pi^0}[\cdots]u_{\pi^0} = B_0(m_u+m_d) = m^2_{\pi^\pm}\tag{8}$$

and

$$u_\eta = \left(0, 1, \frac{\sqrt{2}F_\mathrm{s}}{F_\pi}\right)^T\frac{1}{\sqrt{1+2F_\mathrm{s}^2/F_\pi^2}} \quad \text{with} \quad B_0 u^T_\eta[\cdots]u_\eta = (1+2)\frac{B_0(m_u+m_d)}{1+2F_\mathrm{s}^2/F_\pi^2} \le 3m^2_{\pi^\pm}\,,\tag{9}$$

where the last inequality holds since F_s^2 and F_π^2 are both positive definite. Note that the equality is approximately realized if $F_\mathrm{s}^2 \ll F_\pi^2$ holds, while for $F_\mathrm{s}^2 = F_\pi^2$ the mass-square of the second pseudo-zero mode agrees with the result of the first one, namely with $m^2_{\pi^\pm}$.

The inequality in (9), which is of course the Weinberg bound (2), describes the so-called $U(1)_A$ problem as it manifestly violates the empirical mass relations $m_\pi^2 \ll m^2_{\eta'}$ for $N_\mathrm{f} = 3$ (or $m_\pi^2 \ll m_\eta^2$ for $N_\mathrm{f} = 2$).

The above considerations were in fact classical ones as they referred to the QCD Lagrangian or to the chiral Lagrangian at tree level. The question might arise what happens to this "classical" $U(1)_A$ symmetry at the quantum level?

3. The solution of the first problem: the $U(1)_A$ anomaly

As discussed in Ulf Meißner's lecture in this school, an anomaly is a quantum obstruction to a classical conservation law, see *e.g.* [8]. Especially, there exists an anomaly in the conservation of

[2] The index T stands here for transposition.

[3] The "pseudo-zero modes" were exact zero modes if m_u and m_d would be exactly zero.

the (abelian and continuous) $U(1)_A$ current J_A^μ of QCD which reads in the chiral limit

$$\partial_\mu J_A^\mu = -\frac{g_s^2 N_f}{8\pi^2}\frac{1}{2}G^c_{\mu\nu}\tilde{G}^{c\mu\nu} = -\frac{g_s^2 N_f}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}G^c_{\mu\nu}G^c_{\rho\sigma}\,. \tag{10}$$

It generalizes the usual Adler-Bell-Jackiw $U(1)_A$ anomaly in terms of the following extensions or substitutions: $\mathrm{Tr}_{\mathrm{flavor}}\mathbb{1} = N_f$, $\mathrm{Tr}_{\mathrm{color}}(t^c t^{c'}) = \frac{1}{2}\delta^{cc'}$ with the color generator matrix $t^c \equiv \lambda^c/2$, and the replacement of the electric charge e by the strong coupling g_s. Note that there is no anomaly in the conservation of the octet analog of the $U(1)_A$ current, since $\mathrm{Tr}_{\mathrm{flavor}}(\frac{1}{2}\lambda^a) = 0$ for the $SU(3)$ flavor matrices, as the Gell-Mann matrices are traceless by definition.

In the "path-integral language", the $U(1)_A$ anomaly arises due to the Jacobian in the fermion measure [9, 10] ($\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = J^{-2}\mathcal{D}\psi\mathcal{D}\bar{\psi}$) resulting from the flavor-singlet axial transformation $\psi_f \to \psi'_f = \mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi_f$:

$$\beta\int \mathrm{d}^4x\,\partial_\mu j_A^\mu \stackrel{!}{=} -\mathrm{i}\ln(J^{-2}) = -\beta\, 2N_f\frac{g_s^2}{32\pi^2}\int \mathrm{d}^4x\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu}\,. \tag{11}$$

3.1 The second problem: perturbative considerations of the $U(1)_A$ anomaly

So it seems that the $U(1)_A$ symmetry of the QCD Lagrangian is broken by the $U(1)_A$ anomaly, which acts as a quantum obstruction to the classical symmetry expressed by the conservation of the $U(1)_A$ current at tree-level in the chiral limit. Note, however, that the integrand in Eq. (11),

$$\frac{g_s^2}{32\pi^2}G^c_{\mu\nu}\tilde{G}^{c\mu\nu} = \frac{g_s^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(G_{\mu\nu}G_{\rho\sigma}) = \partial_\mu K^\mu\,, \tag{12}$$

is a total divergence with the Chern-Simons current

$$K^\mu = \frac{g_s^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(A_\nu G_{\rho\sigma} + \mathrm{i}\tfrac{2}{3}A_\nu A_\rho A_\sigma\right) \tag{13}$$

(see *e.g.* Refs. [2, 11] for more details). This means that the $U(1)_A$ anomaly of QCD is irrelevant in the framework of *perturbation theory* when only perturbative (continuous) so-called *small* gauge transformations are applied. This result even holds for the gauge-invariant, Lorentz-invariant, C- and $P\times T$-invariant, but P- and T-breaking theta term of QCD,

$$\mathcal{L}^\theta_{\mathrm{QCD}} = -\bar{\theta}\frac{g_s^2}{32\pi^2}G^c_{\mu\nu}\tilde{G}^{c\mu\nu} = -\bar{\theta}\frac{g_s^2}{32\pi^2}\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^c_{\mu\nu}G^c_{\rho\sigma}\,, \tag{14}$$

added to the usual QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}G^c_{\mu\nu}G^{c\mu\nu} + \sum_{\mathrm{flavor}\,f}\bar{q}_f\left(\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}gA^c_\mu t^c) - m_f\right)q_f\,. \tag{15}$$

Since the Lagrangian term (14) is proportional to a total divergence $\partial_\mu K^\mu$ it would be irrelevant as well – in perturbation theory!

3.2 The solution of the second problem: $U(1)_A$ anomaly and large gauge transformations

However, non-perturbative (*large*) gauge transformations (so-called *instantons*) exist in Euclidean space-time $\mathbb{R}^4$, such that

$$\int_{\mathbb{R}^4} d^4x_E\,\frac{g_s^2}{32\pi^2}G^c_{\mu\nu}\tilde{G}^c_{\mu\nu} = n \in \mathbb{Z} \tag{16}$$

exists which is topologically protected and nonzero in general.

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3.2.1 Instantons in classical Yang-Mills theory

In Euclidean space-time ($t \to -\mathrm{i}\tau$, $\partial_t \to \mathrm{i}\tau$, $g_{\mu\nu} \to -\delta_{\mu\nu}$) the Yang-Mills action is positive,

$$S_\mathrm{E} = -\mathrm{i}S_\mathrm{M}(t \to -\mathrm{i}\tau) = \tfrac{1}{2}\int d^4x_\mathrm{E}\,\mathrm{Tr}(G^\mathrm{E}_{\mu\nu}G^\mathrm{E}_{\mu\nu}) \overset{!}{\geq} 0\,. \tag{17}$$

Rescaling $A^\mathrm{E}_\mu \to A^\mathrm{E}_\mu/g_\mathrm{s}$ and then dropping the superscript E, we get

$$S_\mathrm{E} = \underbrace{\frac{1}{4g_\mathrm{s}^2}\int d^4x_\mathrm{E}\,\mathrm{Tr}\left((G_{\mu\nu}\mp\tilde{G}_{\mu\nu})(G_{\mu\nu}\mp\tilde{G}_{\mu\nu})\right)}_{\geq 0} \pm \underbrace{\frac{1}{2g_\mathrm{s}^2}\int d^4x_\mathrm{E}\,\mathrm{Tr}\left(G_{\mu\nu}\tilde{G}_{\mu\nu}\right)}_{\equiv 8\pi^2 Q/g_\mathrm{s}^2}, \tag{18}$$

where (see *e.g* Ref. [11] for more details)

$$Q = \frac{1}{16\pi^2}\int \mathrm{d}^4x_\mathrm{E}\,\mathrm{Tr}\left(G_{\mu\nu}\tilde{G}_{\mu\nu}\right) = \oint_{S^3} d\sigma_\mu \frac{-1}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left((\partial_\mu\Omega)\Omega^\dagger(\partial_\alpha\Omega)\Omega^\dagger(\partial_\beta\Omega)\Omega^\dagger\right) \tag{19}$$

is the *topological charge* or Pontryagin index or second Chern class, *i.e.* an integer determined by the large gauge transformation $\Omega(n_\mu)$. Here n_μ is a unit 4-vector in Euclidean space which specifies in which direction $|x_\mathrm{E}| \to \infty$ is assumed on the spatial 3-sphere, S^3, at infinity of the Euclidean space $\mathbb{R}^4$. Note that

$$A_\mu \xrightarrow{|x|\to\infty} -\mathrm{i}(\partial_\mu\Omega)\Omega^\dagger \tag{20}$$

corresponds to a *pure gauge* case, such that $G_{\mu\nu} \xrightarrow{|x|\to\infty} 0$ and the action S_E is finite. In this way, Euclidean space-time at infinity becomes isomorphic to an S^3 sphere, and $\Omega(n_\mu)$ specifies the mapping from this S^3 sphere at infinity into the group-valued S^3 sphere which is isomorphic to the color group $SU(2)$ in the two-color scenario or to the subgroup $SU(2) \subset SU(N_c)$ in the general scenario of a color group $G = SU(N_c)$ with N_c colors. Note that the normalization $1/(24\pi^2)$ in Eq. (19) is the inverse of the product $2\pi^2 \times 3! \times \mathrm{Tr}(\mathbb{1}_{2\times2}) = 24\pi^2$ where $2\pi^2$ corresponds to the volume of a unit 3-sphere and 3! to the number of permutations of the pure gauge terms while the $SU(2)$ relation for the three Pauli matrices $\mathrm{i}\tau_1\mathrm{i}\tau_2\mathrm{i}\tau_3 = \mathbb{1}$ is applied under the trace.

The space of mappings of $S^3 \to G$ consists of an countably infinite set of isolated classes of the homotopy $\Pi_3(G) = \mathbb{Z}$, labeled by the winding number Q of Eq. (19) and by the large gauge transformation (20) with Ω_n ($= (\Omega_1)^n$) and $Q = n$. Note that mappings belonging to one class cannot be continuously deformed into those belonging to any other classes.

3.3 Instantons and the solution of the $U(1)_A$ problem

As discussed and stated in Eq. (16), the non-perturbative (large) gauge transformations (so-called *instantons*) exist in Euclidean space-time $\mathbb{R}^4$ such that the topological charge

$$Q = \int_\mathbb{R} \mathrm{d}x_\mathrm{E}\,\frac{g_\mathrm{s}^2}{32\pi^2}G^c_{\mu\nu}\tilde{G}^c_{\mu\nu} = n \in \mathbb{Z}\,, \tag{21}$$

protected by by topology, is nonzero in general, where the integrand is the *topological density* $q(x_\mathrm{E})$. The self-dual field-configurations $G_{\mu\nu} = +\tilde{G}_{\mu\nu}$ give $Q \geq 0$ and are called *instantons* (if $Q \neq 0$). The anti-self-dual configurations $G_{\mu\nu} = -\tilde{G}_{\mu\nu}$ with $Q < 0$ are called *anti-instantons*. In both cases the Euclidean actual $S_{\rm E}$ in Eq. (18) is not only bounded by the last term in this equation, but assumes this bound, *i.e.*

$$S_{\rm E} = \frac{8\pi^2|Q|}{g_{\rm s}^2}\,. \tag{22}$$

Redefining again $g_{\rm s}A^E_\mu \longrightarrow A_\mu$, we see that the expression for the *one-(anti-) instanton amplitude* for QCD,

$$\mathcal{A}_{\rm E}^{I/\bar I} \propto \exp\left\{-\int {\rm d}^4x_{\rm E}\left(\frac{1}{8g_{\rm s}^2}(G^c_{\mu\nu}\mp\tilde G^c_{\mu\nu})^2 \pm \frac{8\pi^2}{g_{\rm s}^2}\frac{1}{32\pi^2}G^c_{\mu\nu}\tilde G^c_{\mu\nu}\right)\right\} \propto {\rm e}^{-8\pi^2/g_{\rm s}^2(\mu)}\,, \tag{23}$$

is nonzero and proportional to ${\rm e}^{-S_{\rm E}}$ for $|Q| = 1$. In the perturbative weak-coupling case, $g_{\rm s}^2(\mu) \ll 1$, the instanton/anti-instanton amplitude is therefore exponentially small. However, in the strong-coupling regime, where $g_{\rm s}^2(\mu) \sim (4\pi)^2$, there is no suppression. So, the $U(1)_A$ axial current is not conserved in the non-perturbative regime of QCD, $\partial_\mu J^\mu_A \neq 0$. Therefore, there cannot exist a pseudo-Goldstone boson related to the spontaneous breaking of the $U(1)_A$ transformation, as the latter is not a symmetry of the theory [12, 13]. Thus the η' meson is *not* a pseudo-Goldstone boson and there are only $N_{\rm f}^2 - 1$ pseudo-Goldstone bosons, π^+, π^-, π^0 for $N_{\rm f} = 2$ and $\pi^+, \pi^-, \pi^0, K^+, K^-, K^0, \bar K^0, \eta$ for $N_{\rm f} = 3$.

4. The QCD vacuum

4.1 The third problem: instantons and non-trivial vacua in QCD

Because of the large gauge transformations $A_\mu \longrightarrow A_\mu^{(n)} = A_\mu - {\rm i}(\partial_\mu\Omega_n)\Omega_n^\dagger$, $n = Q$, there are infinitely many homotopy classes $\Pi_3(SU(3)) = \mathbb{Z}$ and QCD has a topologically non-trivial vacuum structure with infinitely many minima, *i.e. vacua* $|n\rangle$, at $S_{\rm E} = 8\pi^2|Q|/g_{\rm s}^2$, labelled by the winding number (topological charge) $Q = n$. Furthermore, instantons as large gauge transformations do not only solve the $U(1)_A$ problem [12, 13], but also induce transitions $|n\rangle \longrightarrow |n+1\rangle$ etc. between different vacua.

Thus there are further problems on the horizon. Any naively chosen vacuum of QCD, say $|0\rangle_n \equiv |n\rangle$ with n arbitrary but fixed, has the following properties which are in contradiction with the usual vacuum features:

1. $|0\rangle_n$ is unstable under, *e.g.*, the one-instanton action: $\Omega_1 : |0\rangle_n \longrightarrow |0\rangle_{n+1}$ which transforms the chosen vacuum to a different, but *degenerate* state. Thus the vacuum $|0\rangle_n$ is not *unique*.

2. Furthermore, it is not *gauge-invariant* as large gauge transformations show.

3. Finally, it violates the *cluster decomposition* $\langle O_1O_2\rangle \stackrel{!}{=} \langle O_1\rangle\langle O_2\rangle$ which can be traced back to causality, unitarity and locality of the underlying field theory [5, 14]; *e.g.*let the operator O_1 be the the axial charge operator $Q^\dagger(t_{\rm E})$ and the operator O_2 be the corresponding axial charge $Q(0)$ at Euclidean time $t_{\rm E} = 0$, then both $\langle n|O_1|n\rangle = 0$ and $\langle n|O_2|n\rangle = 0$ hold, but because of the chiral anomaly (and more specifically the Atiyah-Singer index theorem, see, *e.g.*, Ref. [8]) there is still

$$\langle n|O_1O_2|n\rangle = \langle n|O_1|n+2\rangle\langle n+2|O_2|n\rangle \neq 0\,, \tag{24}$$

even for $t_{\rm E} \longrightarrow \infty$, manifestly violating the cluster decomposition theorem [15].

4.2 The solution of the third problem: theta vacuum in QCD

A solution to these stated problems of QCD is to assume that the true vacuum of QCD is a superposition of all $|n\rangle$ vacua. This can be achieved with the help of a newly introduced phase parameter θ:

$$|\mathrm{vac}\rangle_\theta = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}n\theta}|n\rangle\,. \tag{25}$$

Then the action of a large gauge transformation, say Ω_1, would only lead to a phase shift of the vacuum,

$$\Omega_1 : |\mathrm{vac}\rangle_\theta \to \mathrm{e}^{-\mathrm{i}\theta}|\mathrm{vac}\rangle_\theta\,. \tag{26}$$

In this way the three problems affecting the trivial QCD vacua are circumvented.

Note that the phase θ is free but unique since

$${}_{\theta'}\langle\mathrm{vac}|\mathrm{e}^{-\mathrm{i}Ht}|\mathrm{vac}\rangle_\theta = \delta_{\theta-\theta'} \times {}_\theta\langle\mathrm{vac}|\mathrm{e}^{-\mathrm{i}Ht}|vac\rangle_\theta \tag{27}$$

has to hold. Thus θ should be viewed as another parameter of strong interaction physics (similar to the quark-mass parameters m_u or m_d etc.). Because of the phase structure $\mathrm{e}^{in\theta}$, it multiplies the topological charge $Q = n$ as given in the Euclidean integral (21). The pertinent integrand of Q, the topological density, multiplied by θ corresponds therefore to a Lagrangian term which manifestly violates P and T and, assuming the CPT theorem to hold, as it should for a local quantum field theory, also CP. This is the so-called theta-term of QCD, and the full QCD Lagrangian reads

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}^{CP}_{\mathrm{QCD}} + \mathcal{L}^{\not{CP}}_{\mathrm{QCD}} = \mathcal{L}^{CP}_{\mathrm{QCD}} - \theta\frac{g_\mathrm{s}^2}{32\pi^2}\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}\,. \tag{28}$$

An axial rotation of any quark field of flavor f, $q_f \to \mathrm{e}^{\mathrm{i}\beta\gamma_5}q_f \approx (1+\mathrm{i}\beta\gamma_5)q_f$, would alter, via the induced chiral $U(1)_A$ anomaly (11) the coefficient of the theta term and affect the QCD Lagrangian in the following way:

$$\mathcal{L} \to \mathcal{L}^{CP} - 2\beta\sum_f m_f\bar{q}_f i\gamma_5 q_f - (\theta + 2N_\mathrm{f}\beta)\frac{g_\mathrm{s}^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu}\,. \tag{29}$$

This has three implications. First, the real parameter of strong interaction physics is not θ, but the so-called $\bar\theta$ given by

$$\bar\theta = \theta + \arg\det\mathcal{M}\,, \tag{30}$$

i.e. the sum of the original θ plus the phase of the determinant of quark mass matrix $\mathcal{M}$. Second, the $\bar\theta$ parameter is an angle, $\bar\theta \in (-\pi,\pi]$, since, *e.g.*, the instanton amplitude in presence of the theta term is proportional to a phase,

$$\mathcal{A}_{\bar\theta} \propto \mathrm{e}^{-S_\mathrm{E}} \propto \mathrm{e}^{-\int \mathrm{d}^4x_\mathrm{E}\left(\frac{1}{8g_\mathrm{s}^2}(G^2\pm\tilde{G})^2\mp(\frac{8\pi^2}{g_\mathrm{s}^2}\mp\mathrm{i}\bar\theta)\frac{1}{32\pi^2}G\tilde{G}\right)} \propto \mathrm{e}^{-\frac{8\pi^2}{g_\mathrm{s}^2(\mu)}\pm\mathrm{i}Q\bar\theta}\,. \tag{31}$$

As $\mathrm{e}^{iQ\bar\theta}$ is periodic and $Q \in \mathbb{Z}$, $\bar\theta$ has to be an angle parameter, $\bar\theta = \bar\theta + 2\pi$. Third, if a quark mass m_f of any flavor f were zero, then the $\bar\theta$ angle could be removed by a suitable axial rotation with $2\beta_f = -\bar\theta$. The obvious candidate for this to happen is the u-quark, the quark flavor with the lightest quark mass. By now, however, it can be excluded that its quark mass is compatible with zero [3].

5. Strong *CP* problem

The resolution of the $U(1)_A$ problem via instantons and the complicated nature of the QCD vacuum effectively adds an extra term to the QCD Lagrangian

$$\mathcal{L}_{\bar\theta} = -\bar\theta \frac{g_s^2}{32\pi^2}\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}\,, \tag{32}$$

which explicitly violates parity P and time-reflection invariance T, since only ϵ^{0123} and any of its permutations are nonzero. As it conserves charge conjugation invariance C, it violates also CP. One of the consequences of these symmetry violations is the prediction of the (possible) existence of permanent electric dipole moments (EDMs) for subatomic particles of strong interaction physics.

5.1 *CP* violation and electric dipole moments

Classically an electric dipole moment vector is given by separated electric charges of an overall charged or neutral entity multiplied by their relative displacement vectors, $\vec d = \Sigma_i \vec r_i e_i$. Since the displacement vectors are a polar vectors, the classical EDM vector $\vec d$ should also be polar. However, for a subatomic particle with non-vanishing mass in its rest-system there does not exist any vector which could serve as a polar candidate, since, *e.g.*, the velocity of the particle is zero there. Only the spin of the particle remains as a vector, but it is an axial vector. This does not matter for the magnetic moment μ of a particle since the scalar product of its spin vector $\vec S$ and the magnetic field $\vec B$ gives a scalar term in the Hamiltonian $\mathcal{H}$, while in the case of the electric dipole moment d such a product with the electric fields leads to a pseudoscalar term,

$$\mathcal{H} = -\mu\frac{\vec S}{S}\cdot\vec B - d\frac{\vec S}{S}\cdot\vec E\,, \tag{33}$$

which breaks both P and T, *i.e.*

$$O_{P,T}\mathcal{H}O^\dagger_{P,T} = -\mu\frac{\vec S}{S}\cdot\vec B + d\frac{\vec S}{S}\cdot\vec E \tag{34}$$

under P or T. It seems that the existence of an electric dipole moment of a subatomic particle (or a quantum mechanical particle in general) is closely connected to the *explicit* breaking of these symmetries. But would *spontaneous* symmetry breaking perhaps also be sufficient to induce EDMs? The answer must be no according to the following well-known theorem (see *e.g.* Refs. [16–19]):

> *In any finite quantum system in the absence of any explicitly broken symmetry there cannot exist a spontaneously broken ground state or stationary state.*

In fact, the question might even come up why this theorem does not apply to the existence of nonzero magnetic moments of the electron, muon, proton, neutron, other hadrons, baryons, nuclei etc. [3]. Well, as stated in Ref. [20], this theorem applies but the solution is trivial: The non-zero value of the total angular momentum (i.e. the spin in the case of subatomic particles) suffices to induce the (rotational) symmetry breaking since it defines an axis and direction (in the laboratory frame) for the projection of the magnetic moment, which shares the same axial-vector properties as the spin. The appearance of a nonzero magnetic moment for a particle without any spin or angular momentum is forbidden and would indeed come as a surprise.

The existence of a non-zero spin alone, however, does not suffice to generate, via spontaneously symmetry breaking, a *permanent* [4] electric dipole moment, as the EDM could align in the direction of the spin or opposite to this direction. This resembles the well-known case of the discrete-symmetry splitting in a double-well potential which will be discussed next.

5.1.1 Double-well potential and spontaneously symmetry breaking

Let us compare what classical physics and what quantum physics predict for the scenario given in Fig. 1. There, we start out with the symmetric case (state c_1 in the classical world and q_1 in the quantum world). Then we introduce *explicitly* a small perturbation of the symmetry (as described by the classical and quantum states c_2 and q_2, respectively). Finally, we restore the symmetry of the underlying dynamics, in this case the double-well potential. In the classical world, the final state c_3 differs from the initial one c_1, although the symmetry of the potential is restored, since the sphere is located in that half of the potential which was preferred by the explicit symmetry breaking at stage 2. In contrast, the original symmetric state q_1 is restored in the quantum scenario. This follows from tunneling in quantum mechanics which restores with finite probability in some given time interval the original symmetric configuration, depending on the height and the width of the tunneling barrier(s). Even if a quantum system is prepared in an asymmetrically and therefore

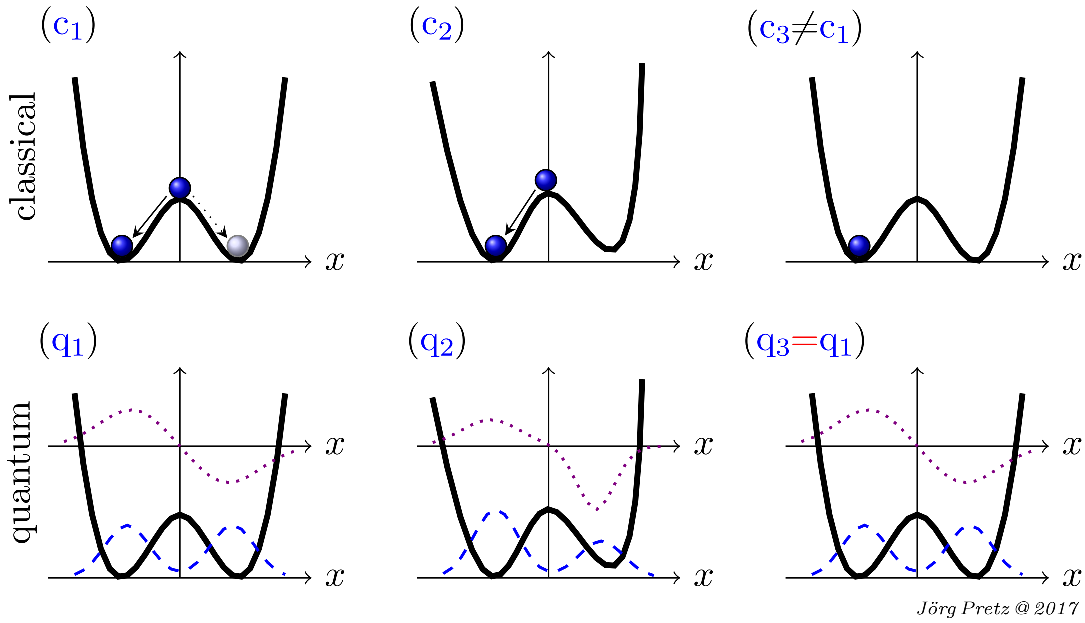

Figure 1: Comparison of classical and quantum-mechanical states (c_i and q_i, respectively) for the double-well potential $V(x)$, centered at $x = 0$, for the scenario $i = 1$: *symmetric case* $\longrightarrow i = 2$: *explicit perturbation of the symmetry* $\longrightarrow i = 3$: restoration of the symmetry of the potential. In the classical case the stationary states are indicated by the position of the sphere and the possible fall directions of the sphere. In the quantum-mechanical case a sketch of the ground state and first excited state are given.

[4] The existence of an *induced* electric dipole moment, however, is compatible with the theorem since the direction and presence of the applied electric field provides the explicit symmetry breaking of the rotational symmetry.

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non-stationary state, it will eventually explore, via tunneling, the full available configuration space, as long as the height and widths of the tunneling barriers are finite.

The message is therefore that in quantum mechanics asymmetric stationary states of finite systems can only exist in the presence of an *explicitly* broken symmetry.

5.2 The fourth problem: a vanishing small electric dipole moment of the neutron

So the *explicit* symmetry breaking of the discrete symmetries P and T is a necessary precondition for the existence of an permanent electric dipole moment of a particle in quantum mechanics and of a sub-atomic particle in particular. Note that the theta-term Lagrangian (32) provides such an explicit symmetry breaking, which because of the coupling to gluons, should be of special importance for hadrons (and of lesser importance due to additional couplings to quark-loops for leptons). The "classical case" is the EDM of the neutron which has been studied since the mid-50s of last century beginning with the work of Smith, Purcell and Ramsey [21]. Purely, by naive dimensional arguments, the following estimate for the size of the EDM of the neutron can be given. Since the spin of the particle is involved, the scale is set by the nuclear magnetic moment of the neutron. However, the fact that there wouldn't be a theta-term if any of the quark flavors f had a vanishing quark mass, *cf.* Eq. (29), should be taken care of as well. An efficient way to so is via the ratio of the *reduced* quark mass of light quarks,

$$m^* = \frac{m_u m_d m_s}{m_u m_d + m_u m_s + m_d ms} \approx \frac{m_u md}{m_u + m_s} \tag{35}$$

to Λ_{QCD}, the scale of long-range, non-perturbative QCD effects. This ratio would vanish if any quark mass were zero. In this way the magnitude of the electric dipole moment of the neutron can be estimated as

$$|d_n| \simeq |\bar\theta| \cdot \frac{m_q^*}{\Lambda_{\mathrm{QCD}}} \cdot \frac{e}{2m_n} \sim |\bar\theta| \cdot 10^{-2} \cdot 10^{-14}\,\mathrm{e\,cm} \sim |\bar\theta| \cdot 10^{-16}\,\mathrm{e\,cm}\,. \tag{36}$$

This estimate should be compared with the present-day experimental bound on the neutron EDM,

$$|d_n^{\mathrm{exp.}}| < 1.8 \cdot 10^{-26}\,\mathrm{e\,cm}\ [90\%\ \mathrm{C.\,L.}], \tag{37}$$

from the nEDM-collaboration [22]. Thus empirically, $\bar\theta$ should be of the size or smaller than 10^{-10}, while a naive dimensional analysis predicts $\bar\theta \in (-\pi, \pi]$ to be of order unity. Note that the other CP-violating phase of the Standard Model, δ_{KM} of the CKM-matrix is indeed of order $O(1)$, *cf.* Ref. [3].

This large mismatch between the naive expectation and the empirical bound on the coefficient of the QCD theta-term is called *the strong CP problem*.

6. Resolutions of the strong CP problem

Various approaches were suggested to solve the strong CP problem, some are listed in Ref. [2]. For instance, *fine-tuning* motivated by many-world scenarios, anthropic principle etc. maybe a way out. Another possibility would be a spontaneously broken CP symmetry such that $\bar\theta = 0$ at the Lagrangian level. However, this does not exclude that a finite non-vanishing $\bar\theta \neq 0$ is reintroduced at the loop level, and anyhow, the already known CP-breaking mechanism in the Standard Model via the KM-mechanism [23] predicts CP-breaking of *explicit* and not spontaneous nature.

The introduction of an additional chiral symmetry might be also a possibility. One way was already discussed, namely the vanishing of the u-quark mass. Modern lattice-QCD calculations exclude this scenario as $m_u = 2.16^{+0.49}_{-0.26}$ MeV [3].

The possibility that is most promising and that we will follow here was introduced by Peccei and Quinn in 1977 [24, 25].

6.1 Peccei-Quinn symmetry and the axion

In fact, a new global chiral $U(1)_{\rm PQ}$ symmetry, namely the Peccei-Quinn (PQ) symmetry, is imposed on the Standard Model (SM). For momentum scales $|p| < f_a$, the global PQ-symmetry is realized non-linearly, *i.e.* spontaneously broken, where f_a is the *order parameter* associated with the spontaneous breaking of the $U(1)_{\rm PQ}$ symmetry [24]. In this way the static angular parameter $\bar\theta$ (mod 2π) is effectively replaced by a *dynamical* pseudoscalar field $a(x)$ which transforms under Peccei-Quinn symmetry as

$$U(1)_{\rm PQ} : f_1^{-1} a(x) \to f_a^{-1} a(x) + \alpha_{\rm PQ} \tag{38}$$

where $\alpha_{\rm PQ}$ is the global $U(1)_A$ group angle. As Weinberg and Wilczek showed in 1978, there necessarily belongs a corresponding Nambu-Goldstone boson to this hidden symmetry, an a priori massless neutral pseudoscalar particle, the so-called axion [26, 27], where the name “axion" was coined by Wilczek.

The SM Lagrangian including the theta-term is then augmented by the axion kinetic term and axion interactions to the standard model matter particles, here represented by the spinor field ψ, where both terms are invariant under the PQ-transformation (38), and finally the coupling of the axion field to the gluons via the topological density:[5]

$$\mathcal{L}_{\text{PQ-SM}} = \mathcal{L}_{\rm SM} - \bar\theta \frac{g_s^2}{32\pi^2} G^c_{\mu\nu}\tilde G^{c\mu\nu} + \frac{1}{2}\partial_\mu a \partial^\mu a + \mathcal{L}^{\rm inv}_{\rm PQ}[\partial^\mu a/f_a, \psi, \bar\psi] + \frac{a}{f_a}\frac{g_s^2}{32\pi^2} G^c_{\mu\nu}\tilde G^{c\mu\nu} \,. \tag{39}$$

The pertinent PQ (Noether) current

$$J^\mu_{\rm PQ} = \frac{\partial \mathcal{L}_{\text{PQ-SM}}}{\partial_\mu a} = \partial^\mu a + \frac{\partial \mathcal{L}^{\rm inv}_{\rm PQ}}{\partial_\mu a} \tag{40}$$

is then anomalous:

$$\partial_\mu J^\mu_{\rm PQ} = \partial_\mu \left(\partial^\mu a + \frac{\partial \mathcal{L}^{\rm inv}_{\rm PQ}}{\partial_\mu a} \right) = \frac{1}{f_a}\frac{g_s^2}{32\pi^2} G^c_{\mu\nu}\tilde G^{c\mu\nu} \,. \tag{41}$$

6.2 The effective potential for the axion field

From the last relation or directly from (39) we can extract the (effective) potential of the axion field as

$$V_{\rm eff}(a) = -\frac{a}{f_a}\frac{g_s^2}{32\pi^2} G^c_{\mu\nu}\tilde G^{c\mu\nu} \tag{42}$$

[5] In Ref. [2] the weight $1/f_a$ in front of interaction term between the axion and the topological density is replaced by ξ/f_a, where ξ is the so-called anomaly coefficient which counts the number of Peccei-Quinn carrying particles. In other publications, $1/f_a$ is replaced by C_G/f_a anticipating the coupling of axion-like-particles to the gluonic term.

in the usual way, such that

$$\frac{\partial V_{\rm eff}(a)}{\partial a} = -\frac{1}{f_a}\frac{g_s^2}{32\pi^2} G^c_{\mu\nu}\tilde{G}^{c\mu\nu}\,, \tag{43}$$

cf. Ref. [2]. In order to get the expectation value of this quantity in the vacuum (actually in the theta vacuum), namely $\left\langle \frac{\partial V_{\rm eff}(a)}{\partial a}\right\rangle$, we apply the path-integral formulation:

$$\langle \hat{O}(\phi)\rangle = \int \mathcal{D}\phi\; O(\phi)\, e^{i\int {\rm d}^4x\, \mathcal{L}[\phi]} \tag{44}$$

where $\hat{O}(\phi)$ is an operator formulated in terms of a field operator $\hat{\phi}$, while $O(\phi)$ is its classical analog. Here we have skipped and will skip the hats on top of $V_{\rm eff}$, the axion field a, etc. In this way we get

$$\begin{aligned}\left\langle \frac{\partial V_{\rm eff}(a)}{\partial a}\right\rangle &= -\frac{1}{f_a}\frac{g_s^2}{32\pi^2}\left\langle G^c_{\mu\nu}\tilde{G}^{c\mu\nu}\right\rangle = -\frac{1}{f_a}\frac{g_s^2}{32\pi^2}\int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu} {\rm e}^{{\rm i}\int {\rm d}^4x\, \mathcal{L}_{\text{PQ-SM}}} \\ &= -\frac{1}{f_a}\frac{g_s^2}{32\pi^2}\int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}a\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu} {\rm e}^{{\rm i}\int {\rm d}^4x\,(\mathcal{L}_{\rm SM}+\mathcal{L}^{\rm inv}_{\rm PQ}+(-\bar{\theta}+\frac{a}{f_a})\frac{g_s^2}{32\pi^2}G^b_{\mu\nu}\tilde{G}^{b\mu\nu})}\,.\end{aligned} \tag{45}$$

Note that the right-hand side of this equation is not zero in general, although

$$G^c_{\mu\nu}\tilde{G}^{c\mu\nu} \overset{P,T}{\longleftrightarrow} -G^c_{\mu\nu}\tilde{G}^{c\mu\nu} \tag{46}$$

under parity P and time reflection T. The reason why the right-hand side can be non-zero is the presence of the theta term $-\int {\rm d}^4x\,\bar{\theta}\frac{g_s^2}{32\pi^2}G^b_{\mu\nu}\tilde{G}^{b\mu\nu}$ in the exponent of the path integral which switches its sign accordingly and which can counter the sign change in (46) when the exponential term is expanded accordingly.

Let us now write the axion field $a(x)$ as

$$a(x) = \langle a\rangle + \delta a(x) \tag{47}$$

where $\langle a\rangle$ is the vacuum expectation value (*i.e.* a constant) and $\delta a(x)$ is the fluctuating field around the vacuum expectation value. In this way we can rewrite Eq. (45) as

$$\begin{aligned}\left\langle \frac{\partial V_{\rm eff}(a)}{\partial a}\right\rangle &= -\frac{1}{f_a}\frac{g_s^2}{32\pi^2}\int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi}\; {\rm e}^{{\rm i}\int {\rm d}^4x\,\mathcal{L}_{\rm SM}}\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu}\, {\rm e}^{{\rm i}\left(-\bar{\theta}+\frac{\langle a\rangle}{f_a}\right)\frac{g_s^2}{32\pi^2}\int {\rm d}^4x\, G^b_{\mu\nu}\tilde{G}^{b\mu\nu}} \\ &\quad\times\int \mathcal{D}\delta a\, {\rm e}^{{\rm i}\int {\rm d}^4x\,(\mathcal{L}^{\rm inv}_{\rm PQ}+\frac{\delta a}{f_a}\frac{g_s^2}{32\pi^2}G^d_{\mu\nu}\tilde{G}^{d\mu\nu})}\,.\end{aligned} \tag{48}$$

Note that this expression is periodic (with period (2π)) in the parameter $\theta_a \equiv -\bar{\theta} + \langle a\rangle/f_a$. In fact, it is an odd function of this parameter, since $\frac{g_s^2}{32\pi^2}\int {\rm d}^4x\, G^{\mu\nu}_b\tilde{G}_{b\mu\nu} = n \in \mathbb{Z}$ switches its sign, if an instanton configuration is replaced by the corresponding anti-instanton configuration with the same absolute value of the winding number, $|n|$. Since also $G^c_{\mu\nu}\tilde{G}^{c\mu\nu}$ switches its sign under this replacement, the overall expression (48) switches then its sign.

Especially, in the one-(anti)-instanton approximation the exponential function in the second but last line of Eq. (48) becomes just $e^{\pm i\theta_a}$. Combined with the implicit sign change of $G^c_{\mu\nu}\tilde{G}^{c\mu\nu}$ when switching from an instanton to an anti-instanton, expression (48) reads then

$$\left\langle \frac{\partial V_{\rm eff}(a)}{\partial a} \right\rangle \propto \sin(\theta_a) = \sin(-\bar\theta + \langle a\rangle/f_a) \tag{49}$$

in the one-instanton approximation, *cf.* Ref. [28].

If in Eq. (48) the parameter θ_a is set to zero, *i.e.*

$$\langle a\rangle := \bar\theta f_a\,, \tag{50}$$

then there isn't any term left in the exponent of the path integral which could counter the sign change in Eq. (46), since the fluctuating field $\delta a(x)$ corresponds to a pseudoscalar particle (*i.e.* the axion) which also changes sign under P or T, such that

$$\int D\delta a \,\exp\left(i\int {\rm d}^4x(\mathcal{L}^{\rm inv}_{\rm PQ} + \frac{\delta a}{f_a}\frac{g_s^2}{32\pi^2}G^d_{\mu\nu}\tilde{G}^{d\mu\nu})\right)$$

is invariant under both operations. Thus, by inserting the condition (50) into (48), we get the extremum (in fact minimum) condition for the expectation value of $V_{\rm eff}(a)$, see Ref. [2] :

$$\left\langle \frac{\partial V_{\rm eff}(a)}{\partial a} \right\rangle\Bigg|_{\langle a\rangle=\bar\theta f_a} = -\frac{1}{f_a}\frac{g_{\rm s}^2}{32\pi^2}\left\langle G^c_{\mu\nu}\tilde{G}^{c\mu\nu}\right\rangle\Bigg|_{\langle a\rangle=\bar\theta f_a} = 0\,. \tag{51}$$

This is compatible with the observation that Eq. (48) is an odd function of the parameter θ_a. This, in turn, is compatible with the vanishing of the argument of the sin-function in the one-instanton approximation (49). So the $\bar\theta$ term is canceled by the $\langle a\rangle$ contribution and the strong CP problem ist solved.

6.2.1 Mass term of the axion

What is about the axion? What is, *e.g.*, its mass?

If one neglects the $U(1)_A$ anomaly of QCD, *i.e.* the term $(-\bar\theta + \frac{a}{f_a})\frac{g_s^2}{32\pi^2}G^c_{\mu\nu}\tilde{G}^{c\mu\nu}$ in the Lagrangian (39), then indeed there are no constraints on the expectation value of the axion field, except that $\langle a\rangle/f_a$ should be interpreted as an angular-valued field:

$$0 \le \frac{\langle a\rangle}{f_a} \le 2\pi\,. \tag{52}$$

The axion would be massless.

However, the inclusion of the QCD anomaly generates a potential for the axion field that is periodic in the effective vacuum angle $\theta_a \equiv -\bar\theta + \langle a\rangle/f_a$. The pertinent generating functional for the effective potential in Euclidean space reads [6]

$$Z(\theta_a) = {\rm e}^{-\int {\rm d}^4x_{\rm E}\,\langle V^{\rm E}_{\rm eff}\rangle} = \left\langle {\rm e}^{{\rm i}\theta_a\frac{g_{\rm s}^2}{32\pi^2}\int {\rm d}^4x\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu}}\right\rangle\,. \tag{53}$$

[6] Note that the $e^{\pm i\theta\cdots}$ contributions in Euclidean space still describe pure phases (as in Minkowski space).

Since for instanton-like field configurations $\frac{g_s^2}{32\pi^2}\int d^4x\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu} = n \in \mathbb{Z}$ holds, the generating functional $Z(\theta_a)$ has to be periodic in the angle θ_a:

$$Z(\theta_a) = \sum_{n=-\infty}^{\infty} e^{in\theta_a} Z_n\,, \tag{54}$$

see *e.g.* Ref. [11]. Moreover, Eq. (53) has to be an even function in θ_a as the odd terms do not survive the transformations (46). Thus $Z(\theta_a) = \sum_{n=-\infty}^{\infty} e^{in\theta_a} Z_{|n|}$, implying $Z_{-n} = Z_n$ and

$$Z(\theta_a) = e^{-\int d^4x_E\,\langle V^E_{\rm eff}\rangle} = \frac{1}{2}\left\langle e^{i\theta_a \frac{g_s^2}{32\pi^2}\int d^4x\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu}} + e^{-i\theta_a \frac{g_s^2}{32\pi^2}\int d^4x\, G^c_{\mu\nu}\tilde{G}^{c\mu\nu}}\right\rangle . \tag{55}$$

Again in the one-instanton approximation (here signaled by $\simeq$), but now in Minkowski space [7] the generating functional is given by

$$\begin{aligned}\langle V_{\rm eff}\rangle &\simeq \ln\left(1 - \cos\theta_a \cdot C_{a_1}\frac{m_q^*}{\Lambda_{\rm QCD}}\cdot\Lambda^4_{\rm QCD}\right)\Big|_{a=\langle a\rangle}\\ &\approx -C_{a_1}\cdot m_q^*\Lambda^3_{\rm QCD}\cos\left(-\bar\theta + \frac{a}{f_a}\right)\Big|_{a=\langle a\rangle} + O\left((m_q^*/\Lambda_{\rm QCD})^2\right)\,,\end{aligned} \tag{56}$$

where the first contribution of the logarithm results from the trivial winding number configuration. This contribution can be absorbed by the normalization of the path integral or, alternatively, corresponds to a finite, θ_a-independent shift of the effective potential by a value proportional to $C_{a_0}\Lambda^4_{\rm QCD}$ for dimensional reasons. The coefficient $C_{a_0} \sim O(1)$ and $\Lambda_{\rm QCD}$ provides the scale in the non-perturbative regime of QCD where g_s is so big that $Z_{|\pm1|} \sim \exp(-8\pi^2/g_s^2)$ can become sizable.

The term $m_q^*/\Lambda_{\rm QCD}$ with the reduced quark mass m_q^* (*e.g.*, $m_q^* = m_u m_d/(m_u + m_d)$ for two flavors) takes into account that the θ_a term can be rotated to zero if any of the quark masses would vanish, *cf.* Eq. (29). The coefficient $C_{a_1} \sim O(1)$ is positive. Thus $\langle V_{\rm eff}\rangle$ indeed takes the minimum value at the extremum condition $\langle a\rangle = \bar\theta f_a$. Then it follows automatically – again in the one-instanton approximation – that

$$\left\langle\frac{\partial V_{\rm eff}}{\partial a}\right\rangle \simeq \frac{1}{f_a}C_{a_1}\cdot m_q^*\cdot\Lambda^3_{\rm QCD}\sin\left(-\bar\theta + \frac{\langle a\rangle}{f_a}\right) \xrightarrow{\langle a\rangle = \bar\theta f_a} 0\,, \tag{57}$$

$$\left\langle\frac{\partial^2 V_{\rm eff}}{\partial a^2}\right\rangle = -\frac{1}{f_a}\frac{g_s^2}{32\pi^2}\frac{\partial}{\partial a}\langle G^c_{\mu\nu}\tilde{G}^{c\mu\nu}\rangle \simeq \frac{1}{f_a^2}C_{a_1}\cdot m_q^*\cdot\Lambda^3_{\rm QCD}\cos\left(-\bar\theta + \frac{\langle a\rangle}{f_a}\right). \tag{58}$$

Thus the (squared) mass of the axion, defined by the second curvature of the effective potential in the ground state (=vacuum), is given by

$$m_a^2 = \left\langle\frac{\partial^2 V_{\rm eff}}{\partial a^2}\right\rangle\Big|_{\langle a\rangle = -\bar\theta f_a} \simeq \frac{1}{f_a^2}C_{a_1}\cdot m_q^*\cdot\Lambda^3_{\rm QCD} > 0\,. \tag{59}$$

In fact, by rotating the complete $\theta_a\frac{g_s^2}{32\pi^2}G^c_{\mu\nu}\tilde{G}^{c\mu\nu}$ term of the Lagrangian (39), with the help of a chiral rotation $q \to e^{i\theta_a\gamma_5/2}q$ and the induced chiral anomaly (resulting from the Jacobian of the path

[7] The θ_a contributions do not change their nature (except sign) under the transition from Euclidean to Minkowski space.

integral measure, *cf.* Eq. (11) or Eq. (29)) into the reduced quark mass term $-m_q^*\bar{q}q \to -m_q^*\bar{q}e^{i\gamma_5\theta_a}q$ with $m_q^* = \frac{m_u m_d}{m_u+m_d}$, one gets, again in the one-instanton approximation,

$$\langle V_{\rm eff}\rangle \simeq \tfrac{1}{2}m_q^* \left\langle \bar{q}e^{i\gamma_5\theta_a}q + \bar{q}e^{-i\gamma_5\theta_a}q\right\rangle = \cos(\theta_a) m_q^* \langle\bar{q}q\rangle\,. \tag{60}$$

Here [29, 30]

$$\lim_{m_q\to 0}\lim_{V_4\to\infty}\langle\bar{q}q\rangle = \lim_{m_q\to 0}\lim_{V_4\to\infty}\frac{1}{V_4}\int dx_E^4\,\langle\bar{q}(x)q(x)\rangle < 0 \quad\text{but}\quad \lim_{m_q\to 0}\lim_{V_4\to\infty}\langle\bar{q}i\gamma_5 q\rangle = 0 \tag{61}$$

was used. Let us now replace $\sum_q m_q\langle\bar{q}q\rangle$ for two flavors, u and d, by the Gell-Mann–Oakes–Renner relation [31]

$$0 < f_\pi^2 m_\pi^2 = -(m_u+m_d)\langle\bar{q}q\rangle + O(m_q^2) \quad\text{with}\quad q = u, d\,. \tag{62}$$

Then we get, in the one-instanton approximation

$$\langle V_{\rm eff}\rangle \simeq -\frac{m_q^*}{m_u+m_d} f_\pi^2 m_\pi^2 \cos(\theta_a)\,, \tag{63}$$

and

$$m_a^2 \simeq \left\langle \frac{\partial^2 V_{\rm eff}}{\partial a^2}\right\rangle\Bigg|_{\theta_a=0} = \frac{1}{f_a^2}\frac{m_q^*}{m_u+m_d} f_\pi^2 m_\pi^2 \tag{64}$$

such that

$$\langle V_{\rm eff}\rangle \simeq -f_a^2 m_a^2\cos(\theta_a) \geq -f_a^2 m_a^2 \quad (=\text{minimum}) \tag{65}$$

$$\left\langle\frac{\partial V_{\rm eff}}{\partial a}\right\rangle \simeq f_a m_a^2 \sin(\theta_a) \quad\text{and}\quad \left\langle\frac{\partial^2 V_{\rm eff}}{\partial a^2}\right\rangle \simeq m_a^2\cos(\theta_a)\,. \tag{66}$$

Equations (62) and (64) show that the square of the axion mass m_a^2 scales as $m_q\cdot\Lambda_{\rm QCD}^3/f_a^2$ and confirm that $C_{a_1}\sim O(1) > 0$.

If the one-instanton approximation is relaxed, the effective potential is rather given by [32]

$$\langle V_{\rm eff}(\theta_a)\rangle = -f_\pi^2 m_\pi^2\sqrt{\frac{m_u^2+m_d^2+2m_u m_d\cos\theta_a}{(m_u+m_d)^2}}\left(1+O\left(m_{u,d}/m_s\right)\right)\,, \tag{67}$$

such that the a-independent term is still the usual vacuum contribution $-f_\pi^2 m_\pi^2$ known from chiral perturbation theory [6, 7] but the axion mass is still given by (64), *i.e.*

$$m_a \approx \frac{\sqrt{m_u m_d}}{m_u+m_d}\frac{m_\pi f_\pi}{f_a}\,, \tag{68}$$

cf. Ref. [3]. Note also that the expressions given in Eq. (67) or Eq. (65) are compatible with the Vafa-Witten theorem [33, 34], since $\langle V_{\rm eff}(0)\rangle \leq \langle V_{\rm eff}(\theta_a)\rangle$.

6.3 The fifth problem: the order parameter f_a in the original Peccei-Quinn model

Not only the axion mass but all the interactions of the axion with standard model particles scale with the inverse $1/f_a$ of the order parameter f_a which is associated with the scale of the spontaneous breaking of the Peccei-Quinn symmetry.

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In the original Peccei-Quinn model [24, 25] the value of the f_a coincided with the usual Higgs vacuum expectation value $v_{\rm EW} \approx 246\,\text{GeV}$ from the electro-weak breaking, where the framework of a two-Higgs model with vacuum expectation values v_1 and v_2 was employed, such that $v_{\rm EW} = \sqrt{v_1^2 + v_2^2}$ and the axion corresponds to a common phase field of the two Higgs fields.

This prediction, namely that $f_a = v_{\rm F}$, was already ruled out by experiment in the 80s of last century. In fact, the original PQ-model predicted the branching ratio [35]

$$\text{BR}(K+ \to \pi^+ + a) \simeq 3 \cdot 10^{-5} \cdot (v_2/v1 + v1/v_2)\,, \tag{69}$$

in contradistinction to the experimental bound [36]

$$\text{BR}_{\rm exp}(K+ \to \pi^+ + \text{nothing}) < 3.8 \cdot 10^{-8}\,. \tag{70}$$

6.4 Invisible axion models

So, the original model with $f_a = v_{\rm EW}$ is empirically excluded, but the case

$$f_a \gg v_{\rm EW} \tag{71}$$

is still viable [2]. This brings us to the so-called invisible axion models. Remember that $1/f_a$ is the scale of the couplings of the axion to the SM matter particles and also to itself. If this scale is very small, observations of the axion are very hard, as all its interactions are strongly suppressed by the tiny $1/f_a$ scale. There is also the gravitational interaction, but that is small to start with. This justifies the name *invisible* axions, as they evade almost all empirical constraints. Another justification for that name are the *light-shining-through-the-wall* experiments, where the appearance and disappearance of axions via their two-photon coupling is tested.

There are basically two classes of invisible axion models: the KSVZ model [37, 38] and the DFSZ model [39, 40], named after the initials of the pertinent authors. In the KSVZ model there is a scalar field σ with $\langle\sigma\rangle \gg v_{\rm EW}$ and a super-heavy quark which carry PQ charge while all quarks and leptons only couple indirectly to the axion. In the DFSZ model there are at least two heavy Higgs doublets and all ordinary quarks and leptons carry PQ charge. In both models at least one electroweak singlet particle acquires a vacuum expectation value and therefore breaks PQ symmetry. These are limiting cases and other models exist which mix the properties of both [3].

6.5 The problem of detecting invisible axions?

As mentioned, the order-parameter of PQ-symmetry breaking f_a should be very large compared with the electroweak scale such that the axion models escape almost all empirical constraints. This on the other hand causes the problem of detecting these so-called invisible axions at all. As mentioned the gravitational interaction of the axions is too small to be detected directly. Of course, if their mass is small enough, axions might contribute or even exhaust the dark matter background. But from the properties of dark matter alone, it is hard to infer the properties of the particles contributing to it, except that the dark matter should be cold and non-thermal. So the dark matter particles should be hardly interacting and most probably non-relativistic. For a direct or indirect detection of axions the suppressed coupling to SM particles is essential. Most experiments focus on the two-photon coupling.

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6.5.1 Two-photon couplings to axions

The QCD anomaly induces also an anomalous axion-coupling to two photons, as the PQ-charge carrying quarks (and leptons) can carry electric charge as well. In the KSVZ models axions do not couple to leptons but only to a heavy quark which, in addition to its Peccei-Quinn charge, might also carry charge e_Q in units of the electric charge e of the proton. The QCD anomaly then induces an anomalous axion-two-photon interaction, *e.g.*:

$$\mathcal{L}^{\rm KSVZ}_{\rm axion} = \frac{a}{f_a}\left(\frac{g_s^2}{32\pi^2} G^c_{\mu\nu}\tilde{G}^{c\mu\nu} + 3e_Q^2 \frac{\alpha_{\rm em}}{4\pi} F_{\mu\nu}\tilde{F}^{\mu\nu}\right), \tag{72}$$

where the factor 3 is the number of colors. The axion-two-photon coupling $a\gamma\gamma = \frac{\alpha_{\rm em}}{4\pi f_a} 3e_Q^2$ is corrected by axion mixing with the lowest pseudoscalar mesons, π^0 and η, by [8]

$$3e_Q^2 \to 3e_Q^2 - \frac{4m_d + m_u}{3(m_u + md)}\,. \tag{73}$$

In general, for KSVZ and DFSZ models the axion-two-photon Lagrangian reads

$$\mathcal{L}_{a\gamma\gamma} = \frac{G_{a\gamma\gamma}}{4} a F_{\mu\nu}\tilde{F}^{\mu\nu} \quad \text{with} \quad G_{a\gamma\gamma} = \frac{\alpha_{\rm em}}{\pi f_a}\left[\frac{E}{2N} - \frac{4m_d + m_u}{3(m_u + md)}\right], \tag{74}$$

where the last term stands again for the axion–π^0 and –η mixings. The factor E/N indicates the relative strength of the electromagnetic (E) to the strong (N) anomaly. For the DFSZ-type models it takes the value $E/N = 8/3$ from $3 \times \left[(\frac{2}{3})^2 + (\frac{-1}{3})^2\right] + (-1)^2$, where the factor 3 indicates again the number of colors and the other terms are the squared charges of the u- and d-quarks and the electron, respectively, *i.e.* the quarks and charged leptons of the first generation in the SM. For KSVZ-type models, the ratio is usually set to zero, *i.e.* $E/N = 0$ assuming $e_Q := 0$.

6.5.2 Window for axion searches

For a detailed status on axion searches, I would like to refer the reader to the Particle Data Group collaboration [3], especially to the included review 90. *Axions and Other Similar Particles*, by A. Ringwald, L. J. Rosenberg, and G. Rybka and to C. O'Hare, "cajohare/AxionLimits" (2020) github.com/cajohare/AxionLimits.

Let me here mentioned only some general classes of searches for axions and axion-like-particles (ALPs) where the latter are still weakly-interacting pseudoscalar particles which couple to SM particles with the strength $1/f_a$. Their mass, however, is not constrained by the axion-mass relation (68) which gives a narrow band (limited by the KSVZ and DFSZ models) in the axion-exclusion plots, but can much larger. In this way, experimentally more accessible regions above the QCD-axion band can be addressed by experimental searches as well:

- In laboratories, involving accelerators of lasers there are the so-called *light-shining-throught-the-wall* experiments which try to detect axions by hitting a wall (or beam dump) with high-intensity laser beams or particle beams and then checking, synchronized with the temporal

[8]The term $-\frac{4m_d+m_u}{3(m_u+md)}$ results as sum from the axion-meson mixing factors, $\frac{1}{2}\frac{m_u-m_d}{m_u+m_d}$ for the π^0 and $\frac{-1}{2}$ for the η times, respectively, their couplings with the pertinent squared quark charges (resulting from the two-photon coupling), namely $3\text{Tr}(\tau_3 Q^2) = 3(4/9 - 1/9) = 1$ for the π^0 and $3\text{Tr}(\mathbb{1}Q^2) = 3(4/9 + 1/9) = 5/3$ for the η, where 3 stands for the loop over three colors, *cf.* Ref. [2].

beam profile, for photons in a cavity with a strong magnet field utilizing the axion/ALP-two-photon coupling.

- Searches for axions from astro sources as the sun use, *e.g.*, helioscopes, *i.e.*. strong magnets that are twice a day aligned in a straight line with the sun.
- Searches for axions from gallactic sources use resonating microwave cavities, again with a strong magnetic field.
- Indirect constraints from astrophysics apply bounds from, *e.g.*, red giants and the supernovae, especially SN 1987a.
- With respect to indirect constraints from cosmology, dark-matter bounds on the oscillations of axion or ALP fields are inferred from the local dark matter background in our galaxy.

All these searches, which mostly for setup reasons are limited to rather narrow but overlapping windows, have so far led to more and more strigent bounds, but not to a positive signal of any axion or ALP [3].

6.6 "The empire strikes back"

Let me end on a note that even the original motivation of axions can be put in question. In Section 6.1 we discussed that through the Peccei-Quinn mechanism the QCD theta-term can be removed and be replaced by the QCD axion. However, this does not exclude that under UV completion of the axion-extended Standard Model, *cf.* Eq. (39), a theta-term might be reintroduced, such that the fine-tune problem is back again.[9] A generic effective Lagrangian for the axion would then read

$$\begin{aligned}\mathcal{L}_{\rm eff}(a) \;=\; & \underbrace{\mathcal{L}_0}_{\text{indep. of } a} + \underbrace{\frac{1}{2}(\partial_\mu a)^2 + \frac{\partial_\mu a}{f_a}\tilde{J}^\mu(\psi,\dots)}_{\text{PQ-invariant}} + \underbrace{\frac{a}{f_a}\frac{g_s^2}{32\pi^2}G^c_{\mu\nu}\tilde{G}^{c\mu\nu}}_{\text{expl. PQ-breaking by QCD anomaly}} \\ & \underbrace{+\Delta\mathcal{L}_{\rm UV}\left(=\epsilon m_{\rm UV}^4\cos(a/f_a+\delta_{\rm UV}\right)}_{\text{explicit PQ-breaking at the UV scale}}, \qquad (75)\end{aligned}$$

such that $\bar{\theta} = \langle a\rangle/f_a$ is calculable in terms of the following CP-violating phases (in the presence of the axion):

- The CP-violating phase generated from the PQ-invariant SM via the Kobayashi-Maskawa mechanism [23] in the CKM-matrix with at least 3 generations of quark-lepton fields: $\delta_{\rm KM}$;
- a CP-violating phase generated from PQ-invariant, beyond the Standard Model (BSM) extensions at a scale $m_{\rm BSM}$: $\delta_{\rm BSM}$;
- a CP-violation phase from the explicit PQ-breaking sector at the UV-scale $m_{\rm UV} \sim M_{\rm Planck}$: $\delta_{\rm UV}$.

[9] I learned about this and the following from Kiwoon Choi's Bethe lectures in Bonn, Germany, in March 2015.

Thus the effective potential of the axion has now four terms

$$V_{\text{eff}}(a) = V_{\text{QCD}}(a) + V_{\text{KM}}(a) + V_{\text{BSM}}(a) + V_{\text{UV}}(a)\,, \tag{76}$$

where the individual contributions are:

- $V_{\text{QCD}} \sim -f_\pi^2 m_\pi^2 \cos(a/f_a)$ describes the explicit PQ-breaking by low-energy QCD, where the minimum of the potential at $a = 0$ ensured that the naively expected $\bar{\theta} \sim 1$ has been removed.

- $V_{\text{KM}} \sim f_\pi^2 m_\pi^2 \times G_{\text{F}}^2 f_\pi^4 \times 10^{-5} \sin\delta_{\text{KM}} \times \sin(a/f_a)$ is the axion-potential resulting from the CP violation by the KM-term of the Standard Model. G_{F} is the Fermi constant, which appears squared as the process has to be flavor-neutral instead of flavor changing and contributes a dimensionless suppression factor $G_{\text{F}}^2 f_\pi^2 \sim 10^{-14}$ in terms of chiral symmetry breaking order parameter f_π. The factor 10^{-5} results from the Jarlskog invariant [41].

- $V_{\text{BSM}} \sim f_\pi^2 m_\pi^2 \times (10^{-2}\text{–}10^{-3}) \times \frac{f_\pi^2}{m_{\text{BSM}}^2} \sin(\delta_{\text{BSM}}) \times \sin(a/f_a)$ is the axion-interaction potential with BSM physics where the factor $(10^{-2}\text{–}10^{-3})$ stands for the loop suppression factor $g^2/(16\pi^2)$ in terms of some coupling constant g to a BSM particle, while δ_{BSM} and m_{BSM} are the pertinent CP-violating phase and the mass scale, respectively, of the BSM physics.

- $V_{\text{UV}} \sim \epsilon m_{\text{UV}}^4 \sin\delta_{\text{UV}} \sin(a/f_a)$ is the axion-interaction to the UV-physics resulting from $\Delta\mathcal{L}_{\text{UV}}$ in Eq. (75).

The three new terms in V_{eff}, Eq. (76), generate a shifted axion vacuum expectation value $\langle a\rangle$ and therefore a new value for $\bar{\theta} = \langle a\rangle/f_a$:

$$\bar{\theta} \sim 10^{-19}\sin\delta_{\text{KM}} + \overbrace{(10^{-2}\text{–}10^{-3}) \times \frac{f_\pi^2}{\text{TeV}^2}}^{10^{-10}\text{–}10^{-11}} \times \left(\frac{\text{TeV}}{m_{\text{BSM}}}\right)^2 \sin\delta_{\text{BSM}} + \epsilon\frac{m_{\text{UV}}^4}{f_\pi^2 m_\pi^2}\sin\delta_{\text{UV}} \tag{77}$$

with ϵ fine-tuned (!) to be $\epsilon < 10^{-10} f_\pi^2 m_\pi^2/m_{\text{UV}}^4 \sim 10^{-88}$ for $m_{\text{UV}} \sim M_{\text{Planck}}$ in order to be compatible with the empirical bound $|\bar{\theta}| \lesssim 10^{-10}$ from the upper experimental limit (37) on the electric dipole moment of the neutron. Therefore, regardless of the existence of BSM physics near the TeV scale, $\bar{\theta} = \langle a\rangle/f_a$ can have any value below the present bound $\sim 10^{-10}$. This in turn leads to the following extension of the naive expression, Eq. (36), for the electric dipole moment of the neutron (where we assume, for the sake of simplicity, that all involved quantities have positive values)

$$\begin{aligned} d_n \;&\sim\; \frac{e}{2m_n}\Bigg[\frac{m_q^*}{\Lambda_{\text{QCD}}}\bar{\theta} + G_{\text{F}}^2 f_\pi^4 \times 10^{-5}\sin\delta_{\text{KM}} + (10^{-2}\text{–}10^{-3}) \times \frac{f_\pi^2}{m_{\text{BSM}}^2}\sin\delta_{\text{BSM}} \\ &\qquad + (10^{-2}\text{–}10^{-3}) \times \frac{f_\pi^2}{m_{\text{UV}}^2}\sin\delta_{\text{UV}}\Bigg] \\ &\sim \left[\frac{m_q^*}{\Lambda_{\text{QCD}}}\bar{\theta}_{\text{UV}} + (10^{-2}\text{–}10^{-3}) \times \frac{f_\pi^2}{m_{\text{BSM}}^2}\sin\delta_{\text{BSM}}\right]. \end{aligned} \tag{78}$$

Note that d_n is dominated by the $\bar{\theta}_{\rm UV}$ contribution induced by the CP-violation in the PQ-breaking sector at the UV scale $m_{\rm UV} \sim M_{\rm Planck}$, $\bar{\theta}_{\rm UV} = \epsilon \frac{m_{\rm UV}^4 \sin\delta_{\rm UV}}{f_\pi^2 m_\pi^2}$, and/or by the BSM contribution near the TeV scale. All other contributions are further suppressed. Namely, all KM contributions are suppressed by at least 10^{-19}, the $\bar{\theta}$ contribution from BSM physics is suppressed relatively to the direct BSM term by the factor $m_q^*/\Lambda_{\rm QCD}$, and the non-theta contribution from the UV physics is suppressed relatively to the BSM physics by $m_{\rm BSM}^2/m_{\rm UV}^2 \ll 1$.

7. Conclusions

Here my story about “hedgehogs" (problems) and “hares" (solutions) comes to a temporary end. The “ball" for further development seems to be in the experimentalists’ court, either by direct or indirect detection of axions or ALPs or by electric dipole moment measurements. In the latter case, one would need more than one measured EDM, *e.g.*for the neutron, proton, electron, deuteron, other light nuclei etc., to disentangle the underlying physics, see *e.g.* Ref. [42].

Only the future might tell whether our “fairy" tale has to be continued, whether axions or non-vanishing electric dipole moments of sub-atomic particles might have be found, eventually.

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